INTRODUCTION

Physics is an empirical science. It is a popular belief that the ultimate judge in physics is experiment and if for any reason a theory contradicts an experiment, it is the theory that is to be blamed. However this is not exactly so. There are a lot of theories which had «survived» although some experiments testified against them. Let us consider an example.

It is well known that Einstein's theory of Brownian motion had become crucial for developing atomic theory of matter since it was later confirmed by brilliant experiments of J. B. Perrin. However the same theory appeared to be refuted by no less brilliant experiments of V. Henri. Why did the confirmation by Perrin turn out to be more important than the refutation by Henri?

Actually, any theory undergoes non-empirical checks and crosschecks before being tested by an experiment. A theory must be consistent, it must not contradict already established theories, and it must be in line with a general wisdom of science, i.e. be simple, elegant, etc. Einstein's theory of Browinian motion was accepted, in particular, because it was in line with the kinetic theory of gases and chemistry. As for the Henry experiments, it was found later that they were incorrectly interpreted.

Thus, an experimental confirmation is necessary but not sufficient condition for accepting a theory. This is always taken into account in confronting a new theory with real data.

Physics is not only empirical but also a theoretical science that employs the language of mathematics. The purpose of the latter is two-fold: it supplies tools of calculation and provides a conceptual framework. Mathematical concepts represent the very essence of physical ideas. The concept of velocity is inconceivable without the concept of derivative. The laws of mechanics cannot be properly formulated without differential equations. Quantum laws require operator equations. Every formal symbol in a physical theory has mathematical meaning. However, despite the fact that a lot of mathematical ideas stemmed from physics, mathematics is an independent discipline. If it so, why is it possible to use the ideas of pure mathematics to describe reality?

The answer is that mathematics studies very general and clear-cut models of natural phenomena — a special way of understanding reality. And so does physics.

Teaching physics can be compared to advancement of scientific knowledge. This viewpoint helps to understand the role of experiment in a general physics course. A founder of experimental method was Galileo Galilei. However experiment per se was not his invention: people relied on experimental evidence from ancient times. We are indebted to Galileo for a method which has become an integral part of physics research.

According to Galileo, a physicist should design an experiment, repeat it several times in order to eliminate or reduce irrelevant factors, conjecture mathematical relationships (laws) between the quantities involved, develop new experimental tests for the conjectured laws using available technics, and, finally, when the laws have been confirmed, make new predictions based on these laws which, in turn, must be experimentally tested.

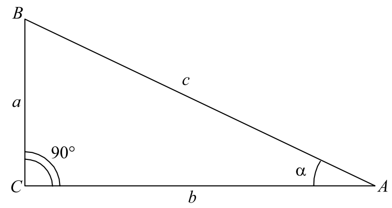

Fig. 1

According to Galileo, observation, working hypothesis, mathematical treatment, and experimental verification are the four stages in a study of natural phenomena.

Consider a simple instructive example. Suppose we have several chunks of a metal sheet (cardboard, plywood, etc.), whose shape is shown in Fig. 1. Assume also that we have tools for measuring weight, length, and angle. By measuring the weight of several triangles cut from the same sheet, one finds a formula for the weight of a triangle (ABC):

$$M_{ABC} = c^2 f(\alpha),$$

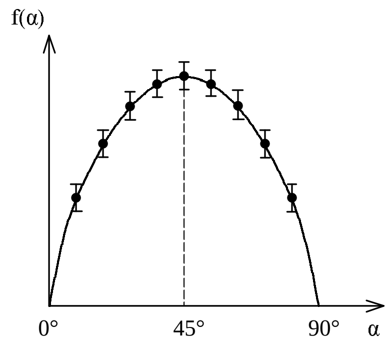

Fig. 2

where $f(\alpha)$ is a universal function plotted in Fig. 2.

Now let us cut a triangle (ABC) in two pieces as in 3 and verify that $\angle BCD = \angle BAC$. It is already found that

$$M_{CBD} = a^2 f(\alpha), \quad M_{ACD} = b^2 f(\alpha).$$

Using the scale one can check that the weights are additive:

$$M_{ABC} = M_{CBD} + M_{ACD}.$$

Then using the assumed universality of function $f(\alpha)$ one finds that

$$c^2 = a^2 + b^2.$$

This equation could be verified by further experiments.

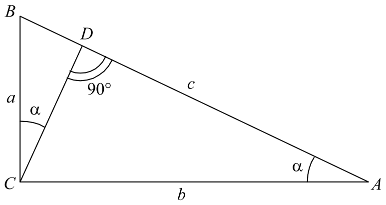

Fig. 3

Does our result contradict Euclidean geometry? Of course, not. Indeed, one can see that

$$M_{ABC} = \rho h S_{ABC},$$

where ρ is the metal density, h is the sheet thickness, and S_{ABC} is the area of the triangle ABC. Obviously,

$$S_{ABC} = \frac{1}{2}ab = \frac{1}{2}c^2 \sin\alpha\cos\alpha = \frac{1}{4}c^2 \sin 2\alpha,$$

i. e.

$$f(\alpha) = \frac{1}{4}\rho h \sin 2\alpha.$$

This thought experiment, in our opinion, is an excellent example of Galileo's experimental method. It is amazing that using measurement instruments and procedures, which by themselves introduce large uncertainties, and only a limited amount of the triangles it is possible to derive an exact mathematical relationship (Pythagoras' theorem). As Einstein said, the greatest mystery of the universe is that it is conceivable.

The main purpose of the laboratory course is to teach students a physical way of thinking. Firstly, they should learn how to reproduce and analyze simple physical phenomena. Secondly, they should get a basic hands-on experience in the laboratory and become acquainted with modern scientific instruments.

A student working in the laboratory should know:
- basic physical phenomena;
- fundamental concepts, laws and theories of classical and modern physics;
- orders of magnitude of the quantities specific for various fields of physics;
- experimental methods

and know how to:
- ignore irrelevant factors, build working models of real physical situations;

- make correct conclusions by comparing theory and experimental data;
- find dimensionless parameters specific for a phenomenon under study;
- make numerical estimates;
- consider proper limiting cases;
- make sure that obtained results are trustworthy;
- see physical content behind technicalities.

A laboratory assignment should be regarded as a research project in miniature. An inclination to doubt and cross-checking is invaluable for any researcher. We hope that our practicum would help to develop this quality.

MEASUREMENTS IN PHYSICS

Measurements in Physics

Numerical value of physical quantity. We say that a quantity x is measured if we know how many units the quantity contains. A number of the units contained is called a numerical value $\{x\}$ of the quantity x. If $[x]$ is a unit of quantity x (e.g. a unit of time is 1 second, a unit of electric current is 1 ampere, etc.), then

$$\{x\} = \frac{x}{[x]}. \tag{1.1}$$

For example, if a current $i = 10$ A, then $\{x\} = 10$ and $[i] = 1$ A. Equation (1.1) can be written as

$$x = \{x\}[x]. \tag{1.2}$$

If a unit is reduced by a factor of α:

$$[x] \to [X] = \frac{1}{\alpha}[x], \qquad \{x\} \to \{X\} = \alpha\{x\}.$$

The physical quantity remains the same because

$$x = \{x\}[x] = \{X\}[X]. \tag{1.3}$$

Too large or too small numerical values are inconvenient. Therefore new units are often used by taking a standard unit with a prefix, e.g. $1\ mm^3 = 1 \cdot (10^{-3}\ m)^3 = 10^{-9}\ m^3$. The decimal prefixes specified by the International System of Units (SI) are listed in Table 1.

It is essential to avoid double or multiple prefixes, e.g. instead of 1 $\mu\mu F$ one should write 1 pF.

Table 1

SI prefixes

Prefix	Symbol		Exponent of 10
	Latin	Cyrillic	
exa	E	Э	18
peta	P	П	15
tera	T	Т	12
giga	G	Г	9
mega	M	М	6
kilo	k	к	3
hecto	h	г	2
deca	da	да	1
deci	d	д	–1
centi	c	с	–2
milli	m	м	–3
micro	μ	мк	–6
nano	n	н	–9
pico	p	п	–12
femto	f	ф	–15
atto	a	а	–18

Dimension. In principle, any physical quantity can be measured using its own units unrelated to the units of other quantities. In this case the equations that express laws of physics would be obscured by many numerical coefficients. The equations would become complicated and difficult to understand. To avoid this issue physicists have long ago abandoned a practice of introducing independent units for all physical quantities. Instead they use systems of units organized according to the following principle. Some quantities are taken as the base ones and the corresponding units are independently established. For instance, in mechanics the system (l, m, t) is used, the base units are length (l), mass (m), and time (t). A choice of the base units (and their number) is conventional. In the international system of units (SI) nine quantities are taken as the base ones: length, mass, time, electric current, thermodynamic temperature, luminous intensity, amount of substance, angle, and solid angle. The units which are not base are called derived units. The latter are derived from the equations used to define them. It is assumed that numerical coefficients in the equations are already fixed. For instance, the velocity v of a point-like object traveling at a constant speed is directly proportional to the distance s and inversely proportional to the time of travel t. If the units for s, t and v are independent, then

$$v = k\frac{s}{t},$$

where k is a numerical coefficient which particular value depends on the choice of the units. For simplicity it is usually set $k = 1$, so that $s = vt$. If the base units are length s and time t, velocity becomes a derived unit. In this case the unit of velocity corresponds to uniform motion when the unit distance is traveled per the unit of time. It is said that the dimension of velocity equals the dimension of length divided by dimension of time. Symbolically,

$$\dim v = lt^{-1}.$$

Similarly, for acceleration a and force F we have:

$$\dim a = lt^{-2}, \qquad \dim F = mlt^{-2}.$$

Now, let physical quantities x and y be related as

$$y = f(x). \tag{1.4}$$

Together with equation (1.3) this equation gives

$$Y = f(X), \tag{1.5}$$

where $X = \alpha x$ and $Y = \beta y$. Let us find the value of β assuming that the argument x and parameter α can take any values. Differentiating Eqs. (1.4) and (1.5) at constant α and β gives

$$\frac{dy}{dx} = f'(x), \qquad \frac{dY}{dX} = f'(X).$$

The second equation can be rewritten as

$$\frac{\beta}{\alpha} \cdot \frac{dy}{dx} = f'(X),$$

i. e.

$$\frac{\beta}{\alpha} f'(x) = f'(X).$$

Since

$$\frac{\beta}{\alpha} = \frac{xY}{yX},$$

it follows that

$$\frac{xY}{yX} f'(x) = f'(X)$$

or

$$x\frac{f'(x)}{f(x)} = X\frac{f'(X)}{f(X)}. \tag{1.6}$$

The right-hand side of Eq. (1.6) depends only on X and the left-hand side depends only on x. This is possible only if both sides are equal to a constant, say c. This observation allows one to write a differential equation:

$$x\frac{f'(x)}{f(x)} = c$$

or

$$\frac{df}{f} = c\frac{dx}{x}.$$

Then

$$f(x) = f_0 x^c,$$

where f_0 is a constant of integration.

Similarly,

$$Y = f_0 X^c,$$

or

$$\beta y = f_0 \cdot (\alpha x)^c.$$

Since

$$y = f_0 x^c,$$

This gives

$$\beta = \alpha^c. \tag{1.7}$$

Thus invariance of a physical quantity with respect to redefinition of its unit (see Eq. (1.3)) results in Eq. (1.7). Let us discuss its physical meaning. Obviously, if quantity x is chosen as a base one, the dimension of quantity y is

$$\dim y = x^c.$$

The above reasoning can be extended to a case when a quantity depends on several base units. Let, for instance, the number of the base units be equal to three and these are length (l), mass (m), and time (t). Then the dimension of any quantity y is

$$\dim y = l^p m^q t^r, \tag{1.8}$$

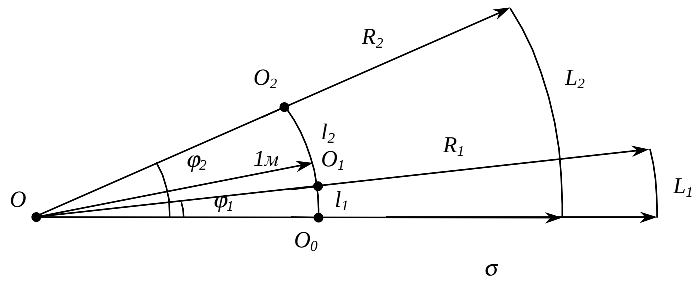

Fig. 1.1. Definition of angle

where p, q, and r are constants. Equation (1.8) shows that if the units of length, mass, and time are reduced by factors of α, β, and γ, respectively, the unit of y will be reduced by a factor of $\alpha^p\beta^q\gamma^r$. Therefore its numerical value will be increased by the same factor. This is a meaning of the concept of dimension. The values p, q, and r are actually rational numbers, which follows from the definition of physical quantities.

Often the dimension of a physical quantity is identified with its unit in some system of units. For example, it is usually said that the dimension of velocity is m/s and the dimension of force is $kg \cdot m/s^2$. Although incorrect this is not a bad mistake.

Units of angles. Angular units require separate consideration. An angle is measured in degrees or using an arc measure. The latter is defined as the length of a segment of a unit circle (see Fig. 1.1). Both units are basically a ratio of arc length to radius:

$$\varphi = \frac{l}{1\ m} = \varphi_2 - \varphi_1 = \frac{l_2}{1\text{м}} - \frac{l_1}{1\text{м}} = \frac{L_2}{R_2} - \frac{L_1}{R_1}.$$

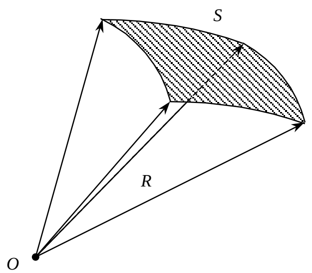

Fig. 1.2. Definition of solid angle

Here the angle φ is measured between two radial vectors OO_1 and OO_2. Here l_1 and l_2 are the arcs of the unit circle and L_1 and L_2 are the arcs of the circles with radii R_1 and R_2, respectively. To emphasize the difference between the arc and degree units, the numerical value φ is called «rad» (radian). For example, if $l = 1$ m then $\varphi = 1\ m/1\ m = 1$ rad which corresponds to $57°17'44{,}80625''$.

Similarly for a solid angle we have (see

T a b l e 2

The base units of SI

Quantity name	Unit name	Quantity symbol
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Temperature	Kelvin	K
Luminous intensity	Candela	cd
Amount of substance	Mole	mol
Angle	Radian	rad
Solid angle	Steradian	sr

Fig. 1.2):

$$\Omega = \frac{S_0}{1\ m^2}.$$

Here S_0 is an area on a sphere (in m^2) which radius is equal to 1 m. If S is an area on sphere of a radius R, then

$$\Omega = \frac{S_0}{1\ m^2} = \frac{S}{R^2}.$$

The unit of solid angle is determined in the following way. For $S_0 = 1\ m^2$

$$\Omega = \frac{1\ m^2}{1\ m^2} = 1\ sr\ (steradian).$$

Thus the total angle (360°) is equal to $\varphi = 2\pi$ rad and the total solid angle (S_0 is the total area of a sphere) is equal to $\Omega = 4\pi$ sr. Often the abbreviations «rad» and «sr» are dropped which sometimes is a source of confusion.

The base units of SI. The base units of the International System of Units are shown in Table 2. The units are defined as follows.

Meter is the length of the path travelled by light in vacuum in 1/299,792,458 of a second.

Kilogram is defined as being equal to the mass of the International Prototype Kilogram. The IPK is made of a platinum alloy known as "Pt?10Ir", which is 90% platinum and 10% iridium (by mass) and is machined into a right-circular cylinder (height = diameter) of 39.17 mm. The chosen alloy provides durability, uniformity, and high polishing quality of the prototype surface (which allows for easy cleaning). The alloy density is 21,5 g/cm^3. The prototype is stored at the International Bureau of Weights and Measures in Sevres on the outskirts of Paris. The relative error of a comparison procedure with the prototype does not exceed $2 \cdot 10^{-9}$.

Second is the unit of time defined as the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of ^{133}Cs atom.

Ampere is the unit of steady electric current that will produce an attractive force of $2 \cdot 10^{-7}$ newton per metre of length between two straight, parallel conductors of infinite length and negligible circular cross section placed one metre apart in a vacuum.

Kelvin is the unit of temperature that is defined as the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.

Mole is the unit of amount of substance defined as an amount of a substance that contains as many elementary entities as there are atoms in 12 grams of pure carbon ^{12}C.

Candela is the unit of luminous intensity that is equal to the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540 \cdot 10^{12}$ Hz and that has a radiant intensity in that direction of 1/683 watt per steradian.

The derivative units of SI are listed in Table 3. The base units listed above together with the derived units constitute the international system of units SI. The units of angle and solid angle can be considered either like the base or the derivative units. In physics radian and steradian are usually regarded as derivative units. However in some fields of physics steradian is considered as the base unit. In that case the symbol «sr» cannot be replaced by 1.

Measurements and data treatment

A goal of the majority of physical experiments is to determine a numerical value of some physical quantity. A numerical value shows how many times a quantity contains a unit. Measured values of different quantities, e.g. time, length, velocity, etc, could be related. Physics finds the relationships and interprets them as equations which can be used to determine some quantities in terms of others.

Getting reliable numerical values is not an easy task because of experimental errors. We consider errors of different types and introduce some

Table 3

SI derived units

Quantity name	Unit name	Symbol	Expression in terms of other SI units
Force	Newton	N	$1\ \mathrm{H} = 1\ kg \cdot m \cdot s^{-2}$
Pressure and stress	Pascal	Pa	$1\ Pa = 1\ N \cdot m^{-2}$
Energy and work	Joule	J	$1\ J = 1\ N \cdot m$
Power	Watt	W	$1\ W = 1\ J \cdot s^{-1}$
Charge	Coulomb	C	$1\ C = 1\ A \cdot s$
Voltage	Volt	V	$1\ V = 1\ W \cdot A^{-1}$
Electric capacitance	Farad	F	$1\ F = 1\ C \cdot V^{-1}$
Electric resistance	Ohm	Ω	$1\ \Omega = 1\ V \cdot A^{-1}$
Electric conductance	Siemens	S	$1\ S = 1\ \Omega^{-1}$
Magnetic flux	Weber	Wb	$1\ Wb = 1\ V \cdot s$
Magnetic flux density	Tesla	T	$1\ T = 1\ Wb \cdot m^{-2}$
Inductance	Henry	H	$1\ H = 1\ Wb \cdot A^{-1}$
Luminous flux	Lumen	lm	$1\ lm = 1\ cd \cdot sr$
Illuminance	Lux	lx	$1\ lx = 1\ lm \cdot m^{-2}$
Frequency	Hertz	Hz	$1\ Hz = 1\ s^{-1}$
Optical power	Dioptre	dpt	$1\ dpt = 1\ m^{-1}$

methods of data treatment. The methods allow one to derive the best approximation to the true values using experimental data, to spot inconsistencies and mistakes, to design a sensible measurement procedure, and to estimate correctly accuracy of a measurement.

Measurements and errors. Measurements are divided into direct and indirect ones.

A direct measurement is performed with the aid of instruments which directly determine a quantity under study. For example, the mass of an object can be found with a scale, the length can be measured with a ruler, and a time interval can be measured with a stopwatch.

An indirect measurement is a measurement of a quantity determined via its relation to the quantities measured directly. For example, the volume of an object can be evaluated if the object dimensions are known, the object density can be found via the measured mass and the volume, and the resistance can be determined via voltmeter and ammeter readings.

A quality of measurement is specified by its accuracy. A quality of direct measurement is determined by the method used, the instrument accuracy, and how reliably the results can be reproduced. The accuracy of indirect measurement depends both on the data quality, and on equations which relate the desired quantity and the data.

The accuracy of a measurement is specified by its uncertainty. The absolute error of a measurement is a difference between the measured and true values of a physical quantity. The absolute measurement error Δx of a quantity x is defined as

$$\Delta x = x_{mes} - x_{true}. \tag{1.9}$$

Besides the absolute error Δx it is often necessary to know the relative measurement uncertainty ε_x which is equal to a ratio of the absolute error to the value of a measured quantity:

$$\varepsilon_x = \frac{\Delta x}{x_{true}} = \frac{x_{mes} - x_{true}}{x_{true}}. \tag{1.10}$$

The quality of measurements is usually specified by the relative error rather than the absolute one. The same 1 mm uncertainty does not matter when it refers to the length of a room but it is not negligible in the length of a table and it is completely intolerable as an uncertainty of the bolt diameter. Indeed, the relative error is $\sim 2 \cdot 10^{-4}$ in the first case, in the second it is $\sim 10^{-3}$, and in the third case the error is about 10 percent or more. Absolute and relative errors are often called absolute and relative uncertainties, respectively. The terms «error» and «uncertainty» when referred to measurement are completely identical and we will use them both.

According to Eqs. (1.9) and (1.10) the absolute and relative errors of a measurement can be determined if the true value of a measured quantity is known. However, if the true value is known no measurement is necessary. The real goal of a measurement is to determine a priory unknown true value of a physical quantity, at least, a value which does not deviate significantly from the true one. As for the errors, they are not calculated, rather they are estimated. An estimate takes into account the experimental procedure, the accuracy of a method, the instrument precision, and other factors.

