BLS WORKING PAPERS

U.S. Department of Labor
U.S. Bureau of Labor Statistics
Office of Prices and Living Conditions

Occupational Hierarchy by Learning Costs and the Equal Elasticity of Labor Demand Puzzle

Gregory Kurtzon, U.S. Bureau of Labor Statistics

Working Paper 460
September 2015

Occupational Hierarchy by Learning Costs and the Equal Elasticity of Labor Demand Puzzle[1]

Gregory Kurtzon
kurtzon.gregory@bls.gov
Bureau of Labor Statistics
2 Massachusetts Ave. NE
Room 3105
Washington DC, 20212
fax: 202-691-6583
September 2015

[1]All views expressed in this paper are those of the author and do not necessarily reflect the views or policies of the U.S. Bureau of Labor Statistics. I would like to thank Christopher Taber, Dale Mortensen, and Gadi Barlevy, and also John Greenlees, Erick Sager, Tim Erickson, Joe Altonji, Daniel Hamermesh, Peter Meyer, Sabrina Pabilonia, Todd Elder, Chris Jepsen, Joshua Pinkston, and Roger Von Haefen for their help, advice, comments, and suggestions. All errors are my own.

Abstract

Empirical studies that show an elastic labor demand from supply shocks such as immigration and an inelastic labor demand from wage shocks such as changes in the minimum wage contradict the typical model's prediction of an equal elasticity. This paper explains this apparent contradiction by generalizing the typical model of complementarity between skill groups and endogenizes that complementarity. Agents choose among complementary occupations on a hierarchy of heterogeneous learning costs. The new choices of low skilled workers to higher cost/wage occupations offset the effects of low skilled supply and wage shocks, making the effects more and less elastic respectively.

JEL codes: J23, J24, J31, J38, J61, J62

Keywords: Occupational Wage Differential, Occupational Choice, Labor Demand, Specific Human Capital, Immigration, Minimum Wage

1. Introduction

There is a seeming inconsistency between the conclusions of studies on the effects of immigration and the minimum wage. The immigration literature, as surveyed by Friedberg & Hunt (1995), tends to show a high elasticity of labor demand with a 10% increase in the share of immigrants, who are mostly low skilled, causing a 1% decline in native wages and roughly a 1% increase in the number of immigrants causing a 1% decline in low skilled wages. As surveyed by Borjas (1999), the effect on native wages has been found to be positive or negative, but certainly not showing a consistently strong negative effect on native wages. The minimum wage literature, as surveyed by Brown (1999), shows that a 10% increase in the minimum wage reduces employment of low wage earners by a mere 1%, implying a low elasticity of labor demand. This paper provides an explanation for this puzzle with a richer model than the usual labor demand structure, and which endogenizes the elasticity of labor demand.

The apparent inconsistency in elasticity estimates can be seen in what is referred to in Acemoglu & Autor (2011) as the "canonical model". Economists generally treat workers of different skill levels as qualitatively different and complementary inputs in production. For example, suppose all workers are divided into high skilled and low skilled, which are usually defined as college and non-college graduates. Let H denote the number of high skilled workers, L denote the number of low skilled workers, Y denote aggregate output, α_1 and α_2 be constants, and σ denote the Allen elasticity of substitution between high and low skilled workers. Aggregate output could be modelled as a CES production function:

$$
\begin{aligned}
Y &= \left[\alpha_1 H^\rho + \alpha_2 L^\rho\right]^{\frac{1}{\rho}} && (1.1)\\
\frac{1}{1-\rho} &= \sigma = \frac{d\ln\left(\frac{H}{L}\right)}{d\ln\left(\frac{W_L}{W_H}\right)} && (1.2)
\end{aligned}
$$

, where W_H and W_L are the wages of high and low skilled workers respectively. When the supply of one kind of worker rises, its wages fall and those of the other kind rise. Wages can rise arbitrarily high and fall arbitrarily close to zero. Many studies, including Katz & Murphy (1992) for example, use similar frameworks to this which are reasonably consistent with empirical evidence.

There is only one elasticity of substitution that measures the degree of complementarity between different worker types; it is constrained to be constant and would be the same for any kind of shock. Therefore, the high measures of σ for immigration contradict the low measures for the minimum wage. In order to explain this, we need a model that explains the observed complementarity of the canonical model and in so doing allows for more flexibility. This model will include the distinction between present and lifetime wages to explain how these shocks can have counterintuitive effects on observable, current wages.

Classifying workers, or agents, as high and low skilled implies that they have different quantities of the same input, whether it's education, productivity, or some other measure of skill. While it is possible for an individual to have many different dimensions to ability, which could result in comparative advantage in certain areas, this paper explores the implications of a one dimensional measure of skill which does not depend on comparative advantage.[1]

This means that high skilled agents would be able to enter the production function in any way that a lower skilled agent can. But agents of different skill levels might act as complements if they perform different occupations. It is easy

[1] The formal definition of comparative advantage used here is the definition used by Satinger (1993) described below.

to see how different occupations would be complementary in production with each other, and would have to be treated as qualitatively different inputs. What might underlie the canonical model is that differently skilled agents sort themselves into different occupations, which are what drives the observed complementarity.

Performing an occupation would require an agent to take time to learn how to do it, which would be a sunk cost. It's reasonable to expect that the occupations would not have the same learning requirement. An occupation with a higher learning time would need to pay higher current wages in order to compensate agents for a shorter working life, so that they would not have lower lifetime wages. This provides an explanation of why education is associated with higher earnings, similar to Willis (1986), who models education as a fixed cost of performing an occupation as an explanation. However, unlike Willis (1986), it is also reasonable to expect that agents who are more productive in general would also be proportionately more productive in learning and thus have shorter learning times. This makes the learning cost more affordable to higher skilled agents, giving them an incentive to choose the higher cost/wage occupations and crowd the lower skilled into the lower cost/wage occupations. This implies that there is an occupational hierarchy from the lowest cost/skill/wage occupations to the highest.

With this hierarchy, there are different effects of skill supply shocks, such as high or low skilled immigration. This is because skill supply shocks change relative wages, which in turn change occupation choices. The addition of high skilled agents will lower wages in high skilled occupations, and because the occupations themselves are complementary, they will raise wages in other occupations. Some agents who would have chosen high skilled occupations will instead choose previously lower skilled occupations, since they would be getting a lower future return for the same learning cost. These new choices will in turn lower wages in those occupations, causing some of those who would have chosen them to move down further. A cascading process will happen all the way down the occupation ladder. A similar effect would happen if low skilled immigrants entered the economy, cascading agents up the job ladder. In general, similarly skilled agents enter nearby occupations, pushing nearby wages down. Only occupations further up and down the hierarchy with agents whose skill is sufficiently higher or lower have rising wages. If the supply shock was large enough, agents who would have chosen an occupation close to what the supply shock chose will move far enough along the occupation ladder so that the supply shock causes their wages to rise, thus switching from being substitutes with the supply shock to being complements. This explains why similarly skilled agents seem to act like substitutes and differently skilled agents act like complements, as in the canonical model, endogenizing the complementarity between them.