Systematic errors and random errors. First of all, we should mention faults which take place because of a human error or instrument malfunctioning. Faults should be avoided. If a fault is detected, the corresponding measurement should be ignored.

Experimental uncertainties which are not related to faults can be either systematic or random.

Systematic errors retain their magnitude and sign during an experiment. They could be due to instrument imperfection (non-uniform scale graduations, a varying spring constant, a varying lead of a micrometer screw, unequal arms of a weighing scale, e t.c.) and to the experimental procedure itself. For example, a low density object is being weighed without taking into account the buoyant force that effectively decreases its weight. Systematic errors could be studied and taken into account by correcting the measurement results. If a systematic error turns out to be too large, it is often simpler to use up-to-date instruments rather than to study uncertainties of the old ones.

Random errors change their magnitude and sign from one measurement to another. Repeating the same measurement many times, one could notice that often the results are not exactly equal but «dance» around some average value.

Random errors could be due to friction (for example, the instrument hand halts and does not point to a correct reading), due to backlash of mechanical parts, due to vibration which is not easy to eliminate in urban settings, due to imperfections of the object under study (for example, when measuring the diameter of a wire it is assumed that it has circular cross-section, which is an idealization), or finally due to the nature of a measured quantity itself (for example, the number of cosmic particles detected by a counter per minute). In the last case one can find that different measurements produce close values distributed randomly around some average value.

Random errors are studied by comparing results obtained in several measurements under the same conditions. If the results obtained in two or three equivalent measurements are identical, further measurements are not necessary. If the results disagree, one should try to understand the reason of the disagreement and eliminate it. If the reason cannot be found, one should perform about 10-12 measurements and treat the results statistically.

The difference between systematic and random errors is not absolute and is related to the experimental procedure. For example, when electric current is measured by different ammeters, the systematic error of the ammeter reading scale becomes a random error which magnitude and sign depend on the particular ammeter. However, one should clearly understand the difference between systematic and random errors for any given experiment.

Systematic errors. It has been already mentioned that systematic errors are due to some permanent factors which, in principle, could be always taken into account and therefore excluded. In practice this task is difficult and requires a lot of skill on the part of an experimenter.

Systematic errors are estimated by analyzing the experimental procedure, accounting for accuracy and precision of the measuring instruments, and doing test experiments. In this practicum we usually account only for the systematic errors due to the instrument inaccuracy. Let us consider some typical cases.

A systematic error of an analog electronic instrument (ammeter, voltmeter, potentiometer, etc.) is determined by its accuracy class which defines the instrument absolute error as a percentage of the maximal value of the scale used. For instance, let a voltmeter scale have a range from 0 to 10 V and a printed sign that shows the figure 1 inside a circle. The figure indicates that the voltmeter has the accuracy class 1 and the allowed uncertainty is 1% of the maximal value of the scale, i.e. in this case the uncertainty is ±0.1 V. Also one should take into account that scale readings are customarily separated by an interval that does not exceed the instrument accuracy by a factor of two.

An accuracy class of analog electronic instruments (and one half of the scale reading as well) determines the maximal absolute uncertainty which is the same along the scale. However a relative uncertainty changes drastically, so an analog instrument provides the best accuracy when the pointer is near the maximal value. Therefore an instrument or its scale should be selected so that the pointer remains on the second half of the scale during the measurement.

Nowadays digital multi-purpose electronic instruments are widely used, they have a high accuracy. Unlike analog devices, the systematic error of a digital instrument is evaluated using the formulas listed in the manual. For example, the relative accuracy of the multi-purpose voltmeter B7-34 with the 1 V scale, can be evaluated as

$$\varepsilon_x = \left[0.015 + 0.002\left(\frac{U_{kx}}{U_x} - 1\right)\right] \cdot \left[1 + 0.1 \cdot |t - 20|\right], \qquad (1.11)$$

where U_{kx} is the maximal value, V,
U_x is a voltage measured, V,
t is the ambient temperature, °C.

When the voltmeter is used to measure a constant voltage of 0.5 V at the ambient temperature of $t = 30$ °C the accuracy is

$$\varepsilon_x = \left[0.015 + 0.002\left(\frac{1}{0{,}5} - 1\right)\right] \cdot \left[1 + 0.1 \cdot |30 - 20|\right] = 0.034\%,$$

that is ±0.00017 V of the measured 0.5 V.

When the voltmeter range is 0-100 or 0-1000 V or it is switched to another kind of measurement (electric current or resistance) the formula remains the same but the numbers are different. The voltmeter accuracy is reliable under the following conditions: an ambient temperature of 5-40 °C, a relative humidity below 95% at 30 °C, and a power supply of ~220±22 V.

Some words should be said about the accuracy of rulers. Metal rulers are relatively precise: the millimeter graduations are engraved with an error less than ±0.05 mm, and the centimeter graduations with an error less than 0.1 mm, so the measurement results can be read with the aid of a hand lens. It is better not to use wooden or plastic rulers since their uncertainties are not known and could be large. A micrometer provides the accuracy of 0.01 mm and the accuracy of a caliper is determined by the accuracy of its vernier scale which is usually 0.1 or 0.05 mm.

Random errors. Random quantities (random error is an example) are studied in the probability theory and mathematical statistics. Below we describe without giving a formal proof the basic properties of random quantities and the rules of statistical treatment of experimental data.

It is not possible to eliminate random errors. However they obey the laws of statistics, so one can always determine the limits in which a measured quantity can be found with a given probability.

The theory that describes the properties of random errors agrees with experiment. The theory is based on the following properties of the normal distribution:

1. In a large pool of random errors, the errors of the same magnitude but of different sign are equally probable.
2. Large errors are less frequent than small. In other words, large errors are less probable.
3. Measurement errors can take continuous values.

To study random errors it is necessary to introduce a concept of probability.

The statistical probability of an event is defined as the ratio of the number n of cases when the event happens, to the number N of all equally possible cases:

$$P = \frac{n}{N}. \qquad (1.12)$$

Let 100 marbles be in a bin and assume that 7 marbles are black and the rest are white. The probability of randomly picking a black marble is $7/100$ and the probability to pick a white one is $93/100$.

Now let us apply the probability concept to estimate the dispersion of random errors.

Suppose n measurements of some quantity (e.g. the diameter of a rod) have been done and assume that *faults and systematic errors are eliminated, so only random errors remain.* The results of the measurement are numerical values x_1, x_2, ..., x_n. If x_0 is the most probable value of the measured quantity (we assume that it is known), the difference Δx_i between a measured value x_i and x_0 is called the absolute random error of the measurement. Then

$$\begin{aligned} x_1 - x_0 &= \Delta x_1 \\ x_2 - x_0 &= \Delta x_2 \\ &\ldots\ldots\ldots\ldots\ldots \\ x_n - x_0 &= \Delta x_n \end{aligned}$$

By summing up the equations we obtain:

$$x_0 = \frac{\sum\limits_{i=1}^{n} x_i - \sum\limits_{i=1}^{n} \Delta x_i}{n}, \tag{1.13}$$

where Δx can be either positive or negative. According to the normal distribution the errors of equal magnitude but of opposite sign are equally probable. Therefore the greater the number of measurements n, the more probable a mutual cancellation of the errors under averaging, so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Delta x_i = 0.$$

Then

$$\lim_{n\to\infty} x_{av} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = x_0. \tag{1.14}$$

Therefore the arithmetic mean x_{av} of the results of different measurements for a very large n (i.e. $n \to \infty$) is the most probable value x_0 of the measured quantity. In practice n is always finite and x_{av} is only approximately equal to the most probable value x_0. The larger the number of measurements n, the closer x_{av} to x_0.

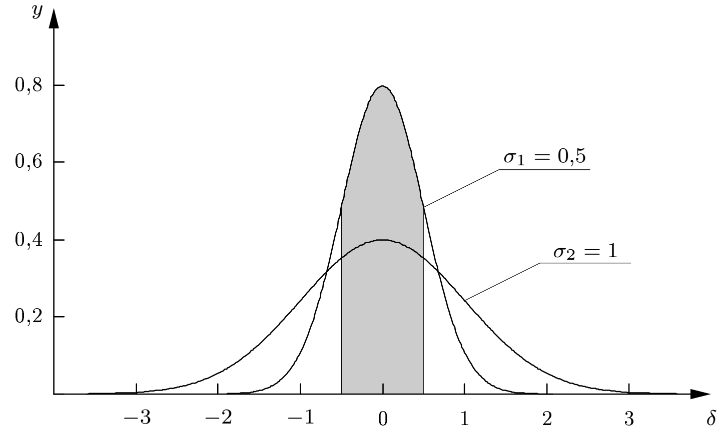

Fig. 1.3. The normal distribution

The arithmetic mean of the obtained results is usually taken as the best approximation to the value of a measured quantity:

$$x_{\text{cp}} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \ldots + x_n}{n}. \tag{1.15}$$

To estimate the reliability of a result it is necessary to examine a distribution of random errors of different measurements. The distribution of errors often obeys the normal distribution (Gaussian distribution):

$$y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x_0)^2}{2\sigma^2}}, \tag{1.16}$$

where y is the probability distribution (probability density function) of the errors:

$$y = \frac{dn}{n \cdot d\delta},$$

where $dn/(n \cdot d\delta)$ is the fraction of the errors in a given infinitesimal interval $d\delta$,

x_0 is the most probable value of the measured quantity,

$\delta = (x - x_0)$ is a random deviation,

σ is the mean of the squared deviation. The quantity σ^2 is also called standard deviation.

The normal distributions corresponding to different σ are plotted in Fig. 1.3.

The points $|\delta| = |x - x_0| = \sigma$ are inflection points of the Gaussian curves. Parameter σ specifies the measure of dispersion of random errors

δ. If the measurement results x are located close to the most probable value x_0 and the values of random deviations δ are small, the value of σ is small as well (curve 1, $\sigma = \sigma_1$). If the random deviations are large and widely dispersed, the curve becomes more widespread (curve 2, $\sigma = \sigma_2$) and $\sigma_2 > \sigma_1$. The quantity σ is a measure of dispersion of the measured quantity.

A ratio of the area under a Gaussian curve between the values $\delta = \pm\sigma$ (the area is shadowed in Fig. 1.3 for $\sigma_1 = 0.5$) to the total area under the curve is 0.68. Therefore the equation $x = x_0 \pm \sigma$ says that the probability to obtain a result x in this interval is 0.68 (68%).

If an equation reads $x = x_0 \pm 2\sigma$, the probability to obtain a result within this interval is 0.95. For $x = x_0 \pm 3\sigma$ the probability is 0.997.

In dealing with experimental uncertainties we always refer to Gaussian distribution. There are serious reasons in favor of using the normal distribution. The most significant one is the central limit theorem: if a net uncertainty is a result of several factors contributing independently to it then the distribution of the net uncertainty will be Gaussian regardless of the particular distribution of each of the factors.

For a finite number of measurements n the deviation of the result from the most probable value x_0 is estimated as the mean of the squared deviation σ_{sep}:

$$\sigma_{sep} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - x_0)^2}. \tag{1.17}$$

In practice this equation is useless since the most probable value of x_0 is unknown. However we get a reasonable estimate for σ_{sep} by replacing x_0 in (1.17) with arithmetic mean x_{av}:

$$\sigma_{sep} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - x_{av})^2}. \tag{1.18}$$

If n is small, x_{av} can differ significantly from x_0 and Eq. (1.18) gives a rough estimate of σ_{sep}. According to mathematical statistics the following equation gives a better estimate:

$$\sigma_{sep} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - x_{\text{cp}})^2}. \tag{1.19}$$

Here σ_{sep} is the mean of the squared deviation of a measurement result and/or the standard deviation derived from the experimental data. The reliability of σ_{sep} improves for a greater number of measurements n.

The uncertainty of the arithmetic mean. In practice we are not usually interested in how the result of any of n individual measurements deviates from the most probable value. Rather the question is what is an uncertainty of the arithmetic mean. To find a reasonable estimate let us perform a series of measurement sets with n measurements of quantity x per set and find x_{av} for every set. The obtained values x_{av} are randomly distributed around some central value x_0, their distribution approaching the normal distribution. The standard deviation of x_{av} from x_0 can be estimated as the mean of the squared deviation σ_{av} (in the same way as we determined σ_{sep} for n values of x.) In the probability theory it is proven that the standard deviation σ_{av} is related to the mean of the squared deviation σ_{sep} as

$$\sigma_{av} = \frac{\sigma_{sep}}{\sqrt{n}} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - x_{av})^2}. \tag{1.20}$$

Therefore the measured quantity x can be presented as

$$x = x_{av} \pm \sigma_{av}. \tag{1.21}$$

This notation says that the probability to find the most probable value x_0 of the measured quantity in the interval $x_{av} \pm \sigma_{av}$ is equal to 0.68 (68%) (assuming n is large).

The uncertainty σ_{av} (or its square) is usually called the standard deviation.

It can be shown that usually the deviation of a measurement exceeds $2\sigma_{av}$ only in 5% of all cases and it is almost always less than $3\sigma_{av}$.

One could naively conclude from above discussion that even using low-quality instruments it is possible to obtain better results by simply increasing the number of measurements. Of course, this is not so. Increasing the number of measurements reduces a random error. Systematic errors related to imperfections of the instruments persist, so one should better choose an optimal number of the measurements.

If the number of experiments is small (less than 8) it is recommended to use more sophisticated estimates. It should be noted that for $n \approx 10$ the value of σ_{av} could be determined with an accuracy of 20–30% Therefore the errors should be calculated with an accuracy of no more than two digits.

Addition of random and systematic errors. In real experiments both systematic and random errors occur. Let the corresponding errors be σ_{sys} and σ_{ran}. The net error is given by

$$\sigma_{net}^2 = \sigma_{sys}^2 + \sigma_{ran}^2. \tag{1.22}$$

This equation shows that the net error is greater than both the random and systematic errors.

An important feature of the equation should be mentioned. Let one of the errors, say σ_{ran}, be less than the other one (σ_{sys}) by a factor of 2. Then

$$\sigma_{net} = \sqrt{\sigma_{sys}^2 + \sigma_{ran}^2} = \sqrt{\frac{5}{4}}\sigma_{sys} \approx 1{,}12\sigma_{sys}.$$

In this example an equality $\sigma_{net} = \sigma_{sys}$ holds with 12% precision. Thus a smaller error almost does not contribute to the net error even if the latter is only twice as large as the former. This observation is very important. If a random error is only one half of the systematic error, it is not practical to repeat the measurements anymore since this will almost not reduce the net error. It would be enough to repeat the measurements two or three times in order to convince yourself that the random error is indeed small.

Treatment of the results of indirect measurements. If a measured quantity is a sum or difference of a couple of measured quantities:

$$a = b \pm c, \tag{1.23}$$

then the expected value of the quantity a is equal to the sum (or the difference) of the expected values of each term: $a_{ex} = b_{ex} \pm c_{ex}$, or, as it was already recommended

$$a_{ex} = \langle b \rangle \pm \langle c \rangle\,. \tag{1.24}$$

Hereinafter the angular brackets (or the bar over a symbol) mean an average: instead of writing a_{av}, we will use the notation $\langle a \rangle$ (or $\bar{a}$).

If the quantities a and b are independent the standard deviation σ_a is given by

$$\sigma_a = \sqrt{\sigma_b^2 + \sigma_c^2}, \tag{1.25}$$

i. e. the squares of the errors or, in other words, the standard deviations of the results are added.

If the measured quantity is equal to product or ratio of two errors

$$a = bc \qquad or \qquad a = \frac{b}{c}, \tag{1.26}$$

then

$$a_{ex} = \langle b \rangle \langle c \rangle \qquad or \qquad a_{ex} = \frac{\langle b \rangle}{\langle c \rangle}. \tag{1.27}$$

The relative standard error for a product or ratio of two independent quantities is given by

$$\frac{\sigma_a}{a} = \sqrt{\left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2}. \tag{1.28}$$

Let us give explicit formulae for the case when

$$a = b^{\beta} \cdot c^{\gamma} \cdot e^{\varepsilon} \ldots \tag{1.29}$$

The expected value of a is related to the expected value of b, c and e, etc. by the same equation (1.29) in which the specific values are replaced by their expected values. The relative standard error of a is expressed in terms of the relative errors of independent b, c, e, ... as

$$\left(\frac{\sigma_a}{a}\right)^2 = \beta^2\left(\frac{\sigma_b}{b}\right)^2 + \gamma^2\left(\frac{\sigma_c}{c}\right)^2 + \varepsilon^2\left(\frac{\sigma_e}{e}\right)^2 + \ldots \tag{1.30}$$

For the reference let us give an explicit general formula. Let

$$a = f(b, c, e, \ldots), \tag{1.31}$$

where f is an arbitrary function of the quantities b, c, e etc. Then

$$a_{ex} = f(b_{ex}, c_{best}, e_{ex}, \ldots). \tag{1.32}$$

Equation (1.32) is valid both for the directly measured b_{ex}, c_{ex} etc. and for the indirectly measured quantities. In the first case the values b_{ex}, c_{ex} etc. are equal to $\langle b \rangle$, $\langle c \rangle$ etc.

The error of a is given by

$$\sigma_a^2 = \left(\frac{\partial f}{\partial b}\right)^2 \cdot \sigma_b^2 + \left(\frac{\partial f}{\partial c}\right)^2 \cdot \sigma_c^2 + \left(\frac{\partial f}{\partial e}\right)^2 \cdot \sigma_e^2 + \ldots \tag{1.33}$$

Here $\partial f/\partial b$ is a partial derivative of f with respect to b, i.e. the derivative with respect to b is calculated provided the rest of the variables (c etc.) are held fixed. The partial derivatives with respect to c, e etc. are defined in the same way. The partial derivatives must be evaluated at the expected values b_{ex}, c_{ex}, e_{ex} etc. Equations (1.25), (1.28) and (1.30) are the specific cases of Eq. (1.33).

The analysis of the equations discussed in this section leads naturally to several recommendations. First of all one should avoid the measurements in which a desired quantity comes out as a difference of two large numbers. For example, it is better to measure directly the thickness of a pipe wall rather than to determine it by subtracting the inner diameter from the outer one (and dividing the result by two). In the latter case the relative error grows significantly since the measured quantity (the wall thickness) is small while its error is determined by adding up the diameter errors and therefore increases. One should keep in mind that the measurement error of 0.5% of the outer diameter could be 5 or more percent of the wall thickness.

The quantities which are treated with the aid of Eq. (1.26) (e.g., when the density of an object is evaluated using its weight and volume) should be measured with approximately the same relative error. For instance, if the volume of an object is determined with an error of 1% and the object weight is known with an error of 0.5%, the object density is determined with an error of 1.1%. Obviously it does not make sense to waste one's time and effort on measuring the object weight with an error of 0.01%.

For measurements which results are treated by means of Eq. (1.29) one should pay attention to the error of the quantity with the greatest exponent.

When planning an experiment one should always remember about a subsequent treatment of the results and write down the explicit expressions for the errors in advance. The equations help to understand which quantities must be measured more carefully than others.

Some laboratory guidelines

Any laboratory experiment should be regarded as a research project in miniature. A lab description provides only a guideline of the experiment. *A specific content, skills, and knowledge which a student would gain from the experiment are mostly due to student' attitude rather than the lab description.* The most valuable skills which a student is able to develop during the laboratory course are: thinking about an experiment, applying theoretical knowledge in the laboratory setting, careful planning of the experiment and avoiding mistakes, and noticing often insignificant little things which could potentially initiate an important research project.

The experimental results are summarized in a lab report which must include the following

1) theoretical motivation of the experiment including a brief derivation of the required equations;

2) a diagram of the experimental setup;

3) a plan of the experiment and tables with experimental data;

4) data treatment: calculations of intermediate quantities, tables, plots, and diagrams of the results, calculations of the final result;

5) comparison of the obtained results with reference data (in handbooks and manuals), discussion of possible mistakes, suggestions of future experiments.

Preparation to experiment. Firstly, it is necessary to read an experiment description and the corresponding theoretical material. It is necessary to have a clear account of the phenomena, physical laws, and orders of magnitude of the quantities under study, as well as the experimental method, instruments, and a measurement procedure.

The lab reports should be written in a sufficiently large workbook so it can be used, at least, during one semester. A report should start with a number and the title followed by a theoretical introduction, a diagram of the experimental setup, and a description of the experiment procedure.

Before an experiment it is necessary to think over the procedure suggested in the lab description and determine a required number of measurements. This will help to prepare the tables for the experimental data.

It is desirable to figure out in advance the range in which the measured quantities will reside and to choose the appropriate units. At least, this must be done at the beginning of the experiment. Also it is necessary to estimate measurement accuracy. If a quantity is expressed in terms of powers of quantities measured directly one should make sure that the relative errors of the quantities with greater exponents are small, i.e. these quantities should be measured with a better accuracy. When possible one should avoid measuring a quantity as a difference between two numerically close quantities . As it was already mentioned, the thickness of a pipe wall should be measured directly rather than calculated as a difference between the outer and inner diameters.

Beginning. At the beginning of the experiment one should carefully examine the experimental setup, figure out how to switch the instruments on and off, how to handle them, and check that the equipment is in order.

Measurement instruments must be handled with care. It it is not a good idea to unscrew the casing of a sensitive instrument and change the settings.

It is necessary to write in the workbook the specifications of the instruments (first of all, an accuracy class, the maximal value on the scale, and the scale graduation) since they are used for data treatment.

When assembling electric circuits a power supply must be connected no sooner than the circuit is completely assembled.

Operation of the experimental setup must be checked before the main measurements. The first measurements are done to make sure that everything is in order and the range and accuracy of the measurements are correctly chosen. If the dispersion of the first results does not exceed a systematic error, multiple measurements are not necessary.

The malfunctions of instruments or the installation must be documented in the workbook and reported to the instructor.

Measurements. The results of the measurements should be written in detail with necessary explanations.

It is useful to plot the measured quantities during the experiment. It helps to see the regions where the values change rapidly. In these regions the quantity must be measured with a better precision (more measurement points) than in the regions where the curve is smooth. If the quantity is assumed to exhibit a priori dependence (e.g. linear) in some interval, the measurements should cover a wider range in order to determine the boundaries of the interval where the dependence holds.

Significant dispersion of the results at the beginning of an experiment should alert the experimenter. Often it is better to interrupt the experiment and try to eliminate the source of the dispersion rather than to do a large number of measurements in order to reach the required accuracy. If a quantity measured depends on some parameter or another quantity that changes gradually, one must make sure that the conditions have not changed during the experiment. To this end the initial measurements should be repeated at the end of experiment or the whole measurement repeated in reversed order.

Before each table one should write down the unit of scale graduations and accuracy class of every measurement instrument. It is better to write down the graduations of an instrument rather than the corresponding value of the measured quantity, e.g. current or voltage. This will spare you some mistakes when writing down the readings. At the end of the day, the data treatment is always possible while repeating the experiment is sometimes difficult or even impossible.

The units should be chosen appropriately so that the results be represented by values in the range from 0.1 to 1000. In this case the tables would be readable and the plots would be convenient to use. For instance, Young' moduli (E) of metals are represented by very large numbers in the SI, so it is convenient to use the unit 10^{10} N/m^2. (For aluminum the numerical value is 7.05.) The corresponding column in the table or a plot axis will be labeled as E, 10^{10} N·m^{-2}. The comma is important: it separates the quantity from its unit. Numerical factors in front of the units can be replaced by words or their abbreviations.

Sometimes another convention is used. A quantity to be displayed in a table or next to a plot axis is measured in ordinary units and represented as a product of the quantity multiplied by some numerical coefficient. For

Young' modulus this convention reads: $E \cdot 10^{-10}$, N·m^{-2}. Although the numerical value listed in the table remains the same (7.05 for aluminum) this convention is less common since the coefficient could be incorrectly referred to the measurement unit.

Evaluation, analysis, and presentation of the results. The results of direct measurements presented as tables and plots are then used for evaluating the desired quantities and their errors and for finding relationships between the quantities. It is convenient to use the same workbook for the calculations and write the results in blank columns of the tables together with raw experimental data. This would help to check and analyze the results of calculations and compare them with the data.

Finally a measured quantity must be presented in the following form: the average, the error, and the number of measurements. The final result of indirect measurements is determined via their functional dependence on the directly measured quantities which are used for evaluating the averages and the errors.

Since an error itself is seldom known with a better accuracy than 20% the numerical value of the error in the final result should be rounded to one or two significant digits. For example, it would be correct to write errors as ± 3, ± 0.2, ± 0.08, and ± 0.14; and incorrect ± 3.2, ± 0.23, and ± 0.084. It is not correct to round the value ± 0.14 to ± 0.1 since the rounding decreases the error by 40%. The last digit of the average value of a quantity and the last digit of the error must be in the same position. For example, a result written as 1.243 ± 0.012 for the error of ± 0.012 takes the form 1.24 ± 0.03 for a larger error of ± 0.03 and 1.2 ± 0.2 for 0.2. Extra significant digits could be kept in intermediate calculations for better rounding of the result. Depending on the chosen units the error could be tens, hundreds, thousands of the units or more. For example, if the weight of an object is 58.3 ± 0.5 kg its expression in grams must be $(583 \pm 5) \cdot 10^2$ g. It would be incorrect to write 58300 ± 500 g.

Finally the obtained results are compared to the tabulated values from reference books in order to estimate their quality.

Plotting graphs. Graphs should be plotted on a special graphing paper: regular graph paper, millimeter paper, or logarithmic paper. The plot size (and the paper size) should not be too large or too small. The optimal size is between a quarter and a full workbook page.

Before starting to plot the graph it is necessary to choose an appropriate scale and the origin on the axes, so that the points are spread over the whole plot area.

Figure 1.4 shows two plots. The experimental points occupy the lower right corner of the plot on the left, which is a poor choice. On the right plot

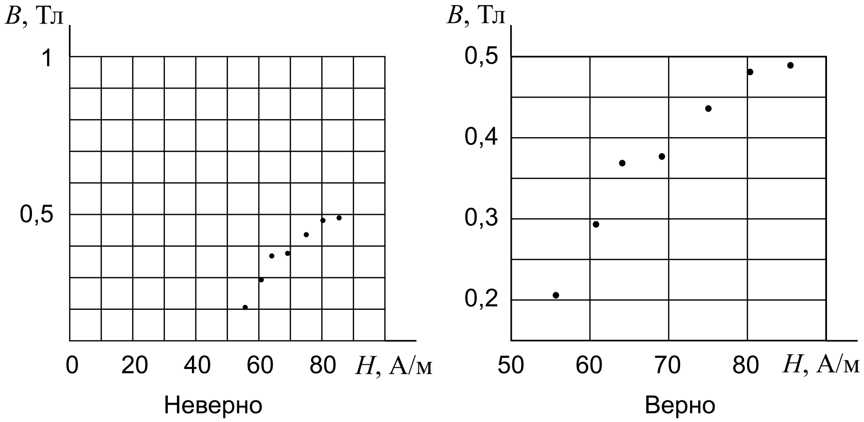

Fig. 1.4. Examples of correct and incorrect plots

a larger scale of the Y axis is chosen and the abscissa origin is displaced, so the points are evenly spread over the whole plot area.

The names and units of the plotted quantities should be clearly written. Labeling all the graduations on the axes is not necessary, there should be enough labels to make the plot comprehensible and easy to use. It is better to place the labels on the outer sides of axes. If a graph paper has a network of lines of different thickness, the solid lines should be used for round values. It is convenient when the network square corresponds to 0.1, 0.2, 0.5, 1, 2, 5, or 10 units of a quantity and it is usually inconvenient when a square corresponds to 2, 5, 3, 4, 7, etc. units. An inconvenient scale of axis graduations makes it difficult to determine coordinates of a point, which leads to frequent mistakes. The name of a quantity on abscissa is usually written below the axis at the right end and the name of quantity on the ordinate is written at the top left to the axis. A unit of measurement is separated by comma.

Points on a plot should be marked clearly. The points should be drawn by pencil, so that possible mistakes could be corrected. Explanatory notes should not obscure the plot; the coordinates of the points written next to them are not necessary. If an explanation is in order the corresponding point or the curve is labeled by a number explained in the text or in the captions. It is advisable to plot the points obtained under different conditions, e.g. heating/cooling or increasing/decreasing a load, by using different marks or colors.

The known errors of experimental points should be drawn as vertical

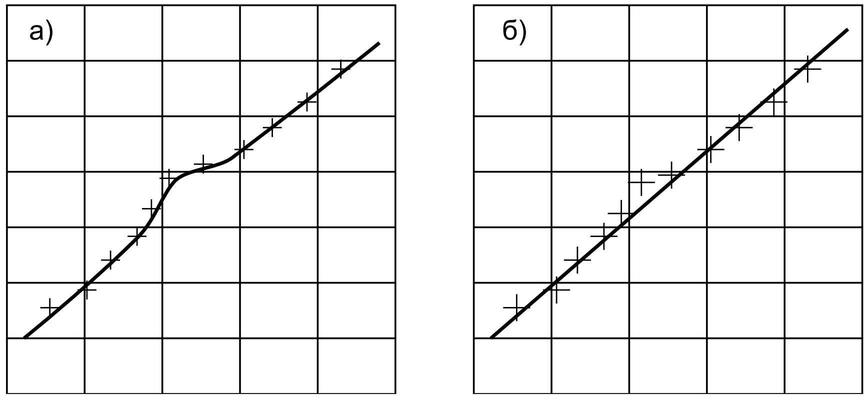

Fig. 1.5. Drawing line through experimental points

and horizontal bars which lengths are proportional to the corresponding errors. In this case a point is represented by a cross. Half of the horizontal bar is equal to an error of abscissa quantity and half of the vertical bar is equal to an ordinate quantity error. If an error is too small to be represented graphically, the corresponding points are drawn as bars $\pm\sigma$ long in the direction where the error is not negligible. Such a representation of experimental points facilitates the analysis of the results. In particular, it would be easier to find the best mathematical relation describing the data and to compare the results with theoretical calculations and other results.

Figures 1.5a, b show the same data points with different errors. The plot in Fig. 1.5a undoubtedly corresponds to a non-monotonous function. The function is shown by a solid curve. The same data set for a larger experiment error (Fig. 1.5b) is well described by a straight line: only a single point deviates from the line by more than one standard deviation (and less than two standard deviations). It is only when the points are drawn with their errors shown explicitly it becomes clear that the data in 1.5a requires a curve to be drawn and the data in 1.5b does not.

Often measurements are performed in order to obtain or confirm a specific relation between the measured quantities. In this case the corresponding curve should be drawn through the experimental points. If necessary, the errors of the measured quantities are then found using deviations of the points from the curve. It is not difficult to draw a straight line through the data points. Therefore if a relation between the plotted quantities is hypothesized or already known from theory it is better to plot some functions of the quantities, so that the relation between the functions becomes

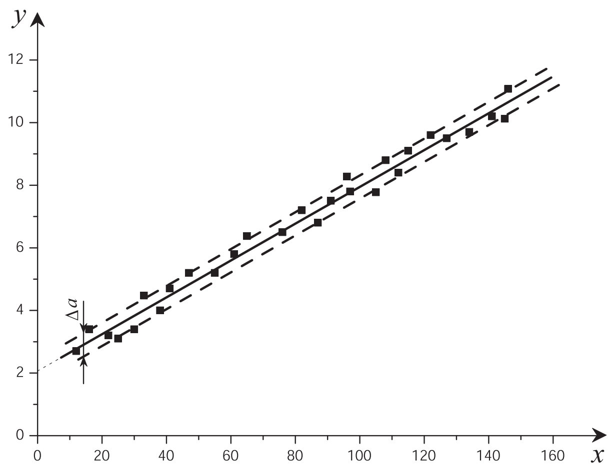

Fig. 1.6. Graphical method of data treatment. Estimating random error of parameter a

linear. For example, consider an experiment that verifies the relation between a time interval it takes an object to fall in the gravitational field and the initial height from which the fall starts. In this case one should plot the height versus the time squared because these quantities are directly proportional to each other if the field is uniform and the air drag is negligible. It would be less convenient to plot the time versus square root of the height although the relation between them is also linear. Notice that logarithms of the time and the height are also proportional in this case but the linearity is significantly violated by relatively small errors of height and time at the beginning of the fall. Logarithmic scale is convenient for power laws and large ranges of changes of variables. In this case a linear dependence allows one to determine the power law exponent.

There are different methods of drawing straight lines through experimental points. The most simple method, which is useful for estimating errors although too rough for getting the final result, requires a transparent ruler or a sheet with a straight line drawn on it. A transparent ruler allows one to determine how many points there are on both sides of the line. The latter should be drawn so that there is an equal number of the points on both sides. The line parameters (a slope and an intercept) are determined from the plot. This gives an analytic expression of the form: $y = a + bx$, which for a nonzero a, does not pass through the origin.

Random errors of the parameters a and b could be estimated from the

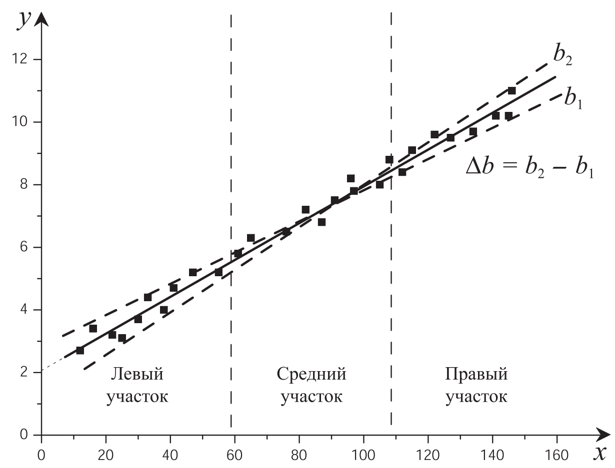

Fig. 1.7. Graphical method of data treatment. Estimating random error of parameter b

plot as follows. To estimate the error of a one determines how much the line is displaced so that the ratio of the numbers of points on both sides becomes 1 : 2 (see Fig. 1.6). Explicitly, the line is displaced upward by Δa_1, so that one third of the points is above the curve and two thirds is below. When the curve is displaced downward by Δa_2, two thirds of the points is above and one third is below. If there are n points, an estimate of the standard deviation a is

$$\sigma_a = \frac{\Delta a_1 + \Delta a_2}{\sqrt{n}}.$$

To estimate the error of the slope b one should divide the whole range of abscissa values x into three equal parts (see Fig. 1.7). The line is then drawn so that the ratio of the numbers of the points on both sides of the line in the external parts is 1 : 2. In other words, increase the slope until the number of points in the left part above the line is twice as large as the number below it and the number of points in the right part below the line is twice the number above, let the corresponding slope be b_1. Then decrease the slope until the number of points below the line in the left part is twice as large as above and in the right part the number above is twice as below, let the corresponding slope be b_2. Then the error of b is estimated as

$$\sigma_b = \frac{b_1 - b_2}{\sqrt{n}}.$$

If the relation is $y = kx$, so that the line goes through the origin, the error of k is estimated as follows. The range of abscissa values x is divided into three equal parts. The points close to the origin are ignored. One should determine the value k_1, for which the number of the points above the line is half the number of the points below (for all the points in the central and right parts), and k_2 for the opposite ratio. The slope k is estimated as

$$\sigma_k = \frac{k_1 - k_2}{\sqrt{n}}.$$

The method of least squares is a more precise and better justified method of drawing a straight line through a set of points. The line is drawn so that the sum of squares of the point deviations from the line is minimal. This means that the coefficients a and b of $y = a + bx$ are found by minimizing the sum

$$f(a,b) = \sum_{i=1}^{n} \left[y_i - (a + bx_i)\right]^2. \tag{1.34}$$

Here x_i and y_i are the coordinates of experimental points.

Now let us give the explicit equations for a, b and their errors in terms of the arithmetic means of x_i and y_i:

$$b = \frac{\langle xy\rangle - \langle x\rangle\langle y\rangle}{\langle x^2\rangle - \langle x\rangle^2}, \tag{1.35}$$

$$a = \langle y\rangle - b\langle x\rangle\,. \tag{1.36}$$

The corresponding errors are given by

$$\sigma_b \approx \frac{1}{\sqrt{n}}\sqrt{\frac{\langle y^2\rangle - \langle y\rangle^2}{\langle x^2\rangle - \langle x\rangle^2} - b^2}, \tag{1.37}$$

$$\sigma_a = \sigma_b\sqrt{\langle x^2\rangle - \langle x\rangle^2}. \tag{1.38}$$

If it is known that the points are described by a linear dependence $y = kx$, the slope k and its error are given by

$$k = \frac{\langle xy\rangle}{\langle x^2\rangle}, \tag{1.39}$$

$$\sigma_k \approx \sqrt{\frac{\langle x^2\rangle\langle y^2\rangle - \langle xy\rangle^2}{n\langle x^2\rangle^2}} = \frac{1}{\sqrt{n}}\sqrt{\frac{\langle y^2\rangle}{\langle x^2\rangle} - k^2}. \tag{1.40}$$

Table 4

Some approximation formulae

Equation	Accuracy of 5%	Accuracy of 1%	Accuracy of 0.1%
	$\lvert a\rvert$ is less	$\lvert a\rvert$ is less	$\lvert a\rvert$ is less
$\frac{1}{1+a} \approx 1 - a$	0.22	0,1	0.032
$\sqrt{1+a} \approx 1 + \frac{1}{2}a$	0.63	0.28	0.09
$\frac{1}{\sqrt{1+a}} \approx 1 - \frac{1}{2}a$	0.36	0,16	0.052
$e^a \approx 1 + a$	0.31	0.14	0.045
$\ln(1+a) \approx a$	0.10	0.02	0.002
$\sin a \approx a$	0.55	0.24	0.077
$\tan a \approx a$	0.4	0.17	0.055
$\cos a \approx 1 - \frac{a^2}{2}$	0.8	0.34	0.11
$(1+a)(1+b)\ldots \approx 1 + a + b + \ldots$			
$\sin(\theta + a) = \sin\theta + a\cos\theta$			
$\cos(\theta + a) = \cos\theta - a\sin\theta$			

This method is the most time consuming but if a calculator or computer is available the method must be preferred.

Sometimes one is not interested in a functional dependence approximating a data set, rather the experimental points are used to find numerical values between them. If so, interpolation methods are employed. In the simplest case a linear interpolation between two neighboring points is used. Interpolating by parabola requires three points.

It should be emphasized that the plots provide a graphical representation of the experimental data. They are very useful for comparing theory and experiment, understanding qualitative features of relations, and for estimating quantity dynamics. However, the final results of any experiment are documented in a table.

Usually the final results are obtained from experimental data by means of calculation. An accuracy of the latter should not exceed an accuracy of the data. Often the calculations are simplified by means of approximation formulae given in Table 4. The numerical entries are the values for which the approximations in the left column provide the accuracy claimed in the table upper row.

It should be noted that our recommendations on data treatment are

Table 5

Synopsis of basic equations

Arithmetic mean of measured quantity	$x_{av} = \langle x \rangle = \dfrac{1}{n}\sum\limits_{i=1}^{n} x_i$
Standard deviation of arithmetic mean of measured quantity	$\sigma = \sqrt{\dfrac{1}{n(n-1)}\sum\limits_{i=1}^{n}(x_i - \langle x \rangle)^2}$
Propagation of (independent) errors	$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots$
Error of calculated result	$A = B \pm C \Rightarrow \sigma_A^2 = \sigma_B^2 + \sigma_C^2$ $\left.\begin{array}{l} A = B \cdot C \\ A = B/C \end{array}\right\} \Rightarrow \left(\dfrac{\sigma_A}{A}\right)^2 = \left(\dfrac{\sigma_B}{B}\right)^2 + \left(\dfrac{\sigma_C}{C}\right)^2$ $A = B^{\beta} \cdot C^{\gamma} \Rightarrow \left(\dfrac{\sigma_A}{A}\right)^2 = \beta^2\left(\dfrac{\sigma_B}{B}\right)^2 + \gamma^2\left(\dfrac{\sigma_C}{C}\right)^2$
Recommended scales	1:1; 1:2; 1:5; 1:10; 1:20 ... 2:1; 5:1; 10:1; 20:1 ...
Drawing the best straight line $y = a + bx$	$b = \dfrac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2}, \quad a = \langle y \rangle - b\langle x \rangle$
Drawing the best straight line $y = kx$	$k = \dfrac{\langle xy \rangle}{\langle x^2 \rangle}$

neither complete nor strict since they are designated for the freshmen whose mathematical background is not sufficient to consider the questions related to mathematical statistics in detail. More elaborated treatment will be possible after first two years of study when enough experience in the lab is gained and sufficient mathematics is learned. Therefore some equations used for data treatment were given without proof, some of them are shown in Table 5.

Finally, several recommendations on the data treatment.

When processing the data it is necessary to consider possible sources of mistakes. Accuracy of intermediate calculations should exceed the data accuracy to eliminate errors related to calculations. Usually it is enough if the accuracy of intermediate calculations will exceed the accuracy of the final result by one significant digit.

Literature

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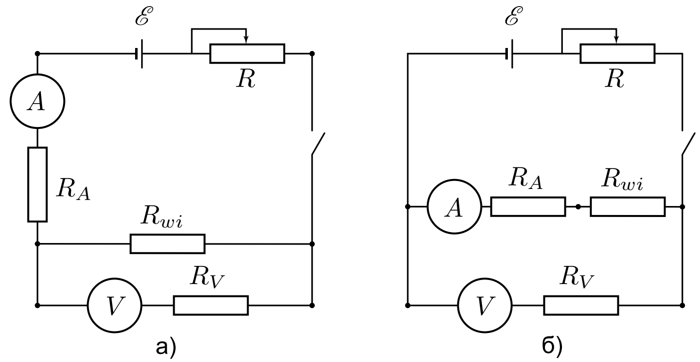

Fig. 1. Circuits for measuring resistance by means of ammeter and voltmeter

Lab 1.1.1

Determination of systematic and random errors in measurement of specific resistance of nichrome.