Because some low skilled native agents would move to higher paying occupations in response to low skilled immigration, they would be earning more in current wages, offsetting or even overcoming the downward effect on nearby wages. Since current wages are what are observed, a large supply shock could have a small total observed effect on current native wages, and the measured elasticity of substitution would be high. Or, the total effect on the observed wages of some agents could even be positive.

The effects of a wage shock, such as a change in the minimum wage, are damped similarly. Agents who cannot make the minimum in their current occupation could still work if they learned a higher wage occupation. Since it wasn't previously worth the cost in lifetime wages to do a higher wage occupation, agents will choose the lowest occupation that allows them to make the minimum. Since other agents will do the same and push down wages in this occupation, agents will cascade up the hierarchy as needed. The only agents who would become unemployed are those that are currently learning their new occupation and the ones that don't have enough skill to learn any occupation that would satisfy the minimum. The small overall effect on employment would cause the measured elasticity of labor demand to be low. Thus a large labor supply shock would have a low effect on native wages at the same time that a large wage shock would have a small effect on employment.

The model of this paper uses an innovative mechanism. Models where wages are determined by the assignment of agents among occupations, or assignment models, are surveyed in Satinger (1993). In all of the models surveyed, the assignment is driven by at least one of three effects: the scale effect, where only one agent can do a particular function at a time; comparative advantage, defined as when the ratios of outputs between two agents differ across occupations; and preferences, where agents choose occupations based on idiosyncratic tastes. In this model, by contrast, agents themselves are perfect substitutes within an occupation - two agents with equal productivity produce the same as one with double the productivity, so no scale effect applies. Also because of this, the ratio of outputs between two agents will always be the same across occupations, so the model does not exhibit comparative advantage. There are also no differences in preferences needed for the model's implications. This means the model of this paper does not fit into the typical types of models: linear programming, differential rents, or the Roy (1951) model.[2] This model also has the benefit of not having to make restrictive functional form assumptions that do not come from theory, and is intuitive and tractable. However, being discrete has the cost of precluding any continuous function for elasticities between skill levels. Two other studies, Teulings (2005) and Costrell & Lowery (2004), modeled a cascading effect of occupation changes due to a supply shock, and endogenize the complementarity between skill levels with it, but without lifetime vs. current wages.[3] While these papers model important aspects of the market and may be relevant for other issues, only the model

[2]The fact that it does not have comparative advantage also separates it from Acemolgu & Autor (2011), who develop an assignment model with three skill groups which is used to study the effects of technological change on the wage distribution.

This is also different from Welch (1979), who also explicitly models occupations as the reason for the observed complementarity between differently skilled agents with experience being the measure of skill, but does not model how agents choose occupations. Instead, occupations are a 1-to-1 function of a agent's experience, so there is no cascading.

[3]Teulings (2005) is based on comparative advantage while Costrell & Lowery (2004) is based on a scale-like effect where occupations differ in ability sensitivity and agents in an occupation diminish (perhaps to zero) the output of other agents in the same occupation.

of this paper can explain the immigration-minimum wage elasticity puzzle.

In section 2, the economy is modelled as the product of many distinct occupations that agents perform. In section 3, the model is used to explain why differently skilled agents appear to be different and complementary inputs while similarly skilled agents act like substitutes, and how skill supply changes such as an immigration shock would move agents across occupations and change wages. It is shown that an immigration shock would have a lower effect on the observed wages of natives. The implications for a wage shock and the damped effect on employment is shown. Section 4 summarizes and discusses.

2. Model

2.1. Model Specification

Labor is divided into different occupations which have diminishing marginal products and are complementary with each other. This is represented by an aggregate production function which uses labor as the primary input and occupation aggregates as intermediate inputs. Let Y denote aggregate output, I_j denote the quantity of occupation j services, and J denote the total number of occupations. Then

$$Y = F\left(I_1, I_2, I_3, ..., I_J\right) \tag{2.1}$$

$$F_j > 0, F_{jj} < 0, F_{ij} > 0 \tag{2.2}$$

for all levels of I_j $\forall j$. Occupation complementarity is modelled by the $F_{ij} > 0$ condition. F has constant returns to scale. All markets are competitive.

There is a continuum of agents indexed by their skill, s, which is a one dimensional measure of productivity, with the lowest and highest skilled agents having skill $\underline{s}$ and $\overline{s}$ respectively,

$$0 \leq \underline{s} \leq \bar{s} \leq \infty \quad . \tag{2.3}$$

The measure of agents with skill less than or equal to s is $N(s)$, while $n(s)$ denotes the measure of agents with skill s. The total population is the integral of $n(s)$ over s, denoted by N. N is finite, as is the expectation of s with respect to $N(s)$, i.e. $\int_{-\infty}^{\infty} s dN(s) < \infty$. $N(s)$ will be treated as exogenous.

Each agent can be in only one occupation. The occupation aggregates are produced from the skill of the agents in that occupation, with the units of each of the occupation aggregates scaled to be the same as units of skill. The skill of each agent is perfectly substitutable with the skill of each other agent within that occupation. Therefore, the amount of each occupation aggregate is simply the sum of the skill that each agent in that occupation contributes and the marginal product of skill in an occupation will be the same for each skill unit in that occupation. Let w_j denote the price of skill in occupation j, let $w_j(s)$ denote the current wages of an agent of skill s in occupation j, $W(s)$ denote the lifetime wages of an agent of skill s, and $W_j(s)$ be the lifetime wages of an agent of skill s conditional on being in occupation j. Then

$$w_j(s) = w_j s \tag{2.4}$$

.

Each agent must spend time learning the occupation before it can be performed. Let t_{sj} denote necessary time devoted to learning occupation j, and let the total time in an agent's career be scaled to 1. Then

$$W_j(s) = w_j s (1 - t_{sj}) \tag{2.5}$$

. Since skill is a measure of how productive agents are at everything they do, let an agent's learning speed be proportionate to skill: an agent with twice the skill of another can learn an occupation in half the time. Then the learning time can be put in terms of a cost in skill units, $E_j \equiv s t_{sj}$, for the fixed entrance cost of learning occupation j. Then

$$W_j(s) = w_j (s - E_j) \tag{2.6}$$

.