Purpose of the lab: determination of specific resistance of nichrome wire and calculation of systematic and random errors.

Tools and instruments: ruler, caliper, micrometer, nichrome wire, ammeter, voltmeter, power supply, Wheatstone bridge, rheostat, switch.

The specific resistance of the material of a uniform wire with a circular cross-section can be determined according to the following equation

$$\rho = \frac{R_{wi}}{l}\frac{\pi d^2}{4}, \tag{1}$$

where R_{wi} is the resistance, l is the length, and d is the diameter of the wire. Therefore to determine the specific resistance of the wire material one should measure the following parameters of the wire: the length, the diameter, and the electrical resistance.

One should take into account that the diameter of a real wire is not constant but varies slightly along the wire. The diameter variation is random. Therefore in equation (1) one should substitute a value of the diameter averaged along the wire and take into account its random error.

The resistance R_{wi} is measured using one of the circuits shown in Fig. 1. In the figure R is a variable resistance (rheostat), R_A is the resistance of an ammeter, R_V is the resistance of voltmeter, and R_{wi} is the wire resistance.

Let V and I be the readings of voltmeter and ammeter, respectively. The values of the wire resistance calculated using these readings, namely, $R_{wi1} = V_a/I_a$ for the circuit (a) and $R_{wi2} = V_b/I_b$ for the circuit (b) will differ from each other and from the true value R_{wi} due to internal resistances of the instruments. However using Fig. 1 one can easily find the relation between R_{wi} and the obtained values R_{wi1} и R_{wi2}. In the first case the voltmeter measures a voltage across the wire correctly, whereas the ammeter does not measure the current through wire, rather it shows the value of the total current flowing through the wire and the voltmeter. Therefore

$$R_{wi1} = \frac{V_a}{I_a} = R_{wi}\frac{R_V}{R_{wi} + R_V}. \tag{2}$$

In the second case the ammeter measures the current through the wire but the voltmeter measures a total voltage across the wire and the ammeter. For this case

$$R_{wi2} = \frac{V_b}{I_b} = R_{wi} + R_A. \tag{3}$$

It is convenient to rewrite equations (2) and (3) as follows. For the circuit (a):

$$R_{wi} = R_{wi1}\frac{R_V}{R_V - R_{wi1}} = \frac{R_{wi1}}{1 - (R_{wi1}/R_V)} \approx R_{wi1}\left(1 + \frac{R_{wi1}}{R_V}\right). \tag{4}$$

For the circuit (b):

$$R_{wi} = R_{wi2}\left(1 - \frac{R_A}{R_{wi2}}\right). \tag{5}$$

The bracketed terms in Eqs. (4) and (5) define corrections which should be taken into account during the measurement. (Although the corrections due to internal resistance of the instruments can be calculated at any time, usually this is not done. In our case the calculation of the corrections turns out to be very simple but for real circuits an accounting for the corrections is time consuming and should be repeated every time the instrument is switched, which seems impossible in practice.) The calculation provides an example of a systematic error due to simplification of the exact equation. For the circuit (a) the resistance R_{wi} turns out to be less than the calculated value and for the circuit (b) it is greater.

The classical method of measuring a resistance with the aid of a dc bridge (Wheatstone bridge) is more precise. The standard bridge P4833 is used for the control measurement of the wire resistance.

In the assembly the nichrome wire stretched between two fixed plane clamping contacts is used as a resistance. The length of a wire section which resistance is measured can be varied by means of a mobile contact.

LABORATORY ASSIGNMENT

1. Get familiar with the operation principles of the measurement instruments. Practice to measure dimensions of different objects with the aid of a caliper and a micrometer.
2. Measure the wire diameter at 8–10 different locations and write down the results in a table. Compare the results obtained by means a caliper and a micrometer. Average out the obtained diameter values. Calculate the cross-sectional area of the wire and estimate an accuracy of the result.
3. Write down into a new table the basic parameters of the ammeter and the voltmeter: the type of an instrument, the accuracy class, the maximal value of the scale x_n, the number of scale graduations n, the scale factor x_n/n, the sensitivity n/x_n, the absolute error Δx_M, and the internal resistance of the instrument (for a given maximal value of the scale).
4. Using the indicated internal resistances of the instruments and the known approximate value of the wire resistance, 5 Ohm, estimate the values of the corrections to R_{wi} corresponding to the circuits shown in Fig. 1 with the aid of Eq. (4) and (5). Choose the circuit that provides a minimal value of the correction.
5. Using a ruler measure the length of a wire section to be explored (between fixed and mobile clamping contacts) and assemble the chosen electrical circuit. Turn on the current. Varying it by means of the rheostat write down in a new table the readings of the ammeter and the voltmeter for 5–6 different values of the current (usually during a direct measurement the readings of the instruments are written directly as the scale graduations):

N_{meas}	1	2	3	4	5	6
V, дел						
I, дел						
V, В						
I, А						

   Repeat the measurement by increasing and decreasing the current. Plot the dependence $V = f(I)$ and calculate the value of R using the plot. Then calculate the resistance R_{wi}. Estimate the error of R_{wi}.
6. Measure the wire resistance using the dc bridge (Wheatstone brigde) P4833. How much does the result differ from the value measured previously? Does the result lie in the error interval of the result obtained with the aid of the ammeter and the voltmeter?
7. Carry out the measurements pp. 5, 6 for three different values of the wire length.

8. Determine the resistivity of the wire material using Eq. (1). Estimate the accuracy of the obtained value. Which accuracy of the wire resistance is required for the attained accuracy of the wire length and the cross-section?
9. Compare the results with the tabulated values.

Literature

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Example of lab report 1.1.1

The instruments used: ruler, caliper, micrometer, nichrome wire, ammeter, voltmeter, power supply, dc bridge (Wheatstone bridge), rheostat, switch.

1. A caliper accuracy is 0.1 mm. A micrometer accuracy is 0.01 mm.

2. Measure a diameter of the wire with a caliper (d_1) and a micrometer (d_2) at 10 different locations (Table 1).

Table 1

Wire diameter

	1	2	3	4	5	6	7	8	9	10
d_1, mm	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4
d_2, mm	0,36	0,36	0,37	0,36	0,37	0,37	0,36	0,35	0,36	0,37
	$\bar{d}_1 = 0{,}4$ mm $\bar{d}_2 = 0{,}363$ mm									

The table shows no random error in the caliper measurements. Therefore the accuracy of the result is due to the caliper accuracy (a systematic error):

$$d_1 = (0.4 \pm 0.1)\ mm.$$

The measurement results obtained with the micrometer contain both systematic and random errors:

$$\sigma_{syst} = 0.01\ mm, \qquad \sigma_{rand} = \frac{1}{N}\sqrt{\sum_{i=1}^{n}(d-\bar{d})^2} = \frac{1}{10}\sqrt{4.1\cdot 10^{-4}} \approx 2\cdot 10^{-3}\ mm,$$

$$\sigma = \sqrt{\sigma_{syst}^2 + \sigma_{rand}^2} = \sqrt{(0.01)^2 + (0.002)^2} \approx 0.01\ mm.$$

Since $\sigma_{rand}^2 \ll \sigma_{syst}^2$ the wire diameter can be considered constant along the wire with an accuracy σ_d totally determined by σ_{syst} of the micrometer:

$$d_2 = \bar{d}_2 \pm \sigma_d = (0.363 \pm 0.010)\ mm = (3.63 \pm 0.10)\cdot 10^{-2}\ cm.$$

3. Determine the cross-sectional area of the wire:

$$S = \frac{\pi d_2^2}{4} = \frac{3.14\cdot(3.63\cdot 10^{-2})^2}{4} \approx 1.03\cdot 10^{-3}\ cm^2.$$

The value of the error σ_S can be calculated as follows

$$\sigma_S = 2\frac{\sigma_d}{d}S = 2\frac{0.01}{0.36}\cdot 1.03\cdot 10^{-3} \approx 6\cdot 10^{-5}\ cm^2.$$

Thus $S = (1{,}03 \pm 0.06)\cdot 10^{-3}\ cm^2$, i. e. the accuracy of the cross-sectional area amounts to 6%.

4. Write down the basic specifications of the instruments in Table 2.

Table 2

Basic specifications of instruments

	Voltmeter	Ammeter
System	Moving-coil	Electromagnetic
Accuracy class	0.5	0.5
Maximal scale value x_l	0.3 V	0.15 A
Number of scale graduations n	150	75
Scale factor $x_п/n$	2 mV/grad	2 mA/grad
Sensitivity $n/x_п$	500 grad/V	500 grad/A
Absolute error Δx_M	1.5 mV	0.75 mA
Internal resistance (for given maximal scale value)	500 Ohm	1 Ohm

5. It is known that $R_{wi} \approx 5$ Ohm, $R_V = 500$ Ohm, and $R_A = 1$ Ohm. Using Eqs. (4) and (5) estimate the corrections for R_{wi}:

for the circuit in Fig. 1a $R_{wi}/R_V = 5/500 = 0.01$, i. e. 1%;

for the circuit in Fig. 1b $R_A/R_{wi} = 1/5$, i. e. 20%.

Conclusion: the circuit in Fig. 1a ensures the better accuracy in a measurement of a relatively small resistance.

6. Assemble the circuit shown in Fig. 1a.

7. Carry out the experiment for three values of the wire length written below: $l_1 = (20.0 \pm 0.1)$ cm; $l_2 = (30.0 \pm 0.1)$ cm; $l_3 = (50.0 \pm 0.1)$ cm.

Repeat the measurement for increasing and decreasing current. Write down the instrument readings in Table 3. Record the results obtained by using the dc bridge (Wheatstone bridge) P4833 in Table 4.

8. Plot the dependencies $V = f(I)$ for all three values of the wire length by drawing straight lines through the experimental points (Fig. 2). From the plots one can conclude that there is no difference between the values obtained for increasing and decreasing current. One can also conclude that the random scatter is negligible and could be ignored.

Table 3

Readings of voltmeter and ammeter

$l = 20$ cm				$l = 30$ cm				$l = 50$ cm			
V, grad $2\frac{mV}{grad}$	I, grad $2\frac{mA}{grad}$	V, mV	I, mA	V, grad $2\frac{mV}{grad}$	I, grad $2\frac{mA}{grad}$	V, mV	I, mA	V, grad $2\frac{mV}{grad}$	I, grad $0.5\frac{mA}{grad}$	V, mV	I, mA
26.0	12.5	52.0	25.0	26.0	8.5	52.0	17.0	34.5	28.0	69.0	14.0
32.5	15.5	65.0	31.0	35.0	11.5	70.0	23.0	44.1	35.6	88.2	17.8
63.2	31.1	126.4	62.2	62.5	20.4	125.0	40.8	67.1	54.5	134.2	27.3
82.8	40.5	165.6	81.0	91.1	30.1	182.2	60.2	98.0	79.6	196.0	39.8
119.5	58.1	239.0	116.2	118.5	38.9	237.0	77.8	127.0	103.3	254.0	51.7
137.8	67.0	275.6	134.0	150.0	49.5	300.0	99.0	147.3	120.0	294.6	60.0
131.0	64.1	262.0	128.2	139.5	46.1	279.0	92.2	142.0	114.6	284.0	57.8
101.5	49.5	203.0	99.0	130.0	42.9	260.0	85.8	116.2	94.0	232.4	47.0
88.1	43.0	176.2	86.0	103.1	34.0	206.0	68.0	85.0	69.2	170.0	34.6
78.2	38.1	156.4	76.2	74.2	24.5	148.4	49.0	61.1	49.5	133.2	24.8
51.0	24.9	102.0	49.8	42.5	14.1	85.0	28.2	41.3	33.2	82.6	16.6
29.1	13.9	58.2	27.8	23.0	7.5	46.0	15.0	31.0	25.2	62.0	12.6

Table 4

Wire resistance

$l = 20$ cm	$l = 30$ cm	$l = 50$ cm
$R_0 = 2{,}080$ Ohm (using P4833)	$R_0 = 3{,}062$ Ohm (using P4833)	$R_0 = 5.010$ Ohm (using P4833)
$R_{wi} = 2.060$ Ohm	$R_{wi} = 3.030$ Ohm	$R_{av} = 4.92$ Ohm
$R_{wi} = 2.068$ Ohm	$R_{wi} = 3.048$ Ohm	$R_{wi} = 4.97$ Ohm
$\sigma_{R_{wi}} = 0.008$ Ohm	$\sigma_{R_{wi}} = 0.014$ Ohm	$\sigma_{R_{wi}} = 0.04$ Ohm

9. Using the plots find the average values of the resistances by calculating the slope of the corresponding straight line: $R_{av} = V/I$, where I and V are the current and the voltage taken at some point of the line close to its end. Write down the results in Table 4.

10. Estimate the accuracy of R_{av} as follows

$$\frac{\sigma_{R_{av}}}{R_{av}} = \sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_I}{I}\right)^2},$$

where I and V are the maximal values of current and voltage obtained in the experiment, whereas σ_V and σ_I are the standard deviations of the measurements by means of the voltmeter and the ammeter. The error σ_V equals half of the

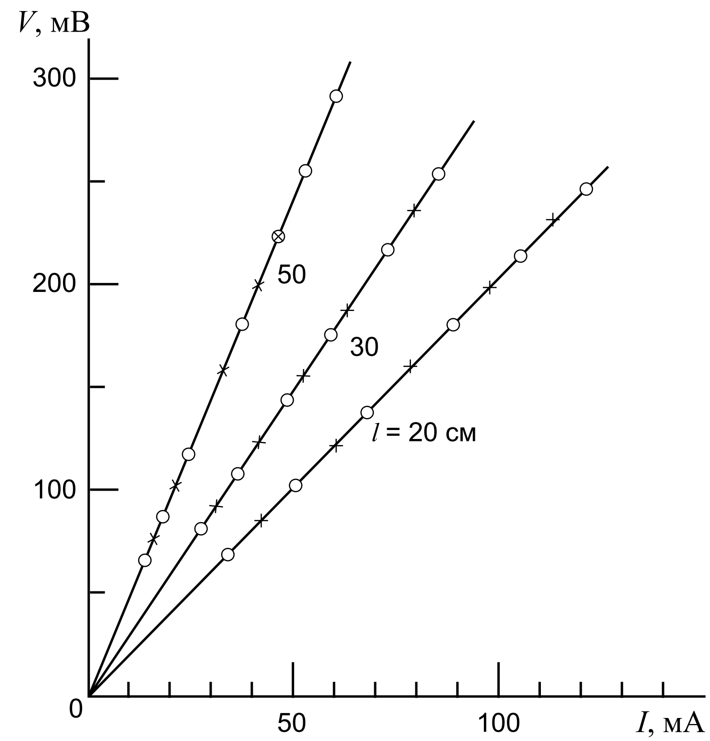

Fig. 2

absolute error of the voltmeter:

$$\sigma_V = \frac{\Delta x}{2} = \frac{1{,}5}{2} \approx 0.75\ mV.$$

For the ammeter the result can be similarly obtained: $\sigma_I = 0.75/2 \approx 0.4\ mA$.

An example of the calculation of $\sigma_{R_{av}}$ for a wire of the length $l = 30$ cm; from Tables 3 and 4 $R_{av} = 3.030$ Ohm, $V = 300$ mV, $I = 99$ mA.

$$\sigma_{R_{av}} = R_{av}\sqrt{\left(\frac{\sigma_V}{V}\right)^2 + \left(\frac{\sigma_I}{I}\right)^2} = 3.03 \cdot \sqrt{\left(\frac{0.75}{300}\right)^2 + \left(\frac{0.4}{99}\right)^2} \approx 1.4 \cdot 10^{-2}\ Ohm.$$

Record the results of the calculations in Table 5.

Table 5

l, sm	20	30	50
R_{Ohm}, Ohm	2.060	3.030	4.92
$\sigma_{R_{av}}$, Ohm	0.008	0.014	0.04

11. For all three values of the length l take into account the measurement correction for the resistance as follows

$$R_{wi} = R_{av} + \frac{R_{av}^2}{R_V}.$$

Due to a relatively small value of the correction one can ignore it: $\sigma_{R_{wi}} = \sigma_{R_{av}}$. The results are written down in Table 4.

12. Compare the wire resistances measured by the voltmeter and the ammeter with the values obtained by using the dc bridge (Wheatstone bridge) P4833. The results coincide within the accuracy of the experiment.

13. Determine the wire resistivity according to equation (1) and find the accuracy σ_ρ as follows

$$\frac{\sigma_\rho}{\rho} = \sqrt{\left(\frac{\sigma_R}{R}\right)^2 + \left(2\frac{\sigma_d}{d}\right)^2 + \left(\frac{\sigma_l}{l}\right)^2}.$$

The results are written in Table 6.

Table 6

l, cm	ρ, 10^{-4} Ohm·cm	σ_ρ, 10^{-6} Ohm·cm
20	1.06	6
30	1.05	6
50	1.02	6

Finally: $\rho = (1.04 \pm 0.06) \cdot 10^{-4}$ Ohm·cm.

A major contribution to the error σ_ρ is due to an uncertainty of the wire diameter; it amounts to ~3%. This error doubles because the diameter is squared in the final formula, so it amounts to ~6%. Therefore it is sufficient to measure the wire resistance with an accuracy about 3–4%.

The obtained value of the resistivity is compared with a tabulated value. For the resistivity of nichrome at 20 °C the reference book (Physical magnitudes. M.:Energypublish, 1991. P. 444) gives the values from $1.12{\cdot}10^{-4}$ Ohm·cm to $0.97{\cdot}10^{-4}$ Ohm·cm depending on the mass ratios of the alloy components. The closest value to that obtained in the lab is $1.06{\cdot}10^{-4}$ Ohm·cm for the alloy: 70÷80% Ni, 20% Cr, 0÷2% Mn (mass ratios).

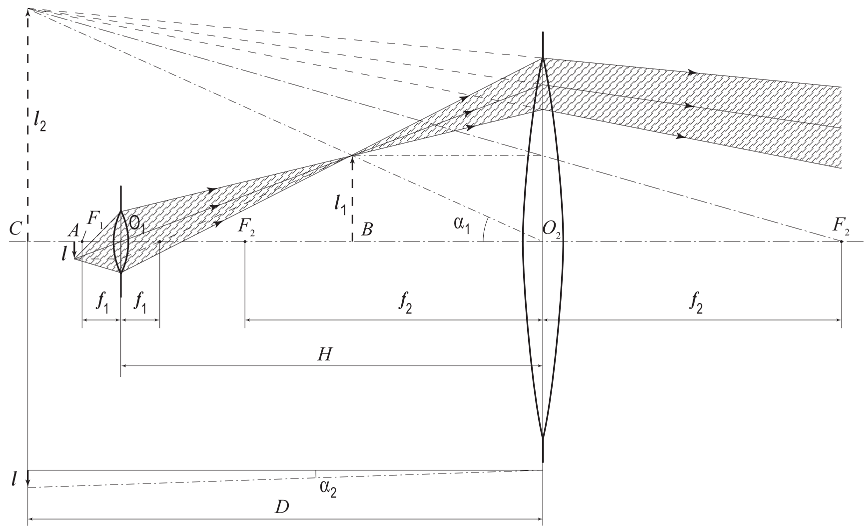

Fig. 1. Optical path in microscope

Lab 1.1.2

Measurement of linear expansion coefficient of a rod with the aid of microscope

Purpose of the lab: to measure the dependence of linear expansion of metal rod versus temperature and to determine its linear expansion coefficient.

Tools and instruments: a microscope, an ocular micrometer, a ruler with millimeter graduations, a quartz tube, a metal rod, an electric heater, a variable transformer, a resistance thermometer, the Wheatstone bridge P4833, a power supply, and a galvanometer.

Microscope. Microscope is an optical instrument designed to magnify images of small objects. The magnifying part of the microscope consists of two sets of lenses called objective and eyepiece (ocular) which are mounted in a tubus about 160 mm apart. We do not intend to study a microscope design in detail, so we concentrate on its operation principle. For simplicity we replace the objective and the eyepiece with two equivalent thin lenses.

Optical path in microscope is shown in Fig. 1. The object l is placed next to the front focal point (just before it) of the short-focus objective

Л_1 which creates the large real image l_1. The image is viewed through the eyepiece Л_2 which serves as a magnifying glass. The eyepiece creates the virtual image l_2 at a convenient distance from observer's eye. The position of l_2 can be varied by changing location of l_1 relative to the front focus of the eyepiece. This is achieved by a small displacement of the microscope with respect to the object.

Microscope magnification is its most important parameter. There are linear and angular magnifications. *Linear magnification* equals the ratio of a transverse size of the image l_2 to that of the object l:

$$\Gamma = \frac{l_2}{l}. \tag{1}$$

Angular magnification equals the ratio of the tangent of the angle α_1 subtended by the image l_2 in the microscope to the tangent of the angle α_2 subtended by the object at the conventional closest distance of distinct vision $D = 25$ cm from unaided eye:

$$\gamma = \frac{\tan\alpha_1}{\tan\alpha_2}. \tag{2}$$

The notations l, l_1, l_2, α_1, and α_2 are those in Fig. 1.