The entrance costs are heterogeneous across occupations: some are easier to learn than others. The occupations are numbered according to

$$0 \equiv E_0 < E_1 < E_2 < E_3 < \cdots < E_J < \bar{s} \tag{2.7}$$

so that the lower numbered occupations have the lower entrance costs. The condition that $E_J < \bar{s}$ insures that it is possible for every occupation to have some agents.

Each agent chooses which occupation to enter in order to maximize lifetime wages given the entrance cost of the occupation. Then,[1]

$$W(s) = \max_j \{W_j(s)\} = \max_j \{w_j(s - E_j)\} \quad . \tag{2.8}$$

If no occupation offers a positive wage, agents can choose occupation 0, which is non-participation. occupation 0 has no entrance cost and pays no wages, so that $E_{0,}W_0(s) \equiv 0$.

The final good, Y, is produced by a single producer, and each occupation aggregate, I_j for occupation j, is produced by an occupation service producer, each of which operate in competitive markets for all goods.

2.2. Equilibrium

Definition 1. *An equilibrium is a set of prices of skill in each occupation,* $\{w_j\}_{j=1}^{J}$, *a set of prices of occupation aggregates* $\{p_j\}_{j=1}^{J}$ *and quantities* $\{I_j\}_{j=1}^{J}$*such that:*
(1) *The final goods producer maximizes profits:*

$$\max_{\{I_j\}_{j=1}^{J}} \left\{ F(I_1, I_2, I_3, ..., I_J) - \sum_j p_j I_j \right\} \quad ; \tag{2.9}$$

[1] Lifetime wages are the only argument in utility and so are treated as equivalent to the agent's utility level.

(2) *occupation aggregate producers maximize profits:*

$$\max_{\{I_j\}_{j=1}^{J}} \{p_j I_j - w_j I_j\} \quad ; \tag{2.10}$$

(3) *Individuals choose an occupation to maximize lifetime wages:*

$$\max_j \{W_j(s)\} = \max_j \{w_j(s - E_j)\} \quad ; \tag{2.11}$$

(4) *Markets clear:*

$$I_j = \int_{\underline{s}}^{\bar{s}} \Pr(j|s)\, s dN(s) \tag{2.12}$$

where $\Pr(j|s)$ *is the probability of an agent choosing occupation j conditional on having skill* s.

Since the profits of each occupation aggregate producer must equal zero, the price of occupation aggregate j, p_j, must equal the price of skill in each occupation, w_j, in any competitive equilibrium. Since $p_j = w_j$ in equilibrium, the price of an occupation and the price of skill in an occupation will be used interchangeably, using the notation w_j to refer to both the price of a unit of skill in each occupation j as well as the price of I_j for each occupation j.

From the first order conditions of (1): $F_j \leq w_j$, and $F_j = w_j$ when $I_j > 0$.

Proposition 2. *If an occupation has a higher entrance cost and a lower skill price than another occupation, all agents would be worse off in it and no agents would choose it. Because of the properties of* F*: (1) any empty occupation must have an infinite skill price and thus, (2) no occupation can be empty in equilibrium.*

Proof. See Appendix. ■

Proposition 3. *In any equilibrium,*

$$0 < w_1 < w_2 < \cdots < w_j < \cdots w_J \quad . \tag{2.13}$$

Proof. See Appendix. ■

This implies that the higher the occupation number, the higher the occupation's skill price as well as entrance cost.

Consider an agent choosing between two occupations, j and k. If $j > k$, the agent will choose occupation j over k if and only if:

$$w_j \left(s - E_j\right) > w_k \left(s - E_k\right) \ , \tag{2.14}$$

or, by rearranging, if and only if:

$$s > \frac{w_j E_j - w_k E_k}{w_j - w_k} \quad . \tag{2.15}$$

This means the choice between any two occupations can be determined by a skill cutoff, such that if an agent's skill is above the cutoff they will choose the higher numbered occupation, and vice versa. If $s = \frac{w_j E_j - w_k E_k}{w_j - w_k}$ the agent is indifferent. Let

$$s_{jk} \equiv \frac{w_j E_j - w_k E_k}{w_j - w_k} \tag{2.16}$$

be the skill cutoff between occupations j and k. However, as is shown in Lemma 8 in the Appendix, only the skill cutoffs between adjacent occupations will matter for characterizing equilibria. For more simple notation, let

$$s_{j+1,j} \equiv s_j \tag{2.17}$$

so that only the J skill cutoffs between adjacent occupations will be considered. In any equilibrium, because $w_0 \equiv 0$, $s_0 = E_1$.

It is first shown that these cutoffs must be ordered, or else there would be an occupation that no agent chooses.

Proposition 4. *Let the set of skill cutoffs,* $\{s_j\}_{j=0}^{J-1}$*, be defined as:*

$$s_j \equiv \frac{w_{j+1}E_{j+1} - w_j E_j}{w_{j+1} - w_j} \quad . \tag{2.18}$$

In any equilibrium $\{s_j\}_{j=0}^{J-1}$ *is ordered from least to greatest:*

$$s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_{J-1} \quad . \tag{2.19}$$

Proof. See Appendix. ■

Occupations with higher costs have higher skill prices and higher skill levels, and thus higher wages, meaning there is a hierarchy of occupations. This equilibrium is proven below.

This is an outline of the proof. Suppose agents sorted themselves according to an arbitrary set of skill cutoffs. Any set of cutoffs implies an allocation of agents into occupations, which determines a set of occupation aggregates and thus skill prices. These skill prices in turn determine the agents' next occupation choices, and so on. In the proof in the Appendix it is shown that the agents' choices will eventually converge on one set of skill cutoffs, a fixed point. It is then shown that this fixed point satisfies the conditions for an equilibrium, so an equilibrium exists.

Theorem 5. *There exists an equilibrium characterized by the set of skill cutoffs:*

$$s_j = \frac{w_{j+1}E_{j+1} - w_j E_j}{w_{j+1} - w_j} \tag{2.20}$$

s.t. each agent of skill s *chooses an occupation s.t.:*

$$s_{j-1} \leq s \leq s_j \quad . \tag{2.21}$$

Proof. See Appendix. ■

2.3. Implications

Consider what would happen if there was a supply increase of agents of a certain skill level, denoted s'. Suppose $s_{j-1} < s' < s_j$, so that initially the new entrants choose occupation j. This would increase the occupation aggregate for j, I_j, and lower the skill price in occupation j, w_j, since $F_{jj} < 0$. All other occupation prices would rise since $F_{jk} > 0 \forall j \neq k$.

All movements of agents across occupations discussed here will be considered to be the choices that agents would have made according to the new skill prices, as opposed to agents who have already paid the entrance cost for an occupation moving to a new one, and so are steady state changes. Agents could choose to pay the cost of another occupation, but that cost would now be a larger portion of their remaining lifetime. However, new agents entering the economy who have not paid a sunk cost would choose a different occupation from what they would have chosen if there had been no shock of s' skilled agents, and if the shocks were anticipated, or if they were trends, agents would have already chosen an occupation according to the effects of the supply change. Therefore the steady state equilibrium can ignore the entrance costs already paid by existing agents. However, as described below, there is good reason to believe that in the cases relevant to the elasticity puzzle, the steady state is reached quickly in any case.

As skill prices change, fewer agents would choose occupations with falling prices and more agents would choose occupations with rising prices. This is because

$$\frac{\partial s_j}{\partial w_j} = \frac{w_{j+1}\left(E_{j+1} - E_j\right)}{\left(w_{j+1} - w_j\right)^2} > 0 \tag{2.22}$$

$$\frac{\partial s_{j-1}}{\partial w_j} = -\frac{w_{j-1}\left(E_j - E_{j-1}\right)}{\left(w_j - w_{j-1}\right)^2} < 0 \quad , \tag{2.23}$$

so that s_j would fall, s_{j-1} would rise, and agents would move from occupation j to occupation j-1 and j+1. All else equal, this would lower w_{j+1} and w_{j-1}, and the same process would tend to happen for w_{j+2}, w_{j-2}, and so on up and down the occupation ladder. Thus, agents added to the economy tend to cause a cascade of higher skilled agents up the occupation ladder and lower skilled agents down the occupation ladder.[2]

While there would be a tendency for this cascading process to happen for any aggregate production function F, it would not necessarily be *monotonic*.[3] The CES production function is used as an example of a case where cascading is monotonic.

Consider the following experiment. Let there be an increase in the mass of agents of skill s' by an amount ε creating a new skill distribution, $N_\varepsilon(s)$. Therefore, the new skill distribution would be defined as:

$$N_\varepsilon(s) = \begin{cases} N(s) \text{ if } s < s' \\ N(s) + \varepsilon \text{ if } s \geq s' \end{cases} . \tag{2.24}$$

Let $\{s_j(\varepsilon)\}_{j=0}^{J-1}$, $\{I_j(\varepsilon)\}_{j=1}^{J}$, $\{w_j(\varepsilon)\}_{j=1}^{J}$, and $Y(\varepsilon)$ denote the new equilibrium skill cutoffs, occupation aggregates, skill prices, and aggregate output for the skill distribution $N_\varepsilon(s)$. The case when $\varepsilon = 0$ refers to the equilibrium before the increase in supply of s' skilled agents.

[2] Upward cascading would also occur if new high cost occupations were added to the economy, increasing J. If job $J+1$ were added, initially the skill price w_{J+1} would be infinite. At that point $s_J = \frac{w_{J+1}E_{J+1} - w_J E_j}{w_{J+1} - w_J}$ would be E_{J+1} and agents would move from J to $J+1$ (if there are agents in the economy with $s > E_{J+1}$), raising w_J, and so causing upward cascading similar to the above. However, this paper only explores shocks to a fixed occupation structure.

[3] In other words, as more and more agents of skill s' were added, some skill cutoffs may move away from s' at first, with relative skill prices acting accordingly. A reason this might happen is that it is possible certain occupations may be very complementary or substitutable with certain other occupations. If this were so, a change in one occupation aggregate due to a supply change of s' skilled agents could cause a large enough change in another occupation's skill price to outweigh any cascading effect. For example, suppose s' skilled agents entered the economy in occupation j. Suppose j was very complementary with job $j+3$, and very substitutable with job $j+4$. If the job price of $j+3$ rose relative to $j+4$, s_{j+3} could rise, and some agents would move in the opposite direction of the cascading. However, this effect is due to the aggregate production function having large asymmetries in the complementarity between occupations. A sufficiently symmetric and well behaved function would not have this issue.

Proposition 6. *If F has a CES form,*

$$Y = [c_1 I_1^{\rho} + c_2 I_2^{\rho} + \cdots + c_J I_J^{\rho}]^{\frac{1}{\rho}} \quad , \tag{2.25}$$

$\rho < 1$, for $\varepsilon > 0$, if $s_{j-1}(\varepsilon) \leq s' \leq s_j(\varepsilon)$:

$$\text{(1)} \quad \frac{\partial s_k(\varepsilon)}{\partial \varepsilon} < 0 \forall k > j, \frac{\partial s_k(\varepsilon)}{\partial \varepsilon} > 0, \forall k < j \quad ; \tag{2.26}$$

$$\text{(2)} \quad \frac{\partial \frac{w_{k+1}(\varepsilon)}{w_k(\varepsilon)}}{\partial \varepsilon} > 0, \forall k > j, \text{ and } \frac{\partial \frac{w_{k+1}(\varepsilon)}{w_k(\varepsilon)}}{\partial \varepsilon} < 0, \forall k < j \quad . \tag{2.27}$$

Proof. See Appendix. ■

This means an increase in skill s' agents would: (1) move all skill cutoffs closer to s' so that agents cascaded only away from the occupation the s' skilled agents first entered; (2) the skill price ratios between higher occupations would rise, and the ratios between lower occupations would fall. The latter point implies that occupations further away from the one that s' skilled agents enter have rising skill prices relative to occupations closer to it.

For any production function F, it can still be shown that the movement of agents up and down the occupation ladder would hold as long as the supply increase was large enough, or in other words, in the limit as $\varepsilon \to \infty$. Consider what would happen as even more s' agents were added. Eventually the economy would approach a limiting state where every occupation would have some s' skilled agents without assuming CES, as described in Proposition 7 below.

Proposition 7. *There exists a $\delta < \infty$, s.t. when $\varepsilon = \delta$, s' skilled agents are in every occupation.*

Proof. See Appendix. ■

All skill cutoffs would equal s', and s' skilled agents be indifferent between each occupation. From (2.18),

$$\frac{w_{j+1}}{w_j} = \frac{s_j - E_j}{s_j - E_{j+1}} \tag{2.28}$$

. Since all cutoffs above s' would fall to s', all relative skill prices $\frac{w_{k+1}}{w_k}$ for all $k > j$ would be higher than before the supply shock, and because all cutoffs below s' would rise to s', all $\frac{w_k}{w_{k-1}}$ for all $k \leq j$ would be lower.