Consider first the linear magnification Γ. Let us write it as

$$\Gamma = \frac{l_2}{l} = \frac{l_2}{l_1}\frac{l_1}{l} = \Gamma_{oc}\Gamma_{ob}. \tag{3}$$

The first factor Γ_{oc} is called *ocular magnification* and the second one $\Gamma_{\text{об}}$ is called *objective magnification*. It should be obvious from Fig. 1 that

$$\Gamma_{\text{об}} = \frac{l_1}{l} = \frac{O_1B}{O_1A}. \tag{4}$$

The distance O_1A is approximately equal to the focal length of the objective and the point B is close to the focal point of the eyepiece, also $f_2 \ll H$, which gives

$$O_1A \approx f_1, \qquad O_1B \approx H - f_2 \approx H. \tag{5}$$

The tubus length H is usually equal to 160 mm. Replacing the numerator and denominator in (4) by their approximate values (5) one obtains:

$$\Gamma_{\text{об}} \approx \frac{H}{f_1}. \tag{6}$$

This value is not exactly equal to the objective magnification, however it is independent of the eyepiece and the microscope adjustment. It is this value which is engraved on the objective casing.

Now consider the eyepiece magnification:

$$\Gamma_{\text{ок}} = \frac{l_2}{l_1} = \frac{O_2C}{O_2B}. \tag{7}$$

It was already mentioned that $O_2B \approx f_2$. The value of O_2C on the other hand depends on the microscope adjustment. Near-sighted observers set $O_2C = 10 - 15$ cm and far-sighted place l_2 at a distance of 40 cm, sometimes even at infinity. When calculating the eyepiece magnification it is customary to set $O_2C = D = 25$ cm, which corresponds to the conventional closest distance of distinct vision for normal human eye. Substituting these values in (7) we get:

$$\Gamma_{oc} = \frac{D}{f_2}. \tag{8}$$

This value is called ocular magnification and it is engraved on its casing.

Now let us consider the *angular magnification*:

$$\gamma = \tan\alpha_1 : \tan\alpha_2 = \frac{l_2}{O_2C} : \frac{l}{D}. \tag{9}$$

For $O_2C = D$ the angular and linear magnifications are equal: $\Gamma = \gamma$.

Equation (3) shows that to get a preliminary estimate of the microscope magnification it would suffice to multiply the eyepiece and objective magnifications. The value obtained is only approximate. A better estimate should be determined experimentally.

In practical measurements the object size is compared to some scale. The scale can be placed in the plane of the object but this is not always possible. More often the scale is located in the plane of the virtual image l_1. In this case both the object and the scale can be viewed simultaneously and therefore be more reliably compared. However, in this setup the scale is compared to the magnified image l_1 rather than to the object itself, so an additional calibration is necessary.

Ocular micrometer. The microscope used in the lab is equipped with an ocular micrometer. It consists of an immobile glass plate with scale graduations and a mobile glass plate with a cross and two parallel marks located in the eyepiece focal plane (see Fig. 2). The mobile plate can move relative to the immobile scale: one turn of the micrometer screw displaces the marks and the cross by one scale graduation (1 graduation = 1 mm). The circumference of

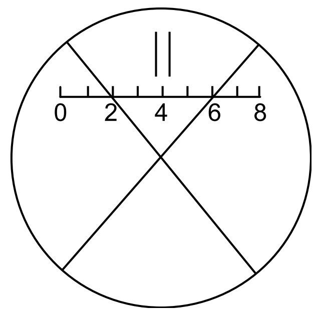

Fig. 2. Scale of ocular micrometer

the screw knob is divided by graduations into 100 parts. Turning the knob by one graduation displaces the cross and the marks by 0.01 mm. Thus the scale in the image plane l_1 (the focal image of the eyepiece) is the scale of the ocular micrometer.

To determine the size of the object l itself it is necessary to calibrate the micrometer scale by using another scale (object scale) placed instead of the object. In so doing the microscope adjustments should not be altered. The object scale is a glass plate with graduations several hundredths of millimeter apart.

Calibration of ocular scale. The ocular scale should be calibrated before using the microscope for the measurements. First of all, the scale should be clearly visible, this is achieved by adjusting the outer lens of the eyepiece. Then the object scale is placed on the microscope stage. To achieve better visibility the object scale must be illuminated at some angle to the glass plane and perpendicular to the marks. Then the clear image of the scale must be obtained. To this end one moves the microscope tubus down almost to the plate by using the focus wheel of coarse adjustment. One should control the distance between the object and the microscope objective by watching from the microscope side when moving the tubus down[1]. Then one should *slowly* lift the tubus until the object scale comes into sight and obtain the sharp image of the scale by using the focus wheel of fine adjustment. Then the scale should be moved to the center of the field of vision. The object scale must be illuminated so that both the object and ocular scale are clearly visible.

The alignment of the ocular and objective scales is checked by the method of parallax. If both images are in the same plane, a small lateral displacement of eye will not result in their mutual displacement. If the displacement is detected the tubus position is corrected by the focus wheel until the parallax is eliminated.

The object scale should be placed on the stage so that the graduations on both scales are parallel. Then the center of the cross is aligned with a graduation on the object scale. The scale graduation and the graduation on the micrometer knob are recorded. Then one should move the cross along the object scale by several millimeters and repeat the procedure for another scale graduation. Using the results it is not difficult to calibrate the ocular scale, i.e. to determine the actual size in the object plane corresponding to one graduation of the ocular scale. The calibration procedure must be repeated three or four times, the results must be tabulated and averaged.

[1] It should be emphasized that moving the tubus down without control is prohibited. Otherwise the objective could press the object and one of them can break down.

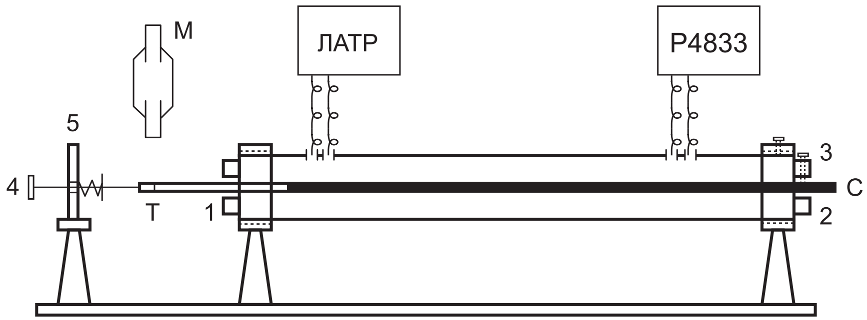

Fig. 3. Experimental setup for measurement of linear expansion coefficient

Laboratory setup. The experimental setup for the measurement of linear expansion coefficient is shown in Fig. 3. The rod under study is placed in a steel tube with electric heater inside. The right end of the tube is firmly attached to a support by a screw. The left end can freely move along the tube axis on the left support. The tube ends are sealed, the rod under study is inserted inside the tube through the openings at the ends. The rod can freely move through the end 1 and it is fixed at the end 2 with the screw 3. A quartz tube T[2] with a mark on it is placed between the end of the rod coming out of the tube end 1 and the spring stopper 4 mounted on the support 5.

The electric heater power supply is controlled by means of the variable transformer. The rod temperature is measured by the resistance thermometer made of copper wire which is wound around the rod and extends between the rod ends.

Usually the rod (and the resistance thermometer as well) is heated from room temperature t_r, the corresponding wire resistance is R_r. The wire resistance depends on temperature as

$$R_t \approx R_r(1 + \Theta(t - t_r)), \tag{10}$$

where Θ is the temperature coefficient of resistance (for copper $\Theta = 4.3 \cdot 10^{-3}$ °C $^{-1}$ at 20 °C), which gives

$$\Delta t = t - t_r = \frac{R_t - R_r}{\Theta R_r}. \tag{11}$$

[2] Coefficient of thermal expansion of fused quartz is negligible compared to that one of metal.

The rod length increases with temperature and the mark on the quartz tube shifts. The displacement is measured with the aid of the microscope equipped with the ocular micrometer. The coefficient of linear expansion of the rod is determined by the equation:

$$\alpha = \frac{L_t - L_r}{L_r(t - t_r)}, \tag{12}$$

where L_t and L_r are the rod lengths at t and t_r respectively. Substitution of the temperature difference $t - t_r$ from eq. (11) finally gives

$$\alpha = \frac{(L_t - L_r)R_r}{L_r(R_t - R_r)}\Theta = \frac{R_r}{L_r}\frac{\Delta L}{\Delta R}\Theta = \frac{R_r}{L_r}\frac{\Delta n}{\Delta R}B\Theta, \tag{13}$$

where B is the ocular scale graduation in millimeters and Δn is the displacement measured in ocular scale graduations.

LABORATORY ASSIGNMENT

1. Make sure that you understand the operation principles of the microscope and the ocular micrometer.
2. Using the object scale calibrate the scale of the ocular micrometer (express ocular graduation in millimeters).
3. Replace the object scale on the microscope stage with the quartz tube T attached to the rod end.

   Obtain the clear image of the mark on T. The initial position of the mark on the ocular scale must be chosen so that the mark remained in the field of vision during the whole experiment. Record the initial position of the mark on the ocular scale at room temperature.
4. Make sure that you understand the operation principle of the Wheatstone bridge P4833 and get it ready for the experiment.
5. Connect the resistance thermometer to the bridge and measure its resistance R_r at room temperature. Record the room temperature t_r.

   Choose the operation mode of the bridge corresponding to the maximum sensitivity.
6. Determine dependence of the rod length on temperature (actually the length vs the wire resistance). To this end connect the electric heater to the transformer output. Set a moderate voltage and wait until the rod is uniformly heated. Measure the thermometer resistance using the bridge P4833 and record the cross position on the ocular scale.

   Gradually increase the output transformer voltage and record the resistances and the corresponding positions of the cross.

7. Plot the experimental points in coordinates n (the cross position) and R (the resistance). Draw the straight line through the points and determine its slope $\Delta n/\Delta R$. Find the error $\delta(\Delta n/\Delta R)$ using the method of least squares (see p. 32).
8. Substitute the value of the slope in Eq. (13) and evaluate the linear expansion coefficient α. The rod length is written on the setup.
9. Evaluate the error of α.

   An example of the lab report is presented in the appendix.

Questions

1. For a given accuracy of ΔL determine the required accuracy of the rod length and the thermometer resistance.
2. Determine the contributions to the error of α: due to calibration of the ocular scale, due to determination of the mark position, due to measurement of the room temperature, and due to the error of the temperature coefficient of resistance.
3. Near-sighted and far-sighted observers adjust the microscope so that the image l_2 is either at small or at large distance, respectively, from the observer's eye. Is it linear or angular magnification that changes less?

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Example of lab report 1.1.2

1. Calibration of the ocular micrometer scale using the object scale. The object scale has a length of 1 mm=100 graduations.

Table 1

n (# of ocular scale graduations)		Δn_i	$\overline{\Delta n}$
for $l = 0$	for $l = 0.5$ mm		
1.44	6.12	4.68	
1.35	6.08	4.73	4,70
1.52	6.21	4.69	

The length of the ocular scale graduation is

$$B = \frac{\Delta l}{\overline{\Delta n}} = \frac{0.50\ mm}{4.70\ grad} = 1.06 \cdot 10^{-1}\ mm/grad.$$

The relative error is

$$\frac{\delta B}{B} = \sqrt{\left(\frac{\delta l}{\Delta l}\right)^2 + \left(\frac{\delta n}{\Delta n}\right)^2},$$

where $\delta l \approx 0.005$ mm (one half of the object scale graduation), and the overall error of the ocular scale,

$$\delta n = \sqrt{(\delta n_1)^2 + (\delta n_2)^2},$$

is determined by the systematic error $\delta n_1 = 0.005$ (one half of the graduation scale of the micrometer) and by the random error

$$\delta n_2 = \sqrt{\frac{1}{m(m-1)} \sum_{i=1}^{m} (\Delta n_i - \overline{\Delta n})^2} = 1.2 \cdot 10^{-2}\ grad.$$

Thus

$$\delta n = \sqrt{(1.2)^2 + (0.5)^2} \cdot 10^{-2} \approx 1.3 \cdot 10^{-2}\ grad,$$

$$\frac{\delta B}{B} = \sqrt{\left(\frac{0.005}{0.5}\right)^2 + \left(\frac{0.013}{4.7}\right)^2} \approx 0.01 = 1\%.$$

Finally the graduation length of the ocular micrometer scale is

$$B = (1.06 \pm 0.01) \cdot 10^{-1}\ mm/grad.$$

2. The thermometer resistance is measured at room temperature $t_r = 22$ °C. The Wheatstone bridge P4833 operates at the ratio $N = 1$; $R_r = 49.29 \pm 0.01\ \Omega$. The position of the mark on the ocular scale is $n_r = 1.88$ grad.

3. The positions of the mark vs the thermometer resistances are tabulated in 2, the plot is shown in Fig. 4.

Table 2

R, Ω	n, grad	R, Ω	n, grad	R, Ω	n, grad
49.25	1.88	52.81	3.65	55.74	5.05
49.85	2.17	53.11	3.73	56.06	5.14
50.15	2.31	53.81	4.08	56.25	5.24
50.93	2.75	54.51	4.46	56.58	5.40
51.50	2.95	55.05	4.74	56.97	5.58
52.18	3.28	55.29	4.82	57.11	5.67

The slope of the curve is determined graphically:

$$\frac{\Delta n}{\Delta R} = \frac{5.67 - 1.88}{57.11 - 49.25} = 0.482\ grad/\Omega.$$

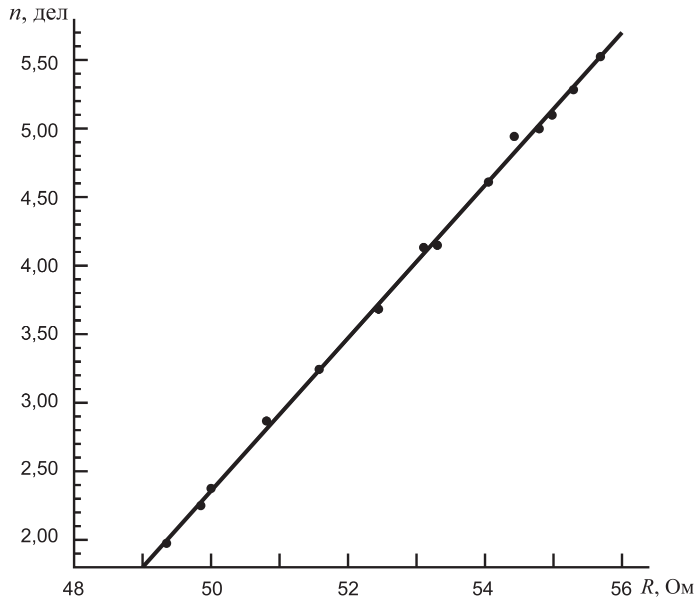

Fig. 4. Position of the mark versus thermometer resistance

The linear expansion coefficient is found from Eq. (13). Since $L_r = (600 \pm 1)$ mm, $\Theta = 4.30 \cdot 10^{-3}$ °C^{-1} at $t_r = 20$ °C, one gets

$$\alpha = \frac{R_r}{L_r}\frac{\Delta L}{\Delta R}\Theta = \frac{R_r}{L_r}\frac{\Delta n}{\Delta R}B\Theta = \frac{49.25 \cdot 0.482 \cdot 0.106 \cdot 4.30 \cdot 10^{-3}}{600} = 1.80 \cdot 10^{-5}\ {}^\circ\mathrm{C}^{-1}.$$

It is impossible to estimate the error of $\Delta n/\Delta R$ using the plot because the straight line fits the points well. Therefore one should use the method of least squares which provides a better accuracy. The goal is to determine the best fit value b in the equation $n_t = a + bR_t$ and the error δb of the coefficient b. The calculation (see (1.35) and (1.37)) gives

$$b = \frac{\langle Rn\rangle - \langle R\rangle\langle n\rangle}{\langle R^2\rangle - \langle R\rangle^2} = 0.477\ grad/\Omega,$$

$$\delta b = \frac{1}{\sqrt{m}}\sqrt{\frac{\langle n^2\rangle - \langle n\rangle^2}{\langle R^2\rangle - \langle R\rangle^2} - b^2} = 0.011\ grad/\Omega.$$

The linear expansion coefficient is determined by the Eq. (13):

$$\alpha = \frac{49.25 \cdot 0.477 \cdot 0.106 \cdot 4.3 \cdot 10^{-3}}{600} = 1.785 \cdot 10^{-5}\ {}^\circ\mathrm{C}^{-1}.$$

The relative error is

$$\frac{\delta\alpha}{\alpha} = \sqrt{\left(\frac{\delta R_r}{R_r}\right)^2 + \left(\frac{\delta L_r}{L_r}\right)^2 + \left(\frac{\delta\Theta}{\Theta}\right)^2 + \left(\frac{\delta B}{B}\right)^2 + \left(\frac{\delta b}{b}\right)^2} \approx$$

$$\approx \sqrt{\left(\frac{1}{106}\right)^2 + \left(\frac{105}{4771}\right)^2} \approx 0.024 = 2.4\%.$$

The absolute error is

$$\delta\alpha = \alpha \cdot 0.024 = 1.785 \cdot 0.024 \cdot 10^{-5} = 0.043 \cdot 10^{-5}\ {}^\circ\mathrm{C}^{-1}.$$

Finally

$$\alpha = (1.79 \pm 0.04) \cdot 10^{-5}\ {}^\circ\mathrm{C}^{-1}.$$

The value of α found directly from the plot agrees with this value.

Lab 1.1.3

Statistical treatment of measurements.

Purpose of the lab: to apply methods of processing experimental data to measurement of electrical resistance.

Tools and instruments: a set of resistors (250–300) and the digital voltmeter V7-23 operating in the mode «Measurement of resistance to direct current».

Industrial production of resistors is a complicated technological process. An actual value of resistance differs from the nominal. The error can be both systematic and random. Inaccurate adjustment of a resistor manufacturing machine results in systematic errors. Random errors are due to non-uniformity of the wire (in width and chemical composition) used in resistor production, random changes of temperature, and machine backlashes.

Measurement of resistance in this lab requires a precise instrument because of relatively small differences from the nominal. An appropriate instrument is «universal digital voltmeter V7-23» used in the «Measurement of resistance to direct current» mode which provides a relative measurement accuracy of hundredths of percent. Exact values can be found in the device manual.

Thus the error due to the measurement instrument is negligible in comparison with the deviations from the nominal arising in the process of resistor manufacturing.

The main part of the lab is measurement of all resistances of a given set (about 250–300) and calculation of the mean value (1.15):

$$\langle R\rangle = \frac{1}{N}\sum_{i=0}^{N} R_i. \tag{1}$$

If the number of resistors is large enough one could obtain a specification of the set that no longer depends on the number of resistors.

To describe random errors arising in resistor production one should plot a histogram. To this end one should find the maximum R_{max} and the minimum R_{min} values of the obtained results. The difference $R_{max} - R_{min}$ is divided into m parts. The obtained value is called the interval of resistance variation:

$$\Delta R = \frac{R_{max} - R_{min}}{m}. \tag{2}$$

The histogram is plotted as follows. The intervals of resistance variation are plotted on the abscissa. The number Δn of the measurements which belong to a given interval is plotted on the ordinate. However it is convenient to divide Δn by the total number of measurements N (which is the absolute probability of occurrence in the corresponding interval) and by the interval width ΔR (which gives probability density). So the quantity plotted on the ordinate is

$$y = \frac{\Delta n}{N\Delta R}.$$

It is interesting to observe how the histogram changes as the number of partitions m increases. In the process m must remain much less than N.

One should also plot the mean value of the resistance on the abscissa and notice how it is located relative to the histogram.

Standard deviation specifies dispersion of a random quantity (1.18):

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(R_i - \langle R\rangle)^2}. \tag{3}$$

It is instructive to plot the points $\langle R\rangle - \sigma$ and $\langle R\rangle + \sigma$ on the abscissa and notice how the histogram is located relative to these points.

The value of σ defines the Gaussian (normal) distribution (1.16):

$$y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(R-\langle R\rangle)^2}{2\sigma^2}}. \tag{4}$$

One should plot this function on the histogram.

LABORATORY ASSIGNMENT

1. Read carefully the brief manual «universal digital voltmeter V7-23» and pay special attention to the section «Measurement of resistance to direct current».
2. Turn on the voltmeter power supply and wait for 15–20 minutes until the voltmeter warms up.
3. Measure resistances of the given set of $N = 250–300$ resistors.
4. Plot the histogram (follow instructions in the text) for $m = 10$ and $m = 20$.
5. Calculate $\langle R\rangle$ and compare it with the nominal value. Plot the values on the abscissa and compare them with the position of maximum of the histogram. Plot the values $\langle R\rangle - \sigma$ and $\langle R\rangle + \sigma$ on the abscissa. Compare the histogram width with these values.
6. Calculate the number of the resistances which belong to the interval between $\langle R\rangle - \sigma$ and $\langle R\rangle + \sigma$ and between $\langle R\rangle - 2\sigma$ and $\langle R\rangle + 2\sigma$.
7. Plot the Gaussian distribution and compare it with the histograms corresponding to different numbers of partitions n.

Literature

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Example of lab report 1.1.3

The following equipment is used: a set of 270 resistors with the nominal of 560 Ohm and the universal digital voltmeter V7-23 operating in the mode «Measurement of resistance to direct current».

The measured resistances of 270 resistors (in Ohm) are listed in Table 1 in ascending order.

Using the tabulated resistances we plot the histograms for $m = 20$ and $m = 10$. To compare the histogram with the normal distribution we plot the number of results Δn in a given interval divided by the total number of results N and by the interval width ΔR on the abscissa, instead of plotting the number Δn itself. The values of Δn and $w = \Delta n/(N\Delta R)$ versus the group number k are listed in Tables 2 and 3, respectively. The histograms are shown in Figs. 1 and 2. We calculate the mean value of the resistance according to Eq. (1):

$$\langle R\rangle = \frac{1}{N}\sum_{i=1}^{N} R_i = 560{,}7\ Ohm.$$

The standard deviation is determined according to Eq. (3):

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(R_i - \langle R\rangle)^2} \approx 9\ Ohm.$$

Table 1

Measured resistances of 270 resistors

539.7	540.7	541.5	542.3	542.8	543.4	543.9	544.3	545.0
545.4	545.5	545.9	546.0	546.1	546.1	546.5	546.8	546.9
547.6	547.9	548.0	548.4	548.7	548.9	549.0	549.1	549.2
549.3	549.3	549.3	549.4	549.6	549.7	549.7	549.9	550.0
550.1	550.8	551.8	552.0	552.1	552.3	552.3	552.7	553.0
553.2	553.3	553.6	553.7	553.9	554.2	554.2	554.2	554.2
554.3	554.3	554.5	554.7	554.8	555.0	555.1	555.1	555.1
555.2	555.3	555.3	555.3	555.3	555.3	555.3	555.5	555.6
555.7	555.7	555.7	555.7	556.0	556.1	556.1	556.4	556.4
556.4	556.5	556.5	556.6	556.6	556.7	556.8	556.8	556.9
557.0	557.0	557.0	557.1	557.1	557.1	557.2	557.2	557.3
557.3	557.4	557.4	557.4	557.5	557.5	557.7	557.7	557.8
557.8	557.9	558.0	558.0	558.0	558.1	558.1	558.4	558.4
558.5	558.5	558.5	558.5	558.6	558.7	558.8	558.8	558.8
558.8	558.9	558.9	559.0	559.0	559.1	559.1	559.3	559.3
559.4	559.4	559.4	559.6	559.7	559.7	559.7	559.7	559.8
559.8	559.8	559.8	559.9	560.0	560.0	560.0	560.0	560.0
560.2	560.2	560.3	560.4	560.4	560.4	560.6	560.7	560.9
561.0	561.1	561.1	561.1	561.4	561.5	561.5	561.6	561.9
562.0	562.0	562.0	562.3	562.3	562.5	562.5	562.6	562.6
562.6	562.7	562.7	562.7	562.8	562.8	563.0	563.1	563.1
563.2	563.6	563.6	563.6	564.0	564.2	564.3	564.4	564.5
564.5	564.5	564.8	565.0	565.1	565.1	565.2	565.3	565.3
565.7	565.8	565.9	566.1	566.1	566.2	566.8	566.9	567.6
567.8	568.0	568.1	568.3	568.7	568.9	569.1	569.7	569.8
570.2	570.3	570.6	570.7	571.0	571.1	571.1	571.5	572.1
572.3	572.4	572.6	572.9	573.0	573.1	573.2	573.3	573.5
574.0	574.0	574.8	575.1	576.0	576.3	578.1	578.8	578.8
579.0	579.2	579.7	579.8	580.0	580.5	580.6	581.1	581.4
582.7	582.9	583.1	584.1	584.3	586.6	586.7	587.3	589.0

Table 2

k	1	2	3	4	5	6	7	8	9	10
Δn	4	4	10	17	8	16	44	45	28	25
$w \cdot 1000$	6	6	15	25	12	24	65	67	41	37
k	11	12	13	14	15	16	17	18	19	20
Δn	14	10	11	10	3	7	5	5	3	1
$w \cdot 1000$	21	15	16	15	4	10	7	7	4	1

Table 3

k	1	2	3	4	5	6	7	8	9	10
Δn	8	27	24	89	53	24	21	10	10	4
$w \cdot 1000$	6	20	18	66	39	18	16	7	7	3

The intervals between $\langle R\rangle-\sigma$ and $\langle R\rangle+\sigma$ and between $\langle R\rangle-2\sigma$ and $\langle R\rangle+2\sigma$ contain 46% and 93% of the total number of the results, respectively. Normal distribution is defined by Eq. (4):

$$y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(R-\langle R\rangle)^2}{2\sigma^2}}.$$

This function is shown in Figs. 1 and 2. One can see that the histograms agree well with the normal distribution. According to the normal distribution a resistance belongs to the interval between $\langle R\rangle - \sigma$ and $\langle R\rangle + \sigma$ with the probability of 68% and to the interval between $\langle R\rangle - 2\sigma$ and $\langle R\rangle + 2\sigma$ with the probability of 95%.

The experiment shows that the resistance of a resistor chosen randomly belongs to the interval 560 ± 9 Ohm with the probability of 46%, to the interval 560 ± 18 Ohm with the probability of 93%, and to the interval 560 ± 27 Ohm with the probability of 99%.

Thus all the resistances belong to 5-percent interval ($\langle R\rangle \pm 3\sigma$).

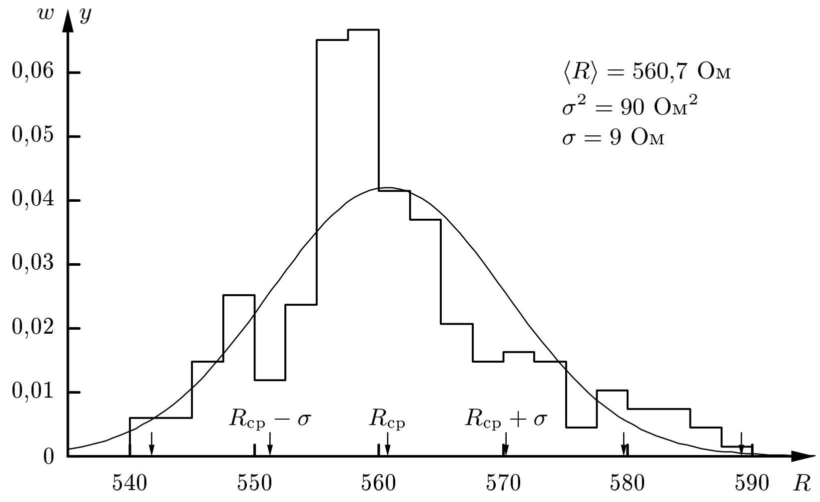

Fig. 1. Histogram for $m = 20$

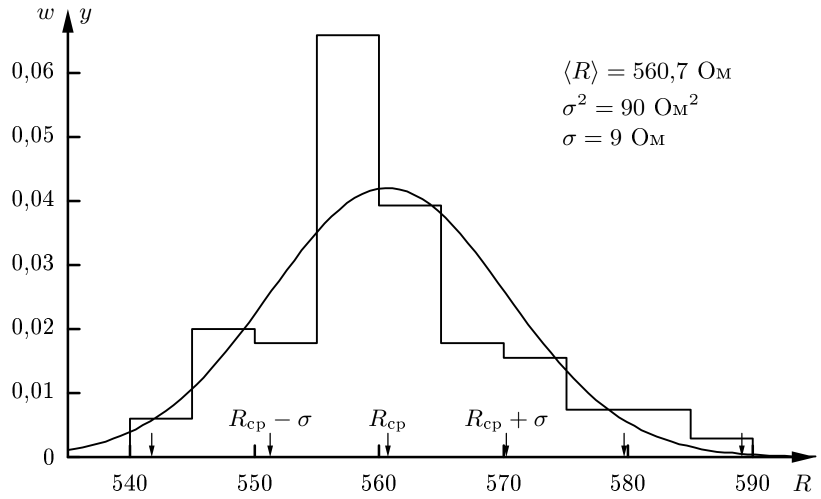

Fig. 2. Histogram for $m = 10$

Lab 1.1.4

Measurement of radiation background intensity.

Purpose of the lab: to apply methods of experimental data processing and to study statistical laws in measurement of radiation background intensity.

Tools and instruments: Geiger-Müller counter CTC-6, a power unit, and a computer connected to the counter via the interface.

As it was stated in discussion of random errors the random dispersion of experimental results could be due to both systematic errors and random variations of the measured quantity. A flux of cosmic rays, which considerably contribute to radiation background, randomly varies with time. If variations take place near a definite value one says that the flux fluctuates. In this case the random variable can be characterized by the mean value and the standard deviation from this mean value. To determine the mean value and standard deviation one employs the same methods which are used in calculations of the mean values and random errors of measurements. Cosmic rays are divided into the primary ones reaching the Earth orbit from outer space and the secondary rays arising due to interaction of the primary rays with the Earth atmosphere. The secondary rays constitute the major part of the rays at the sea level. The main part of the primary rays comes to the Earth from the Galaxy; the rest arises due to solar activity and has lower energies. The origin of the galactic rays is a subject of debate. A part of cosmic radiation is emitted by stars of the Galaxy during chromospheric flares in the same way as on the Sun. More energetic rays are apparently due to supernova outbursts and pulsars. It is hypothesized that acceleration of space particles can be attributed to high-velocity clouds of plasma originated in supernova explosions and to galactic magnetic fields. The primary cosmic rays form the flux of stable particles with a high kinetic energy which in the appropriate units lies in the range from 10^9 to 10^{21} electron-volt (or shortly eV 1 electron-volt = $= 1{,}6 \cdot 10^{-12}\ erg = 1{,}6 \cdot 10^{-19}$ J). It is found that in outer space the particle flux is independent of direction (isotropic). The basic quantity specifying the amount of particles in the cosmic rays is intensity I. By definition intensity is the number of particles passing through the unit area perpendicular to the direction of observation per unit of spatial angle (steradian) and per unit of time. The unit of measurement is

$$\frac{number\ of\ particles}{cm^2 \cdot sr \cdot s}.$$

For the isotropic distribution of cosmic rays that takes place outside the Earth atmosphere the density F of particle flux coming from the upper hemisphere equals

$$F = 2\pi \int_0^{\pi/2} I \cos\theta \sin\theta \, d\theta = \pi I \quad \left(\frac{amount\ of\ particles}{cm^2 \cdot s}\right).$$

The density of particles with absolute velocity V equals:

$$n = \frac{4\pi I}{V} \quad \left(\frac{amount\ of\ particles}{cm^3}\right).$$

Notice that the majority of particles outside the Earth atmosphere moves at speeds close to the speed of light c, therefore to estimate n one can substitute c for V. Also note, that the intensity of the secondary cosmic rays near the ground is proportional to $\cos^2\theta$, where θ is the angle between the velocity and the vertical.

Particle flux density is equal to the number of particles crossing the area of 1 cm^2 per 1 second. The density is 1 particle/(cm^2·s) at the distance about 50 km from the Earth surface. The majority of the particles has the energy of 10 GeV. Particles with energies less than 1 GeV are absent in the flux, which is apparently due to magnetic fields of the Earth and the Sun.

Generally the primary cosmic rays consist of protons (92%) and helium nuclei (6.6%) also called α-particles. Heavier nuclei (up to nickel) are also detected, they constitute about 0,8% of the net flux. Electrons and positrons constitute about 1%, the positron flux is ten times less than the electron one. γ-quanta with energies greater than 10^8 eV amount to only 0,01%. Time variation of the flux of primary cosmic rays is not significant. The most variable part consists of the particles with energies about 1 GeV; the variations are due to changing magnetic fields of the Solar system, 11-year cycles of solar activity, the 27-day period of the Sun revolution around its axis, chromospheric bursts of the Sun (5—13 bursts during an active year), and magnetic storms in the Earth magnetosphere.

When traversing the Earth atmosphere the primary cosmic rays interact with the atomic nuclei of atmosphere gases and produce the secondary cosmic rays. Only one of 100,000 protons of the primary rays reaches the ground. However there are a lot of secondary protons; together with muons (also called μ-mesons) and neutrons they form the so called hard (high-energy) component of the secondary cosmic rays. A radiation is called hard if it passes through the lead plate of 10 cm thick. The soft (low-energy) component of cosmic rays (shielded by a lead plate of 10 cm thick) mostly consists of electrons, positrons, and photons. The soft component in the atmosphere close to the ground is produced by the hard component. The flux density of soft component grows with height more rapidly than the hard component flux. The density of vertical flux of the soft component at the sea level is approximately half of the flux density of the hard component which equals $1{,}7{\cdot}10^{-2}$ particles/(cm^2·s). However the flux density of the soft component 15 km above the Earth is 4–5 times greater than that of the hard component. The net flux density of cosmic rays is maximum at the height of 17 km. Overall, the flux of cosmic rays at the sea level is about 100 times less than at the upper boundary of the Earth atmosphere and two thirds of the flux consist of muons. Analysis of silt on the ocean floor has revealed that the average flux density of cosmic rays remained approximately constant during the last 35 thousand years.

The flux density of secondary rays close to the ground strongly depends on direction. It has its maximum in the vertical direction and minimum in the horizontal one. The flux is approximately proportional to the square of the cosine of the angle between the flux and the vertical, which is due to increasing the length of the path of the rays in the Earth atmosphere. Small time variations of the flux density of secondary rays are caused by variations in pressure, temperature, and magnetic field in the Earth atmosphere.

Although the powerful particle accelerators are in operation nowadays, the cosmic rays remain the sole source of particles of ultrahigh energies. However such particles do not come frequently. A particle with the energy of 10^{19} eV crosses the area of one square meter only once in two thousand years. Of course the area of 10 square kilometers reduces the waiting period to several days. High energy particles are detected via the generated fluxes of secondary particles called air showers. The total number of particles in a shower originating about 20–25 km above the ground can reach several millions and covers the area of several square kilometers. The simultaneous detection of a large number of particles on a significant area proves their common origin and makes it possible to determine the energy of the parent particle.

Cosmic rays and natural radioactivity of the Earth and the atmosphere are primary sources of ions in the lower part of the Earth atmosphere (up to a height of 60 km). Ionization in the atmosphere initially decreases with height but higher than 1 km it starts to increase, the increase accelerates at the height of 3 km. The number of ions per unit volume is 3–4 times greater at the height of 5 km than at the sea level, but at the height of 9 km it is already 30 times greater.

Cosmic rays can be detected and their intensity can be measured via ion-

ization they produce. To this end a special device, namely, Geiger-Müller counter is used. The counter consists of a gas-filled vessel with two electrodes. Several types of such counters exist. The counter used in the lab (CTC-6) consists of a thin-walled metal cylinder operating as an electrode (cathode). The other electrode (anode) is a thin wire stretched along the cylinder axis. To use the counter in the particle count mode one should apply the voltage of 400 V on the electrodes. The particles of cosmic rays ionize the gas in the counter and also knock out electrons from its walls. These electrons are accelerated by the strong electric field between the electrodes and knock out secondary electrons in their collisions with the gas molecules. The secondary electrons in turn are accelerated and ionize gas molecules. This results in electron avalanche and the current through the counter sharply increases. The electric circuit of the counter is shown in Fig. 1.

A direct voltage is supplied to the counter by a power unit through resistor R. In the initial state the electrodes of the counter and capacitor C_1 are charged to 400 V, whereas the resistance of R is much less than leakage resistances of the counter and C_1. The capacitor C_2 blocks the direct voltage from being applied to the computer interface.

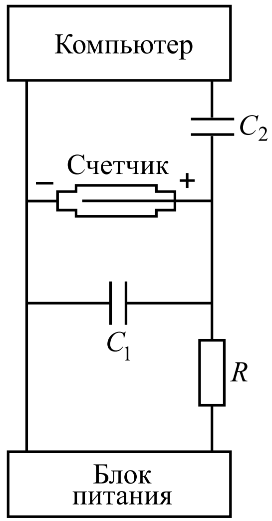

Fig. 1. Electric circuit of Geiger counter

A small current through the counter initiates a rapid electron avalanche of the charge accumulated in CTC-6 and capacitor C_1. The energy of the discharge is supplied by the capacitor C_1 which is connected in parallel with the counter. The discharge stops when the voltage across the counter becomes low and does not support the avalanche anymore (the potential difference across the electron free path is less than the ionization potential). The circuit returns to initial state in several RC_1. During this process a short pulse of current passes through the capacitor C_2 in the electronic circuit of computer interface.

Capacitance C_1 should be neither too high nor too small. The accumulated energy should be high enough to initiate the avalanche but the charging time of the capacitor ($\tau \sim RC_1$) called the dead time should not be too large because during this time the counter is not able to detect particles (usually the dead time is about several microseconds). In CTC-6 counter the capacitance of the Geiger tube serves as C_1, so the extra capacitor is not necessary.

The resistance R should also be neither too great (it increases the counter dead time), nor too small, otherwise the capacitor accumulates enough charge during the discharge and the avalanche would not terminate. Usually $R \sim 1$ MOhm.

The number of detected particles depends on the time of measurement, the counter size, the gas composition and its pressure, and also on the material of the counter walls. The major portion of detected particles is due to the natural radiation background.

Variations of particle flux, which are significant in the laboratory measurement, are related to short-time variations of physical conditions of the particle production and propagation in the Earth atmosphere. As it was already mentioned, the random variable measured in the lab is the particle flux density changing with time in a random way. The methods of data processing are the same as those of random errors. An estimate shows that the measurement error due to Geiger–Müller counter is negligible in comparison with variations of the flux itself (flux fluctuations). The measurement accuracy is mostly determined by the time required to restore the initial state of the counter after detection of a particle. This period is called the resolution time. The size of the counter must be chosen so that the time period between the particles passing through the counter exceeds the resolution time.

The quantity measured in the lab is the number of particles passed through the counter during time intervals of 10 and 40 seconds. Different time intervals are chosen to demonstrate that the standard distribution works better for larger time intervals and the histogram is more symmetric. Random values obtained for smaller time intervals should be treated by means of the Poisson distribution (see the Appendix).

The standard deviation of the number of counts measured for some period of time is equal to the square root of the mean number of counts for the same period: $\sigma = \sqrt{n_0}$ (see Eq. (10) of the Appendix). However the true value of the measured quantity is unknown (otherwise the experiment would be unnecessary). Therefore when evaluating the error of a particular measurement one has to substitute the measured value n rather than the true mean value n_0:

$$\sigma = \sqrt{n}. \tag{1}$$

Equation (1) shows that usually (with the probability of 68%) the variation of the measured number of particles n from the mean value is less than $\sqrt{n}$. The result of measurement is written as:

$$n_0 = n \pm \sqrt{n}. \tag{2}$$

Now consider the following important problem. Suppose one carries out a set of N measurements and obtains the number of particles n_1, n_2, ..., n_N. So far we used these numbers to determine how much the result of a particular measurement differs from the true mean value. As it was already mentioned this problem addresses reliability of the result obtained in a single measurement. But if one carries out several measurements the results can be used to solve another problem: they allow one to determine the mean value of the measured quantity better than for a single measurement. If N measurements have been carried out the mean value of the number of particles detected in one measurement equals obviously

$$\bar{n} = \frac{1}{N}\sum_{i=1}^{N} n_i, \tag{3}$$

whereas the standard error of the single measurement can be estimated according to Eq. (1.18), i. e. by substitution $n_0 = \bar{n}$ in Eq. (1.17) :

$$\sigma_{sep} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(n_i - \bar{n})^2}. \tag{4}$$

According to Eq. (1) one expects that this error is close to $\sqrt{n_i}$, i. e. $\sigma_{sep} \approx \sigma_i = \sqrt{n_i}$, where one could substitute any measured value n for n_i. Since n_i are different, one obtains different estimates of σ_{sep}. All of them differ from the more reliable estimation of σ_{sep} given by Eq. (4). This is to be expected. When processing measurement results, we always get approximate values of the measured quantity and the errors which could more or less coincide with the true values. The value $\sqrt{\bar{n}}$ is the closest one to σ_{sep} defined by Eq. (4), i. e.

$$\sigma_{sep} \approx \sqrt{\bar{n}}. \tag{5}$$

Of course, the value $\bar{n}$ from Eq. (3), which is obtained by averaging the results of N measurements, does not exactly coincide with the true value n_0, it is essentially a random quantity. Probability theory shows that the standard deviation of $\bar{n}$ from n_0 can be determined by Eq. (1.20):

$$\sigma_{\bar{n}} = \frac{1}{N}\sqrt{\sum_{i=1}^{N}(n_i - \bar{n})^2} = \frac{\sigma_{sep}}{\sqrt{N}}. \tag{6}$$

Here Eq. (4) is used in the second equation.