2.3.1. Contrast with Canonical Model

This limiting case shows that even if this model and the canonical model described in section 1 act similarly for small shocks, for large shocks differences would be observed. When s' agents are in all occupations, skill prices would be frozen, and all other agents would be in the highest and/or lowest ranked occupation. Since the skill prices determine wages, wages would also be at a finite limit, and the elasticity of substitution between high and low skilled agents would be zero. Even if the cascading was not continuous as in the CES case, at some point as more s' agents entered the economy the elasticity between high and low skilled agents would have to fall.

But in the canonical model the elasticity is fixed, and as the relative quantity of one skill group goes to 0, that group's marginal product/wages rise to infinity. For the high skilled group (which would be all agents with skill s greater than a given level),

$$\lim_{H \to 0} W_H = \alpha_1 \left(\frac{H}{L}\right)^{\rho-1} \left[\alpha_1 \left(\frac{H}{L}\right)^{\rho} + \alpha_2\right]^{\frac{1}{\rho}-1} = \infty \tag{2.29}$$

, and similarly for the low skilled group.

The cascading effect also endogenizes complementarity between differently skilled agents. Agents with close enough skill levels act like substitutes because they are within the range where supply increases of one type of agent cause skill prices and thus wages to fall. Because of the constant returns to scale, they would have to push up skill prices in other occupations. Agents with different enough skill levels are outside of that range, and cause each other's wages to rise. This complementarity effect happens despite the agents themselves being perfect substitutes in producing any occupation aggregate.

In fact, as more new agents enter the economy, the original substitutes could shift to being complements with the new agents. As nearby agents move further up or down the occupation ladder, they could eventually move into occupations that have rising skill prices. As one example, if s' agents were the highest in the economy, eventually all other agents would move into occupation 1. Before s' skilled agents entered occupation 1, w_1 would continue to rise. Even agents with skill close to s' would see at least a brief rise in wages in occupation 1 if not while they were in other low ranked occupations.

2.4. Implications for Observed Elasticities

Only the long run steady state has been considered so far, but there is reason to believe the long run is reached very quickly for the lowest skilled agents. This is because the entrance costs at the bottom of the hierarchy could be quite low. Some evidence for this effect comes from Converse et al. (1981), which provides survey evidence that low wage earners do work in occupations that require significant but small training time.[4] Agents near the bottom who's wages are affected by low skilled immigration or a change in the minimum may be able to learn new occupations within months or less. This time frame could fit into many of the empirical studies of labor demand. It could also be that not all of the entrance cost is sunk for nearby occupations. If similar occupations required similar knowledge, the marginal entrance of moving up a few occupations on the hierarchy could be small.

The canonical model predicts that a shock of low skilled agents such as immigrants will reduce wages for all low skilled agents. But here, because natives who

[4] Converse et al. (1981), which studies a survey of employers of minimum wage workers, show that employers often provide training for low wage occupations. On average, they find that 51.7 hours of formal training were required, and 24.6 days of on-the-job training. Therefore it is reasonable to assume that low wage occupations would often require training that is done at the worker's expense, which is what the entrance costs E_j are.

are higher skilled than the low skilled immigrants will cascade up the occupation ladder and earn higher current wages, the observed effects of immigration on native wages could be lower or even ambiguous. Suppose a shock of immigrants had skill $s' < s_1$, so that they initially entered occupation 1. Consider natives with a skill of $s_n < s_1$, so that they were also in occupation 1. As s' immigrants enter, s_1 and all other skill cutoffs fall toward s'. At least some natives would make different occupation choices and choose higher occupations, including switching from occupation 1 to 2. Skill prices in occupations close to 1 would fall relative to higher occupation's skill prices.

Let w_j' denote the new skill price of w_j. If $w_2' < w_1$, so that the skill price in occupation 2 fell lower than what occupation 1's skill price used to be, then even those natives who moved from 1 to 2 would have lower current wages, $w_2' s_n < w_1 s_n$. But if w_2 only fell to $w_2' > w_1$, so that it was still higher than w_1 was before the shock, those natives that moved from 1 to 2 would have higher current wages, $w_2' s_n > w_1 s_n$. A similar argument can be applied to those natives that moved from any occupation j to a higher occupation $k > j$. This would offset or even reverse the effects of falling skill prices on the average low skilled (or overall) native wages.

For the effects of a wage shock, the canonical model predicts that every agent who earns less than the minimum will become unemployed. In this model, suppose a minimum wage w_m is mandated. This is applied to each agents' current, not lifetime wages. An agent could not stay in their current occupation j if they couldn't make the minimum in it: $w_j s < w_n$, meaning $s < \frac{w_m}{w_j}$. Those agents would have to choose another occupation k with a high enough skill price to allow them to meet the minimum, $w_k s \geq w_m$, so that $s \geq \frac{w_m}{w_k}$. They could only do this if they could pay the entrance cost E_k, $s > E_k$.[5]

By Proposition 4 and (2.15), each agent prefers occupations closer to the orig-

[5]The survey of Converse et al. (1981) shows 12.6% of employers of workers earing at or near the minimum gave employees more responsibilities (pg. 281), with 6.1% requiring new training. It is therefore reasonable to suppose that many workers might pay for their own training and move to higher occupations.

inal choice j over those further away. Since $s_{j-1} < s < s_j$, then $s < s_{j+l}$ for all $l \geq 1$, and the agent would prefer every occupation $j+l$ to $j+l+1$. But the lower an agent's skill, the higher the occupation they would need to switch to in order to satisfy the minimum. Therefore the agent would choose

$$k = \begin{cases} \min\left\{k \text{ s.t. } s > \max\left\{\frac{W_m}{w_k}, E_k\right\}\right\} & \text{if } s > \frac{W_m}{w_k}, E_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases} . \tag{2.30}$$

If $w_1 s_1 < w_m$, so that agents in higher occupations than 1 are affected, then by Proposition 2 agents would leave occupation 1 and w_1 would rise until $w_m < w_1 s_1 < w_2 s_1 < w_3 s_2 < ... < w_J s_{J-1}$. Therefore in equilibrium the minimum is binding only for agents who would choose occupation 1 and the new $s_0 = \frac{w_m}{w_1}$.

By the agent optimization condition and the Theorem, since those who switched were forced to switch from their optimal choice, their lifetime wages would be lower - the higher skill price would not compensate for the higher entrance cost. Only if there was no such occupation that they could pay the higher entrance cost E_k for, would they be in occupation 0 and be unemployed. Therefore the measured elasticity of labor demand from a change in the minimum would be lower than in the canonical model.