Usually it is not the absolute but the relative error of measurement which is of great interest. For the considered set of N measurements (10 s each) the *relative error* of a measurement (i. e. the expected difference between n_i and n_0) is

$$\varepsilon_{sep} = \frac{\sigma_{sep}}{n_i} \approx \frac{1}{\sqrt{n_i}}.$$

The relative error of the mean value $\bar{n}$ is determined similarly:

$$\varepsilon_{\bar{n}} = \frac{\sigma_{\bar{n}}}{\bar{n}} = \frac{\sigma_{sep}}{\bar{n}\sqrt{N}} \approx \frac{1}{\sqrt{\bar{n}N}}. \qquad (7)$$

The value σ_{sep} from Eq. (5) is substituted in the last equation of (7).

Thus the relative error of $\bar{n}$ is determined only by the *total* number of counts $\bar{n}N$ and it is independent of the set partitioning (10, 40 or 100 s). This is to be expected, because all the measurements constitute the single measurement, which registers $\sum n_i = \bar{n}N$ counts. As we can see the relative accuracy of a measurement gradually improves as the number of counts grows (and the time of the measurement increases).

Using Eq. (7) we have found that to attain an accuracy up to 1% of the measurement of intensity of cosmic rays one should obtain at least 100^2=10 000 counts, the accuracy of 3% requires only 1000 counts, the accuracy of 10% is reached at 100 counts, etc. The accuracy is the same regardless of the way the net number of counts (1000 or 10 000) is obtained: in a single or several independent experiments.

A specially designed computer code is used to measure the intensity of cosmic rays and treat the experimental data. Using this code one can obtain the specifications of the experimental assembly and carry out a numerical experiment which simulates the real one. The simulated data are generated by a special code (random-number generator). In real experiment the code allows one to follow real-time variations of the quantity under study, its mean value, the standard deviation, the histogram, and to verify the theoretical formulae concerning measurements and errors. Data analysis can be performed for various durations of the interval and the number of counts. The code also contains the main definitions and formulae used in data treatment.

LABORATORY ASSIGNMENT

1. Study the sections of the manual concerning measurements before the experiment.
2. Study the experimental setup.

3. Turn on the computer and the assembly. After computer booting the code `STAT` is loaded and the experiment begins. Study the manual of `STAT` which is available in the laboratory.
4. Carry out the demonstration experiment in which the data is produced by the random-number generator. Study how the following values vary depending on the number of measurements:
   1) the measured quantity,
   2) its mean value,
   3) the error of individual measurement,
   4) the error of the mean value.
5. After the main experiment is completed copy the experimental data from the computer monitor to the workbook.
6. Using the data plot the histogram $w_n = f(n)$ of the distribution of the number of counts for 10 s. To this end plot the integers n on the abscissa and the fraction of the events corresponding to the number of counts equal to n on the ordinate. The fraction of events w_n which is the probability of getting n counts is determined according to the obvious formula:

$$w_n = \frac{\textit{number of events with outcome } n}{\textit{total number of measurements}(N)}.$$

7. Combine the measurement results for $\tau = 20$ s bins in pairs and plot the histogram of the distribution of the number of counts for 40 s bins. The histograms of the distributions of the number of counts for 10 and 40 s bins should be plotted on the same graph; this makes visual comparison easier. The abscissa graduations on the second graph should be chosen so that the positions of the mean values $\bar{n}$ coincide. How does the histogram change when the period of the measurement increases? What determines the width of the histogram peak?
8. Determine the mean number of particles for 10 and 40 second bins and the corresponding standard deviations for individual and the mean values. Verify that the standard deviation of individual measurement is related to the mean number of particles as $\sigma = \sqrt{\bar{n}}$.
9. Determine the fraction of the events for which a deviation from the mean value does not exceed σ, 2σ. Compare the results with theoretical estimates.

Literature

1. *Лабораторные* занятия по физике / Под ред. Л.Л. Гольдина. — М.: Наука, 1983. С. 40–52.
2. *Laboratory* practice on general physics. V. 3 / Edited by Yu.M. Tsipenjuk. — M.: MIPT edition, 1998. P. 159–166, 367–372.

3. *Sivukhin D.V.* Course of general physics. V. V. Part 2. P. 354–370.

Example of lab report 1.1.4

Lab equipment: Geiger-Müller counter (CTC-6), a power unit, and a computer.

1. Turn on the computer. (Accumulation of data for the main measurement begins.)

2. In the course of the demonstration experiment we verify that when the number of measurements increases

1) the quantity to be measured fluctuates;

2) the fluctuations of the mean value of the measured quantity decrease and the mean value tends to a constant;

3) the fluctuations of the error of individual measurement decrease and the error of individual measurement (the systematic error) tends to a constant;

4) the fluctuations of the error of the mean value and the value itself decrease.

3.Perform the main experiment: the measurement of the density of the cosmic rays flux for 10 seconds (the results have been accumulated since turning on the computer). Using the computer code process the results similarly to the demonstration experiment. The results are recorded in tables 1 and 2.

4. Combine the measurement results from Table 1 in pairs, which corresponds to $N_2 = 100$ measurements for the time interval of 40 s. The results are recorded in Table 3.

5. Represent the results of the last measurement in a special form which is suitable for plotting the histogram (Table 4). The histograms of distributions of the mean number of counts for 10 and 40 s are plotted on the same graph (see Fig. 2). The abscissa graduation is 4 times greater for the second distribution to make the maxima coincide.

6. Using Eq. (3) calculate the mean number of counts for 10 s:

$$\bar{n}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1} n_i = \frac{2896}{400} = 7.24.$$

7. Find the standard deviation of individual measurement using Eq. (4):

$$\sigma_1 = \sqrt{\frac{1}{N_1}\sum_{i=1}^{N_1}(n_i - \bar{n}_1)^2} = \sqrt{\frac{2934}{400}} \approx 2.7.$$

8. Verify Eq. (5):

$$\sigma_1 \approx \sqrt{\bar{n}_1}; \qquad 2.7 \approx \sqrt{7.24} = 2.69.$$

9. Determine the fraction of the events for which deviations from the mean value are less than σ_1, $2\sigma_1$, and compare them with the theoretical estimates (see Table 5).

Table 1

Number of counts for 20 s

# опыта	1	2	3	4	5	6	7	8	9	10
0	20	16	20	16	16	15	13	16	13	14
10	17	22	14	12	15	17	20	16	16	17
20	16	15	28	15	19	5	14	17	14	15
30	11	6	14	11	16	12	18	14	14	25
40	10	21	18	14	13	20	18	15	17	11
50	10	7	6	21	23	19	10	13	14	15
60	10	12	13	9	18	19	17	11	9	16
70	16	15	12	16	12	20	6	11	13	19
80	22	17	19	17	10	13	10	20	16	10
90	12	10	19	16	14	15	5	14	13	13
100	12	14	12	14	13	13	17	7	18	15
110	13	13	22	12	15	14	10	16	15	10
120	17	19	27	13	16	16	13	15	15	13
130	6	18	8	14	16	17	13	15	19	16
140	17	13	15	19	16	14	20	18	16	12
150	16	12	14	12	11	8	12	10	13	20
160	11	10	10	10	20	16	15	15	11	10
170	13	12	15	14	15	13	12	17	15	11
180	11	13	15	14	11	10	16	14	14	22
190	10	16	20	18	11	11	10	22	15	11

Footnote: Table is composed so that, e.g. the result of the 123-rd event is on the intersection of the 120-th row and the 3-rd column.

10. Using Eq. (3) determine the mean number of counts for 40 s:

$$\bar{n}_2 = \frac{1}{N_2}\sum_{i=1}^{N_2} n_i = \frac{2896}{100} \approx 29.0.$$

11. Find the standard deviation of individual measurement using Eq. (4):

$$\sigma_2 = \sqrt{\frac{1}{N_2}\sum_{i=1}^{N_2}(n_i - \bar{n}_2)^2} = \sqrt{\frac{3210}{100}} \approx 5.7.$$

12. Verify Eq. (5):

$$\sigma_2 \approx \sqrt{\bar{n}_2}; \qquad 5.7 \approx \sqrt{29.0} = 5.4.$$

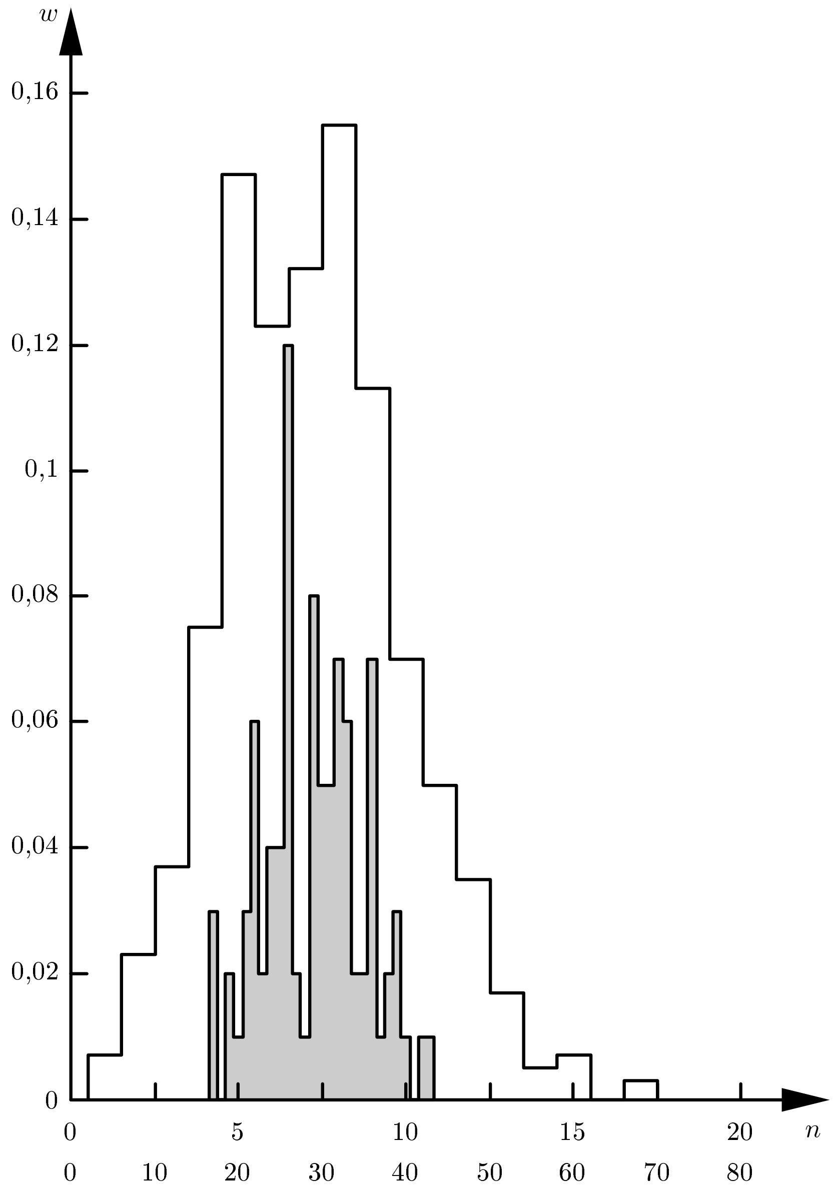

Fig. 2. Histograms for $\tau = 10$ s and $\tau = 40$ s

Table 2

Data for histogram of distribution of number of counts for 10 s

Number of pulses n_i	0	1	2	3	4	5
Number of events	0	3	9	15	30	59
Fraction of events w_n	0	0.007	0.023	0.037	0.075	0.147
Number of pulses n_i	6	7	8	9	10	11
Number of events	49	53	62	45	28	20
Fraction of events w_n	0.123	0.132	0.155	0.113	0.070	0.050
Number of pulses n_i	12	13	14	15	16	17
Number of events	14	7	2	3	0	1
Fraction of events w_n	0.035	0.017	0.005	0.007	0	0.003

Table 3

Number of counts for 40 s

# of sample	1	2	3	4	5	6	7	8	9	10
0	36	36	31	29	27	39	26	32	36	33
10	31	43	24	31	29	17	25	28	32	39
20	31	32	33	33	28	17	27	42	23	29
30	22	22	37	28	25	31	28	32	17	32
40	39	36	23	30	26	22	35	29	19	26
50	26	26	26	24	33	26	34	29	26	25
60	36	40	32	28	28	24	22	33	28	35
70	30	34	30	38	28	28	26	19	22	33
80	21	20	36	30	21	25	29	28	29	26
90	24	29	21	30	36	26	38	22	32	26

13. Compare the standard deviations of individual measurements for two distributions: $\bar{n}_1 = 7.4$; $\sigma_1 = 2.7$ and $\bar{n}_2 = 29$; $\sigma_2 = 5{,}7$. One can easily see that although the absolute value σ of the second distribution is greater ($5.7 > 2.7$), the relative half-width of the second distribution is smaller:

$$\frac{\sigma_1}{\bar{n}_1} \cdot 100\% = \frac{2.7}{7.24} \cdot 100\% \approx 37\%, \qquad \frac{\sigma_2}{\bar{n}_2} \cdot 100\% = \frac{5.7}{29} \cdot 100\% \approx 20\%.$$

This can be also seen in Fig. 2.

14. Determine the standard error of the quantity $\bar{n}_1$ and the relative error of the estimate $\bar{n}_1$ using $N = 400$ measurements for 10 s bins. According to Eq. (6)

$$\sigma_{\bar{n}_1} = \frac{\sigma_1}{\sqrt{N_1}} = \frac{2.7}{\sqrt{400}} \approx 0.13.$$

Table 4

Data for histogram of distribution of number of counts for 40 s

Number of pulses n_1	17	18	19	20	21	22	23	24	25
Number of events	3	0	2	1	3	6	2	4	4
Fraction of events w_n	0.03	0	0.02	0.01	0.03	0.06	0.02	0.04	0.04
Number of pulses n_1	26	27	28	29	30	31	32	33	34
Number of events	12	2	10	8	5	5	7	6	2
Fraction of events w_n	0.12	0.02	0.01	0.08	0.05	0.05	0.07	0.06	0.02
Number of pulses n_1	35	36	37	38	39	40	41	42	43
Number of events	2	7	1	2	3	1	0	1	1
Fraction of events w_n	0.02	0.07	0.01	0.02	0.03	0.01	0	0.01	0.01

Table 5

Error	Number of events	Fracton of events, %	Theoretical estimate
$\pm\sigma_1 = \pm 2.7$	268	67	68
$\pm 2\sigma_1 = \pm 5.4$	384	96	95

Find the relative error according to the first Eq. (7):

$$\varepsilon_{\bar{n}_1} = \frac{\sigma_{\bar{n}_1}}{\bar{n}_1} \cdot 100\% = \frac{0.13}{7.24} \cdot 100\% \approx 1.8\%;$$

and according to the last Eq. (7):

$$\varepsilon_{\bar{n}_1} = \frac{100\%}{\sqrt{\bar{n}_1 N_1}} = \frac{100\%}{\sqrt{7.24 \cdot 400}} \approx 1.9\%.$$

Finally,

$$n_{t=10s} = \bar{n}_1 \pm \sigma_{\bar{n}_1} = 7.24 \pm 0.13.$$

15. Determine the standard error of the quantity $\bar{n}_2$ and the relative error of the estimate $\bar{n}_2$ using $N_2 = 100$ measurements for 40 s bins. According to Eq. (6)

$$\sigma_{\bar{n}_2} = \frac{\sigma_2}{\sqrt{N_2}} = \frac{5.7}{\sqrt{100}} = 0.57.$$

The relative error according to the first Eq. (7) is

$$\varepsilon_{\bar{n}_2} = \frac{\sigma_{\bar{n}_2}}{\bar{n}_2} \cdot 100\% = \frac{0,57}{29} \cdot 100\% \approx 2.0\%;$$

and according to the second Eq. (7):

$$\varepsilon_{\bar{n}_2} = \frac{100\%}{\sqrt{\bar{n}_2 N_2}} = \frac{100\%}{\sqrt{29 \cdot 100}} \approx 1.9\% = \varepsilon_{\bar{n}_1}.$$

Finally,

$$n_{t=40s} = \bar{n}_2 \pm \sigma_{\bar{n}_2} = 29.0 \pm 0.6.$$

Appendix

The Poisson distribution. In physics the measurement results are often represented by integers. For example, a discrete (usually large) number of particles passes through Geiger counter during the time of measurement. A nucleus undergoing fission splits into integer number of parts. Statistical patterns in these cases possess some general features.

Consider a counter which detects cosmic rays. Whereas the number of counts for any period of time is an integer, the flux density ν (i. e. the average number of counts per one second per unit area) is usually non-integer.

Let's find the probability that for a given flux density ν the counter triggers n times during a given time interval. For the sake of simplicity we will assume that the counter has unit area, which does not influence the final result.

Since we calculate probabilities one should imagine a great number of similar simultaneously operating counters. Some of them trigger exactly n times. The ratio of the number of these counters to the total number of counters is the probability of the event that a counter triggers n times during the given time interval.

Let the net number of counters be N. On average $N\nu$ particles pass through them per second and $N\nu dt$ particles pass for the time dt. If dt is small enough none of the counters detects more than one particle during this time, therefore the counters can be divided into two groups: those which triggered and those which did not. The last group is, of course, the largest one. Obviously the number of triggered counters is equal to the number of counted particles, i. e. approximately $N\nu dt$, so their ratio to the net number of counters is $N\nu dt/N = \nu dt$.

Therefore the probability of a particle passing through a counter for dt equals νdt. This argument is valid only if dt is very small.

Let us calculate now the probability $P_0(t)$ that no particle passes through a counter for t. By definition the number of such counters at t equals $NP_0(t)$ and at $t+dt$ it is equal to $NP_0(t+dt)$. The last number is less than $NP_0(t)$ because during dt the number of the counters decreases by $NP_0(t)\nu dt$. Therefore

$$NP_0(t+dt) = NP_0(t) - NP_0(t)\nu dt,$$

or

$$P_0(t+dt) - P_0(t) = -P_0(t)\nu dt.$$

Dividing this equation by dt and taking the limit of infinitesimal dt we obtain

$$\frac{dP_0}{dt} = -\nu P_0.$$

Integrating this equation we obtain

$$P_0(t) = e^{-\nu t}. \tag{8}$$

The constant of integration is determined by the obvious condition that initially the probability to find a counter which has not triggered equals unity.

Now let us calculate the probability $P_n(t+dt)$ of the event of exactly n particles passing through a counter for the time $t+dt$. These counters are divided into two groups. The first group includes the counters which triggered exactly n times for the period t and not triggered for the period dt. The second group includes the counters which triggered exactly $n-1$ times for the time t and triggered once during the period dt. The number of counters in the first group equals $NP_n(t)(1-\nu dt)$ and the number of counters in the second group equals $NP_{n-1}(t)\nu dt$. (Each expression consists of two multipliers. The first determines the probability that a counter triggers a given number of times during the time t and the second specifies the probability to trigger or not to trigger during the time dt.) Thus we obtain:

$$NP_n(t+dt) = NP_n(t)(1-\nu dt) + NP_{n-1}(t)\nu dt.$$

Now move $NP_n(t)(1-\nu dt)$ into the left part of the equation and divide it by Ndt:

$$\frac{dP_n}{dt} + \nu P_n = \nu P_{n-1}.$$

Applying the recurrence relation for $n=1$, $n=2$ etc., and using (8) we obtain

$$P_n = \frac{(\nu t)^n}{n!} e^{-\nu t}.$$

Notice that νt denoted as n_0 equals the mean number of particles passing through a counter for the time t. Then our formula can be written as

$$P_n = \frac{n_0^n}{n!} e^{-n_0}. \tag{9}$$

It is the final formula which is known as *the Poisson distribution law*. It determines the probability that for a given mean number of counts n_0 (not necessarily integer) exactly n counts take place (n is integer).

The Poisson distribution law is specified by the single parameter: the mean number of counts. Neither the time of measurement nor the counter area matters. Similarly the law is not limited by a Geiger counter detecting cosmic rays. The law applies to the number of telephone calls passing through central station or to any other problem in which the number of counts is an integer and independent of the number of counts detected previously (independent events).

Consider some properties of Eq. (9). First of all let us calculate the probability to find any number n:

$$\sum_{n=0}^{\infty} P_n(n_0) = \sum_{n=0}^{\infty} \frac{n_0^n}{n!} e^{-n_0} = e^{-n_0} \sum_{n=0}^{\infty} \frac{n_0^n}{n!} = e^{-n_0} e^{n_0} = 1.$$

Of course this result is evident because any value of n could be found in experiment, therefore we have calculated the probability of a certain event.

Now calculate the mean value of n:

$$\langle n \rangle = \sum_{n=0}^{\infty} n P_n(n_0) = \sum_{n=1}^{\infty} n \frac{n_0^n}{n!} e^{-n_0} = e^{-n_0} n_0 \sum_{n=1}^{\infty} \frac{n_0^{n-1}}{(n-1)!} =$$
$$= n_0 e^{-n_0} \sum_{n=0}^{\infty} \frac{n_0^n}{n!} = n_0 e^{-n_0} e^{n_0} = n_0.$$

The obtained result is predictable since we started from the assumption that the mean value of n equals n_0.