3. Conclusions and Discussion

It has been shown that a model of agents choosing among complementary occupations on a hierarchy determined by heterogeneous learning costs can explain an observed elastic labor demand from immigration and an inelastic labor demand from changes in the minimum wage. The movement to higher cost/wage occupations could offset the effects of low skilled immigration on low skilled natives and the unemployment effects of the minimum wage. This model generalizes the canonical model and endogenizes the complementarity between workers of different skill levels that is empirically observed. It does not depend on the typical assumptions of comparative advantage, scale, and preferences, and doesn't depend upon a particular production function used. It is also relatively intuitive and tractable.

This model does not necessarily provide the entire explanation of the effects of supply and wage shocks, but could add to other potential dampers of those shocks discussed in the literature cited above. The magnitude of the cascading effects will depend on how large the entrance costs are. Since they represent how costly it is to learn an occupation, this will hinge on how much future wages are discounted. If the relevant discount rate is very low, the entrance costs will be low, the skill prices in higher occupations will only be slightly higher, and shifts to higher occupations won't make much of a difference on wages. Alternatively, if economic growth raises wages quickly relative to the relevant discount rate, there will also be a fairly flat hierarchy, since the wage cost of learning an occupation today would be a small fraction of total lifetime wages, which would include high wages in the future.

4. References

1. Acemoglu, Daren and David Autor. "Skills, Tasks and Technologies: Implications for Employment and Earnings." in *The Handbook of Labor Economics Vol. 4b* (2011): 1043-1171.

2. Borjas, George J. "The Economic Analysis of Immigration." *Handbook of Labor Economics, Volume 3A*(1999): 1697-1760.

3. Brown, Charles, "Minimum Wages, Employment, and the Distribution of Income." *Handbook of Labor Economics, Volume 3B* (1999): 2102-2134.

4. Converse, Muriel, Richard Coe, Mary Corcoran, Maureen Kallick, and James Morgan. "The minimum wage: an employer survey." *Report of the Minimum Wage Study Commission, Vol. VI* (1981): 241-341.

5. Costrell, Robert M. and Glenn C. Loury. "Distribution of Ability and Earnings in a Hierarchical Job Assignment Model." *Journal of Political Economy* 112 (2004): 1322-1363.

6. Friedberg, Rachel M. and Jennifer Hunt. "The Impact of Immigrants on Host Country Wages, Employment and Growth." *The Journal of Economic Perspectives* 9 (1995): 23-44.

7. Katz, Lawrence F. and Kevin M. Murphy. "Changes in Relative Wages, 1963-1987: Supply and Demand Factors." *The Quarterly Journal of Economics* 107 (1992): 35-78.

8. Roy, A. “Some Thoughts on the Distribution of Earnings.” *Oxford Economic Papers* 1 (1951): 135-146.

9. Sattinger, Michael. "Assignment Models of the Distribution of Earnings." *Journal of Economic Literature* 31 (1993): 831-880.

10. Teulings, Coen. "Comparative Advantage, Relative Wages, and the Accumulation of Human Capital." *Journal of Political Economy* 113 (2005): 425-461.

11. Willis, Robert J. "Wage Determinants: A Survey and Reinterpretation of Human Capital Earnings Functions." in *The Handbook of Labor Economics Vol. 1* (1986): 525-602.

5. Appendix

5.1. Proofs and Lemmas for the Results of the Model Section

Proof of Proposition 2 (1).

Proof. Suppose F_j is finite when $I_j = 0$. Multiply each occupation aggregate by a constant, $\alpha > 1$. F is homogenous of degree 1, so F_j is homogenous of degree zero. Thus

$$F_j\left(\alpha I_1, \alpha I_2, ..., \alpha \cdot 0, ..., \alpha I_J\right) = F_j\left(I_1, I_2, ..., 0, ..., I_J\right) \ . \tag{5.1}$$

However, $\frac{\partial^2 F}{\partial I_j \partial I_k} > 0$, so when each aggregate is multiplied by α,

$$F_j\left(\alpha I_1, \alpha I_2, ..., \alpha \cdot 0, ..., \alpha I_J\right) > F_j\left(I_1, I_2, ..., 0, ..., I_j\right) \ . \tag{5.2}$$

This is because all I_k for all $k \neq j$ have risen, so F_j must rise. For any finite F_j, (5.1) and (5.2) contradict. Since $w_j = F_j$, $w_j = \infty$. ■

Proof of Proposition 2 (2).

Proof. Suppose there were some occupations $j \in \Xi$ which were empty. Then by (1) $w_{j\in\Xi} = \infty$. Then

$$W_{j\in\Xi}\left(s\right) = w_{j\in\Xi}\left(s - E_{j\in\Xi}\right) = \infty$$

for those agents with $s > E_{j\in\Xi}$, which have nonzero measure since $\bar{s} > E_J$. For

$j \notin \Xi$, Therefore

$$W_{j\in\Xi}(s) > W_{j\notin\Xi}(s) \ ,$$

contradicting agent optimization. ■

Proof of Proposition 3

Proof. Suppose not. Then for some j, k, s.t. $k > 0$, $w_{j+k} < w_j$. By construction, $E_{j+k} > E_j$. Then

$$w_j(s - E_j) > w_{j+k}(s - E_{j+k})\,\forall s \tag{5.3}$$

and all agents would prefer j over $j+k$. Thus no agents would be in occupation $j+k$ contradicting Proposition 2. ■

Proof of Proposition 4

Proof. Suppose not. Then for some occupation j, $s_j < s_{j-1}$. By the definition of s_{j-1}, and by(2.15) every agent with $s < s_{j-1}$ would prefer $j-1$ to j. Similarly, every agent with skill $s \geq s_{j-1}$ would also have a higher skill than s_j, and would choose $j+1$ over j and $j-1$. Since every agent must have skill $s \lesseqgtr s_{j-1}$, every agent would prefer another occupation over j. Thus, no agents would be in occupation j, contradicting Proposition 2. ■

Many intermediate steps must be made to prove Theorem 5. First it is shown that any equilibrium can be characterized by a set of skill cutoffs.

Lemma 8. *Any equilibrium allocation of agents into occupations can be characterized by the set of skill cutoffs,* $\{s_j\}_{j=0}^{J-1}$*, as defined by*

$$s_j \equiv \frac{w_{j+1}E_{j+1} - w_jE_j}{w_{j+1} - w_j} \quad . \tag{5.4}$$

Proof. Any agent would have a skill level s such that

$$s_0 \leq \cdots \leq s_{j-2} \leq s_{j-1} \leq s \leq s_j \leq s_{j+1} \leq \cdots \leq s_{J-1} \tag{5.5}$$

for some occupation j. First consider all agents with skill not equal to any cutoff,

s s.t. $s \neq s_j$ for any j. Then,

$$s_0 \leq \cdots \leq s_{j-2} \leq s_{j-1} < s < s_j \leq s_{j+1} \leq \cdots \leq s_{J-1} \tag{5.6}$$

Because $s < s_j$, the agent would prefer occupation j to $j+1$. Because $s < s_j \leq s_{j+1}$ the agent would prefer $j+1$ at least as much as $j+2$, and similarly $j+2$ would be at least as preferred as $j+3$, and so on. By induction, j is preferred to $j+k$ for all $k \geq 1$. Likewise for $j-1$, $j-2$, and $j-k$ for all $k \geq 1$. Thus, any agent would prefer occupation j s.t. $s_{j-1} < s < s_j$ to all higher and lower occupations, and would have a unique occupation choice depending only on the set $\{s_j\}_{j=0}^{J-1}$. Now consider the case where an agent's skill is equal to a cutoff, $s = s_j$ for some j. Agents with skill $s = s_j$ are indifferent between occupations j and $j+1$. In any equilibrium, these agents would choose between j and $j+1$ in a unique allocation consistent with $s = s_j$. In other words, I_j and I_{j+1} would be consistent with $s = \frac{w_{j+1}E_{j+1} - w_j E_j}{w_{j+1} - w_j}$. To see that this allocation is unique, consider two different allocations, denoted by $'$ and $''$, s.t. $I'_j > I''_j$ and $I'_{j+1} < I''_{j+1}$.

$$F''_j \left(..., I''_j, I''_{j+1}, ...\right) \quad > \quad F'_j \left(..., I'_j, I'_{j+1}, ...\right) \tag{5.7}$$

$$F''_{j+1} \left(..., I''_j, I''_{j+1}, ...\right) \quad < \quad F'_{j+1} \left(..., I'_j, I'_{j+1}, ...\right) \tag{5.8}$$

because $F_{jj} < 0$ and $F_{j,j+1} > 0$. Thus the skill prices are different in each case, and from equation (2.22)

$$s''_j = \frac{w''_{j+1}E_{j+1} - w''_j E_j}{w''_{j+1} - w''_j} < \frac{w'_{j+1}E_{j+1} - w'_j E_j}{w'_{j+1} - w'_j} = s'_j \quad . \tag{5.9}$$

Therefore, there can be only one allocation of skill $s = s_j$ agents across occupations j and $j+1$ that is consistent with $s = s_j$. The same argument can be applied for any $s = s_k = s_{k+1} = \cdots$ for any occupation k. ■

Proof of Theorem 5

Proof. Let

$$\bar{I} \equiv \int_{\underline{s}}^{\bar{s}} s dN(s) \tag{5.10}$$

be a finite number, since $\int_{-\infty}^{\infty} s dN(s) < \infty$ by assumption. Let I be defined as a vector of occupation aggregates, $\{I_j\}_{j=1}^{J}$, on the space S, s.t.

$$I \in S = \left\{ I \in R^J | 0 \leq I_j \leq \bar{I} \forall j \right\} \quad . \tag{5.11}$$

Therefore, S is compact and convex. Let $w \in R^J$ be the vector of skill prices, $\{w_j\}_{j=1}^{J}$. Let $\Omega : S \to R^J$, $\Omega(I) = w$, be the mapping of I into w defined by

$$\Omega_j(I) = w_j \equiv F_j(I_1, I_2, \ldots, I_j, \ldots, I_J) \forall j \quad . \tag{5.12}$$

Let $\sigma \in R^J$ be a vector of skill cutoffs, $\{s_j\}_{j=0}^{J-1}$. Let $\Psi : R^J \to R^J$, $\Psi(w) = \sigma$, be defined as

$$\Psi_j(w) = s_j \equiv \begin{cases} \frac{w_{j+1}E_{j+1} - w_j E_j}{w_{j+1} - w_j} \text{ if } w_{j+1} > w_j \\ \infty \text{ otherwise} \end{cases} \quad . \tag{5.13}$$

Let $\Phi : R^J \to S$, $\Phi(\sigma) = I$, be defined by

$$\Phi_j(\sigma) = I_j \equiv \begin{cases} \int_{s_{j-1}}^{s_j} s dN(s), \text{ if } s_j \geq s_{j-1} \\ 0 \text{ otherwise} \end{cases} \quad . \tag{5.14}$$

Let $\Gamma : S \to S$, $\Gamma(I) = I$, be defined by

$$\Gamma(I) = \Phi(\Psi(\Omega(I))) \quad . \tag{5.15}$$

Since F is continuous, $\Omega(I) = w$ is continuous in I. Since s_j for any j is continuous in w_j and w_{j+1}, $\Psi(w) = \sigma$ is continuous in w, and $\Psi(\Omega(I))$ is continuous in I. Since I_j for any j is continuous in s_{j-1} and s_j, $\Phi(\sigma)$ is continuous in σ, and thus $\Phi(\Psi(\Omega(I)))$ is continuous in I. Thus $\Gamma(I)$ is continuous. By the definition of $\Phi(\sigma)$, and the fact that all skill levels are non-negative, any $s_j \geq s_{j-1}$ yields

$I_j \geq 0$, and for anything else $I_j = 0$, so every $I_j \geq 0$. Since $\int_{-\infty}^{\infty} sdN(s) = \bar{I}$, $I_j \leq \bar{I} \forall j$. Therefore, $\Gamma(I) = I$ maps S onto itself,

$$\Gamma : S \to S \quad . \tag{5.16}$$

By the above and Brouwer's fixed point theorem, $\Gamma(I)$ has a fixed point in S. Finally, it is shown that this fixed point must be an equilibrium as defined by the model. Suppose for any occupation set of adjacent occupations $\{I_j\}_{j=l}^{l+k} = 0$. From Proposition 2 (1), $\{w_j\}_{j=l}^{l+k} = \infty$. From Ψ, $s_{l-1} = E_l$ and $s_{l+k} = \infty$. Because there are agents with skill between E_l and ∞ by equation (2.7), Φ implies $I_j > 0$ for some j, $l \leq j \leq l+k$, which is a contradiction. Thus, for any fixed point,

$$I_j \neq 0 \forall j \quad . \tag{5.17}$$

By the definition of Ω and the final goods producer's FOCs, the fixed point satisfies producer optimization. By the definition of Ψ, Lemma 8, and since $I_j \neq 0$ for any j, any fixed point of quantities I satisfies agent optimization. By the definition of Φ, markets clear. This implies that any fixed point is an equilibrium. Because a fixed point for the system exists, and any fixed point is an equilibrium, an equilibrium exists. ■

The intuition behind (1) of the proof of Proposition 6 is that in order for a higher cutoff to rise, the corresponding higher occupation aggregate must rise relative to the corresponding lower one. For that to happen, the next higher skill cutoff must rise or the next lower one must fall. That pattern must continue up to the highest occupation aggregate, which cannot rise when the highest cutoff rises, and also down to cutoff s_j, which must rise. Thus there is a contradiction. The reverse holds for lower occupations. Part (2) follows from (1) since changes in skill cutoffs imply changes in relative skill prices.

Proof of Proposition 6

Proof. When F has a CES form,

$$w_k(\varepsilon) = c_k I_k(\varepsilon)^{\rho-1} Y(\varepsilon)^{1-\rho} \tag{5.18}$$

and so depends only on occupation aggregate k and aggregate output. The skill cutoff,

$$\begin{aligned} s_k(\varepsilon) &= \frac{c_{k+1} I_{k+1}(\varepsilon)^{\rho-1} Y(\varepsilon)^{1-\rho} E_{k+1} - c_k I_k(\varepsilon)^{\rho-1} Y(\varepsilon)^{1-\rho} E_k}{c_{k+1} I_{k+1}(\varepsilon)^{\rho-1} Y(\varepsilon)^{1-\rho} - c_k I_k(\varepsilon)^{\rho-1} Y(\varepsilon)^{1-\rho}} \\ &= \frac{c_{k+1}\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)^{\rho-1} E_{k+1} - c_k E_k}{c_{k+1}\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)^{\rho-1} - c_k} \end{aligned} \tag{5.19}$$

only changes as the ratio of occupation aggregates, $\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}$, changes, according to:

$$\frac{\partial s_k(\varepsilon)}{\partial\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)} = \frac{(\rho-1)\, c_{k+1} c_k \left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)^{\rho-2} (E_k - E_{k+1})}{\left[c_{k+1}\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)^{\rho-1} - c_k\right]^2} > 0 \quad . \tag{5.20}$$

For the proof of (1), suppose it is not true. Then there exists an occupation k s.t. $\frac{\partial s_k(\varepsilon)}{\partial\varepsilon} > 0$ when $s_k(\varepsilon) > s'$, or $\frac{\partial s_k(\varepsilon)}{\partial\varepsilon} < 0$ when $s_k(\varepsilon) < s'$. If $\frac{\partial s_k(\varepsilon)}{\partial\varepsilon} > 0$ when $s_k(\varepsilon) > s'$, because

$$\frac{\partial s_k(\varepsilon)}{\partial\varepsilon} = \frac{\partial s_k(\varepsilon)}{\partial\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)} \frac{\partial\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)}{\partial\varepsilon} \tag{5.21}$$

and from equation (5.20), $\frac{\partial s_k(\varepsilon)}{\partial\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)} > 0$, then $\frac{\partial \frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}}{\partial\varepsilon} > 0$. For that to happen while $\frac{\partial s_k(\varepsilon)}{\partial\varepsilon} > 0$, either (a) $\frac{\partial I_{k+1}(\varepsilon)}{\partial\varepsilon} > 0$, and/or (b) $\frac{\partial I_k(\varepsilon)}{\partial\varepsilon} < 0$.

If (a), since $\frac{\partial s_k(\varepsilon)}{\partial\varepsilon} > 0$, then $\frac{\partial s_{k+1}(\varepsilon)}{\partial\varepsilon} > 0$. Thus $\frac{\partial s_{k+2}(\varepsilon)}{\partial\varepsilon} > 0$, and so on to $\frac{\partial s_{J-1}(\varepsilon)}{\partial\varepsilon} > 0$. But if $\frac{\partial s_{J-1}(\varepsilon)}{\partial\varepsilon} > 0$, then from equation (5.20), $\frac{\partial I_J(\varepsilon)}{\partial\varepsilon} < 0$. But since the $\frac{\partial \frac{I_J(\varepsilon)}{I_{J-1}(\varepsilon)}}{\partial\varepsilon} < 0$, from equation (5.20) this contradicts $\frac{\partial s_{J-1}(\varepsilon)}{\partial\varepsilon} > 0$.

If (b), since $\frac{\partial s_k(\varepsilon)}{\partial \varepsilon} > 0$, then $\frac{\partial s_{k-1}(\varepsilon)}{\partial \varepsilon} > 0$, and so on, so that $\frac{\partial I_{j+1}(\varepsilon)}{\partial \varepsilon} < 0$ and $\frac{\partial s_{j+1}(\varepsilon)}{\partial \varepsilon} > 0$. But this contradicts $\frac{\partial I_j(\varepsilon)}{\partial \varepsilon} > 0$.

The reverse arguments hold for $\frac{\partial s_k(\varepsilon)}{\partial \varepsilon} < 0$ when $s_k(\varepsilon) < s'$.

For the proof of (2), consider occupation $k > j$. Since $\frac{\partial s_k(\varepsilon)}{\partial \varepsilon} < 0$, then $\frac{\partial \frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}}{\partial \varepsilon} < 0$. Because

$$\frac{w_{k+1}(\varepsilon)}{w_k(\varepsilon)} = \frac{c_{k+1} I_{k+1}(\varepsilon)^{\rho-1} Y^{1-\rho}}{c_k I_k(\varepsilon)^{\rho-1} Y^{1-\rho}} \tag{5.22}$$

$$= \frac{c_{k+1}}{c_k}\left(\frac{I_{k+1}(\varepsilon)}{I_k(\varepsilon)}\right)^{\rho-1} \tag{5.23}$$

it must be that

$$\frac{\partial \frac{w_{k+1}(\varepsilon)}{w_k(\varepsilon)}}{\partial \varepsilon} > 0 \quad . \tag{5.24}$$

The reverse argument holds for $k < j$. ■

Proof of Proposition **??**

Proof. Suppose not. Then there is at least one occupation at $\varepsilon < \infty$ where there are no s' skilled agents. Let k denote this occupation, and l denote an occupation that s' skilled agents are in. Suppose $l > k$. Then

$$s_{lk}(\varepsilon) = \frac{w_l(\varepsilon) E_l - w_k(\varepsilon) E_k}{w_l(\varepsilon) - w_k(\varepsilon)} \quad . \tag{5.25}$$

As ε increases, $w_l(\varepsilon)$ falls and $w_k(\varepsilon)$ rises without bound, so from equation (2.22), $s_{lk}(\varepsilon)$ rises without bound. Let δ be the value of ε where

$$s' = \frac{w_l(\delta) E_l - w_k(\delta) E_k}{w_l(\delta) - w_k(\delta)} \quad . \tag{5.26}$$

At this point, s' skilled agents begin in to enter occupation k, creating a contradiction. ■