Now let us find the standard deviation of n. To this end we calculate the variance of n (the mean value of the deviation squared):

$$\left\langle (n-n_0)^2 \right\rangle = \left\langle n^2 - 2nn_0 + n_0^2 \right\rangle = \left\langle n^2 \right\rangle - 2\langle n \rangle n_0 + n_0^2 = \left\langle n^2 \right\rangle - n_0^2.$$

To calculate $\left\langle n^2 \right\rangle$ it is convenient to find $\langle n(n-1) \rangle$ at first and then make use of the following expression $\langle n(n-1) \rangle = \left\langle n^2 \right\rangle - \langle n \rangle = \left\langle n^2 \right\rangle - n_0$:

$$\langle n(n-1) \rangle = \sum_{n=0}^{\infty} n(n-1) P_n(n_0) = \sum_{n=2}^{\infty} n(n-1) \frac{n_0^n}{n!} e^{-n_0} =$$
$$= e^{-n_0} n_0^2 \sum_{n=2}^{\infty} \frac{n_0^{n-2}}{(n-2)!} = n_0^2 e^{-n_0} \sum_{n=0}^{\infty} \frac{n_0^n}{n!} = n_0^2 e^{-n_0} e^{n_0} = n_0^2.$$

Hence: $\left\langle n^2 \right\rangle = n_0^2 + n_0$ and

$$\sigma^2 \equiv \left\langle (n-n_0)^2 \right\rangle = \left\langle n^2 \right\rangle - n_0^2 = (n_0^2 + n_0) - n_0^2 = n_0.$$

Finally,

$$\sigma \equiv \sqrt{\langle (n-n_0)^2 \rangle} = \sqrt{n_0}. \tag{10}$$

Gaussian distribution. When the parameter n_0 tends to infinity the Poisson distribution takes the form of Gaussian distribution. Many other distribution laws have the same limit. This is explained by the central limit theorem which states that a distribution of the sum of a large number of independent random values tends to Gaussian distribution. For example, the number of particles passing through a counter for n seconds (random quantity, the Poisson distribution) could be treated as the sum of n numbers of particles passing through the counter per second.

Consider the Poisson distribution for large n_0 and n. Discreteness of the distribution is no longer significant in this limit because n varies almost continuously. We will specify the deviation of n from n_0 by ε defined by the following relation

$$n = n_0(1+\varepsilon) \qquad or \qquad \varepsilon = \frac{n-n_0}{n_0}.$$

Using Stirling's formula

$$\ln n! = \ln\sqrt{2\pi n} + n\ln n - n$$

and Eq. (9) we obtain

$$\ln P_n = n\ln n_0 - n_0 - \ln\sqrt{2\pi n} - n\ln n + n =$$
$$= n\ln\frac{n_0}{n} + (n - n_0) - \ln\sqrt{2\pi n} \approx -\ln\sqrt{2\pi n_0} - \frac{n_0\varepsilon^2}{2},$$

then

$$P_n = \frac{1}{\sqrt{2\pi n_0}} e^{-\frac{(n-n_0)^2}{2n_0}}. \tag{11}$$

The probability distribution P_n can be extended to continuous quantities. To this end notice that $n - n_0$ is equal to the deviation of experimental value n from the mean value n_0. Let us denote this deviation as x:

$$x = n - n_0.$$

Using Eq. (10) we substitute the standard deviation σ for n_0. Finally, notice that P_n could be treated as the probability to find the value n in the interval between $n - 1/2$ and $n + 1/2$. This interval corresponds to $\Delta x = 1$. Making the substitutions and changing the notation from P_n to $P(x)$ we obtain

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}. \tag{12}$$

Function $P(x)$ is the probability that the value x belongs to the unit interval Δx around the central value x. Choosing the infinitesimal interval dx instead we find

$$dP = \rho(x)dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx. \tag{13}$$

Equation (13) determines the probability that the random value is between $x - dx/2$ and $x + dx/2$. The quantity $\rho(x)$ is called probability density. For the random value which has a non-zero mean value μ the probability density (13) is

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{14}$$

The distribution (14) is called Gaussian distribution.

Using Eq. (13) it is easy to find the probability that the random value lies between x_1 and x_2, where x_1 and x_2 are any numbers. Obviously,

$$P(x_1 \leqslant x \leqslant x_2) = \int\limits_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx. \tag{15}$$

The integral (15) cannot be expressed via primitive integrals. It is called the error function $\mathrm{erf}(x)$:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-t^2}\,dt. \tag{16}$$

One can easily show, that

$$P(x_1 \leqslant x \leqslant x_2) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{x_2}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{x_1}{\sqrt{2}\sigma}\right)\right]. \tag{17}$$

The function $\mathrm{erf}(x)$ is antisymmetric relative to the origin $x = 0$:

$$\mathrm{erf}(-x) = -\mathrm{erf}(x). \tag{18}$$

Using the tables of $\mathrm{erf}(x)$ one can easily find the probability that a random value lies between $-\sigma$ and σ, between -2σ and 2σ, and between any other values:

$$P(-\sigma \leqslant x \leqslant \sigma) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) - \mathrm{erf}\left(-\frac{1}{\sqrt{2}}\right)\right] = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) \approx 0{,}68,$$
$$P(-2\sigma \leqslant x \leqslant 2\sigma) \approx 0{,}95,$$
$$P(-3\sigma \leqslant x \leqslant 3\sigma) = 1 - 0{,}0044.$$

The probability to find x between two values quickly approaches unity as the width of the interval increases.

Indeed they are met not so rarely. It takes place, because real error distributions are various and never strictly obey Gauss law. Such distributions are treated as Gauss for the lack of better. In the area of small deviations from mean value Gauss law mostly correctly estimates probabilities of different meeting in practice deviations, but in the area of large deviations describes them badly, and more the deviations — worse the description.

Lab 1.1.5

Study of elastic proton-electron collisions

Purpose of the lab: to calculate momenta and scattering angles of protons and electrons using photographs of particle tracks; to treat the results using non-relativistic and relativistic theory and to decide which theory applies.

Tools and instruments: slides with photographs of particle tracks in a hydrogen bubble chamber; a slide projector with a coordinate grid for viewing the film.

One of the most efficient methods of studying atomic nuclei and elementary particles is to investigate their collisions with energetic particles and register the particles originated in the collisions. In these experiments the following techniques are used: 1) creating beams of particles used as projectiles, 2) preparing targets containing nuclei or other particles, and 3) detecting properties of the outgoing particles.

Energies of outgoing particles originated in the most radioactive sources are limited by several MeV's[1]. Particles, which carry electric charge, can be accelerated in special machines called particle accelerators. Particle energy of a commercial accelerator ranges from several MeV to tens of GeV. All sources of nuclei and elementary particles are divided into radioactive sources (primary and secondary particles), accelerators (primary, secondary, and tertiary beams), and nuclear reactors and cosmic rays.

A list of available targets is also limited. It includes all stable nuclei and electron.

The major problem with particle detection stems from the fact that possible macroscopic effect on matter due to a particle is very small. The most prominent effect of this kind is ionization of matter by an electrically charged particle. Some detectors employ electromagnetic radiation of charged particles passing through matter. Neutral particles are registered by secondary effects. The main part of a detector is a physical system in unstable state: superheated vapor or liquid, gas in a pre-discharge state, and so on. A micro-particle entering such a system causes macro-catastrophe.

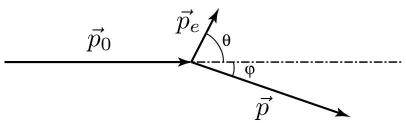

Fig. 1. Elastic collision between proton and electron at rest

Elastic collisions between protons and electrons is the subject of this lab; the experimental data are photographic images of particle tracks in a hydrogen bubble chamber. Working substance in the chamber is a superheated liquid. A track due to a charged particle is formed by vapor bubbles. The detailed mechanism of bubble formation is still to be understood.

Consider an elastic collision between a proton and an electron at rest. Figure 1 shows: the proton momentum $\vec{p}_0$ before the collision, the proton momentum $\vec{p}$ after the collision, the electron momentum $\vec{p}_e$, and the scattering angles φ and θ of the proton and the electron with respect to the direction of incoming proton, respectively.

The law of conservation of momentum reads (see Fig. 1):

$$\begin{aligned} p_0 &= p\cos\varphi + p_e\cos\theta, \\ p\sin\varphi &= p_e\sin\theta. \end{aligned} \quad (1)$$

[1] 1 eV (electron-volt) = $1.6 \cdot 10^{-19}$ J.

Excluding the angle φ we get

$$(p_0 - p_e\cos\theta)^2 + p_e^2\sin^2\theta = p^2$$

or

$$p_0^2 - 2p_0p_e\cos\theta + p_e^2 = p^2. \quad (2)$$

This relation follows from the law of conservation of momentum and it is valid both in relativistic and non-relativistic mechanics.

Using the law of conservation of energy one must be careful since relativistic and non-relativistic expressions for particle energy are different. In classical (non-relativistic) mechanics kinetic energy is expressed in terms of mass, velocity, and momentum:

$$E_{\text{к}} = \frac{mv^2}{2} = \frac{p^2}{2m}. \quad (3)$$

By introducing the notations M and m for the mass of proton and electron, respectively, and using the notations for the momenta introduced above (see Eq. 1)), the law of conservation of kinetic energy in non-relativistic approximation can be written as:

$$\frac{p_0^2}{2M} = \frac{p^2}{2M} + \frac{p_e^2}{2m}. \quad (4)$$

Excluding the proton momentum after the collision from Eqs. (2) and (4) one obtains:

$$p_e\left(1 + \frac{m}{M}\right) = 2p_0\frac{m}{M}\cos\theta \quad (5)$$

or

$$\cos\theta = \frac{M+m}{2m}\cdot\frac{p_e}{p_0}. \quad (6)$$

It is evident that the momentum of the electron after the collision is directly proportional to the cosine of its scattering angle. The momentum increases as the angle decreases. Taking into account that $M/m \approx 2000$, one gets

$$p_e \approx 2p_0\frac{m}{M}\cos\theta. \quad (7)$$

This implies that the maximum electron momentum is

$$p_{e\max} \approx 0.001p_0. \quad (8)$$

Then it follows from Eq. (1) that $p \approx p_0$ and $\theta \gg \varphi$.

Relativistic mechanics requires the modified expression for energy and momentum in order for the laws of conservation of momentum and energy be valid in different reference frames.

$$p = \frac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \tag{9}$$

$$E = \frac{mc^2}{\sqrt{1 - \dfrac{v^2}{c^2}}}. \tag{10}$$

Here v is the particle velocity, c is the speed of light, and m is the particle mass.

Introducing the notations

$$\beta = \frac{v}{c} \tag{11}$$

and

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \tag{12}$$

one can rewrite eqs. (9) and (10) as

$$E = \gamma mc^2, \tag{13}$$

$$p = \frac{E}{c^2} v = \gamma\beta mc, \tag{14}$$

$$E^2 = p^2c^2 + m^2c^4. \tag{15}$$

In relativistic mechanics the total energy γmc^2 of a free particle is the sum of the kinetic energy $(\gamma - 1)mc^2$ and the rest energy mc^2.

Let the proton energy before and after the collision be E_0 and E, respectively. The energy of the electron after the collision is E_e and before the collision was equal to the electron rest energy mc^2. Conservation of the proton and electron energy gives:

$$E_0 + mc^2 = E + E_e. \tag{16}$$

Notice that before and after any elastic collision the particles are the same. Therefore, kinetic energy of the system which equals the difference between the total and the rest energy for each particle is also conserved. For electron

$$K = E_e - mc^2. \tag{17}$$

Now take p from (2) and E from (16), substitute in (15):

$$(E_0 + mc^2 - E_e)^2 = (p_0^2 - 2p_0p_e\cos\theta + p_e^2)c^2 + M^2c^4$$

and simplify this expression taking into account that $E_0^2 = p_0^2c^2 + M^2c^4$ and $E_e^2 = p_e^2c^2 + m^2c^4$,

$$m^2c^4 + E_0mc^2 - E_0E_e - mc^2E_e = -p_0p_ec^2\cos\theta,$$

which gives the relation between the electron momentum p_e and the angle θ:

$$\cos\theta = \frac{E_0E_e + mc^2E_e - E_0mc^2 - m^2c^4}{p_0p_ec^2} = \frac{(E_0 + mc^2)(E_e - mc^2)}{p_e^2c^2}\frac{p_e}{p_0} =$$

$$= \frac{(E_0 + mc^2)(E_e - mc^2)}{E_e^2 - (mc^2)^2}\frac{p_e}{p_0} =$$

$$= \frac{E_0 + mc^2}{E_e + mc^2}\frac{p_e}{p_0} = \frac{M + m + K_0/c^2}{2m + K_e/c^2} \cdot \frac{p_e}{p_0}. \tag{18}$$

Kinetic energy is negligible compared to rest energy for velocities small compared to the speed of light, then Eq. (18) becomes Eq. (6).

Using the relation (15) between electron energy and momentum one gets the following relation between the scattering angle of the electron and its momentum:

$$\cos\theta = \frac{E_0 + mc^2}{p_0} \cdot \frac{p_e}{\sqrt{p_e^2c^2 + m^2c^4} + mc^2}. \tag{19}$$

It is evident that the relation between the momentum and the cosine is nonlinear. The cosine grows slower with the momentum than in the non-relativistic case.

It is convenient to rewrite Eq. (19) using the dimensionless parameter

$$z = \frac{p_ec}{E_e + mc^2} = \frac{p_0c}{E_0 + mc^2}\cos\theta \approx \frac{p_0c}{E_0}\cos\theta = \beta\cos\theta. \tag{20}$$

This parameter is directly proportional to $\cos\theta$. A plot of the function $z(\cos\theta)$ can be used to determine the initial momentum of the protons.

It has already been mentioned that the elastic collisions between protons and electrons were observed in the bubble chamber placed in a uniform magnetic field. The bubble chamber is a cylinder filled with a liquid which temperature is close to the boiling point. The liquid does not boil because it is pressurized by a piston or a membrane used as a cylinder base. The pressure drops when the proton beam enters the chamber, the liquid becomes superheated and remains unstable for some time. If during this time (several milliseconds) a charged particle passes through the chamber, the liquid will boil along the particle track which becomes visible as a chain of vapor bubbles. The working liquid serves as the target and the detector at the same time. Liquid hydrogen is often used as the working liquid, which allows one to observe interaction of energetic particles with protons (the hydrogen nuclei) and with electrons (from the hydrogen electron shells). The chamber operates at the temperature of liquid hydrogen of 29 K and at the pressure of 5 atm.

Bubble chamber is superior compared to the Wilson chamber in having a greater density of the working medium, which lessens particle free path and enables to detect more interaction events in the same volume. Nowadays bubble chambers are not used, they have been superseded by spark chambers.

The bubble chamber in which the particle tracks have been photographed was placed in a uniform magnetic field $\vec{B}$ perpendicular to the photographic plane. Recall that the particle with electric charge e which is moving with the velocity $\vec{v}$ in the magnetic field $\vec{B}$ is subjected to the Lorentz force:

$$\vec{F} = e\,\vec{v} \times \vec{B}. \qquad (21)$$

In our case it would be safe to assume that $\vec{v}$ and $\vec{B}$ are orthogonal. The Lorentz force is perpendicular to the velocity, so the particle executes circular motion. The circle radius r and the particle momentum p are related as

$$\frac{mv^2}{r} = evB, \qquad (22)$$

or

$$p = eBr. \qquad (23)$$

This equation is valid both in classical and relativistic mechanics.

In what follows $B = 2$ T. If pc and r are measured in megaelectronvolts (MeV) and centimeters, respectively, then

$$pc = 6r. \qquad (24)$$

Work with the photographs begins with installing the film in the slide projector and obtaining a sharp image on a screen. The direction in which the film is moving is considered as the direction of abscissa of the coordinate grid. Then the film is examined and suitable images are selected.

The photograph shows tracks of protons passing through the chamber. The protons collide both with atomic nuclei (hydrogen nuclei in our case, i.e. protons) and with electrons. In the first case either elastic scattering or a nuclear reaction occurs, the latter often results in pion creation. The path of incoming proton has a sharp cusp.

In the case of proton-electron collision a proton path is smooth since proton is much heavier than electron. The trajectories of the recoiled electrons, which are usually called δ-electrons, are curved by the magnetic field. As it follows from Eq. (23) the curvature radius of a trajectory is proportional to the particle momentum and so it is much smaller for electrons than for protons. Deceleration of electron due to its interaction with the environment results in decreasing its momentum and therefore the curvature radius of its path which becomes a spiral (see Fig. 2).

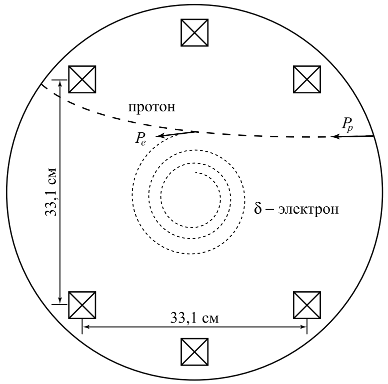

Fig. 2

The crosses (in the squares) on the photographs are the labels placed on the bubble chamber window, through which the shots are taken, to determine the image scale.

Besides the tracks of δ-electrons one can also see the tracks of the electrons which are not related to the proton trajectories. Such electrons, which seemingly appear out of nothing, are due to scattering of γ-quanta (energetic electromagnetic radiation) on electrons. The photographs also show the tracks of the pairs e^+ and e^- originated at the same point and bent in the opposite directions. Such electron-positron pairs are created by γ-quanta in the field of a nucleus.

Not all the photographs can be used for the measurements. One should select the images on which the centers of the consecutive spiral revolutions are not significantly displaced with respect to each other and the diameter of the first spiral revolution exceeds 8-10 mm. The photographs on which an electron recoils at the angle less than 2–3° must be discarded. The reason is that the angle visible on the photograph is not the whole story, there is always a component perpendicular to the film. The error of the measurement arising due to the undetectable perpendicular component increases if the angle is small. Also one should take into account that probability for a δ-electron to emerge is inversely proportional to the square of its kinetic energy, therefore the majority of δ-electrons have small energies and their trajectories have small radii. This circumstance complicates the measurements. It is advisable to select both «narrow» and «wide» spirals.

The measurements are performed with the aid of a magnifying glass (×28). The distance between the crosses on the bubble chamber window is known and it is used to determine the size of a trajectory. In our case the radii R measured on the projector screen must be multiplied by the coefficient $K = 0.427$ in order to obtain the corresponding radii r in the bubble chamber. Figure 2 shows the whole photograph which can be observed by means of a magnifying glass with a less magnification.

A photograph allows one to determine the angle between the proton trajectory and the initial segment of electron spiral. The electron momentum is determined by the curvature radius of the spiral. In so doing the experimental relation between electron momentum and scattering angle can be found. Comparing the relation with Eqs. (6) and (18) one could infer whether relativistic effects should be taken into account.

The curvature radius R of electron trajectory and the scattering angle θ are determined as follows. The selected image of the collision is centered on the projector screen (see Figs. 3 and 4). The coordinates are chosen so that the abscissa is directed along the proton trajectory. The origin is placed at the initial point of δ-electron trajectory which coordinates are $(x_1,\, y_1)$. We assume that the initial segment of the spiral is well approximated by a circle:

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \tag{25}$$

Here x_0 and y_0 are the coordinates of the circle center and R is its radius.

Figures 3 and 4 show two possible directions in which an electron can recoil. One can see that the circle center is located either on the left or on the right of the ordinate. In both cases the angle α between the ordinate and the radius drawn from the center $(x_0,\, y_0)$ to the origin $(x_1,\, y_1)$ equals θ which can be determined providing R and y_0 are known. Then

$$\cos\theta = \frac{y_0}{R}. \tag{26}$$

The radius of electron trajectory R measured on the screen is used to calculate the radius in the bubble chamber, $r = 0.427R$. The electron momentum is then determined from Eq. (24).

Radius and coordinates of the center of a circle can be determined from the coordinates of three points of the circle. One of the points is the origin $(x_1,\, y_1)$. Two more points are shown in Fig. 3: the point $(x_3,$ $y_3)$ of the trajectory intersection with the ordinate and some intermediate point $(x_2,\, y_2)$. Substitution of the point coordinates in Eq. (25) gives three

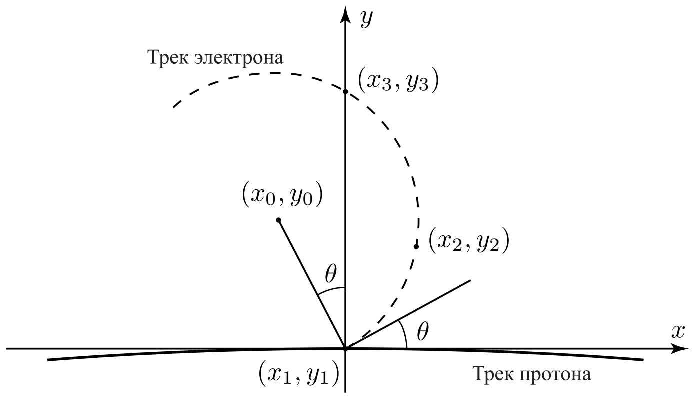

Fig. 3

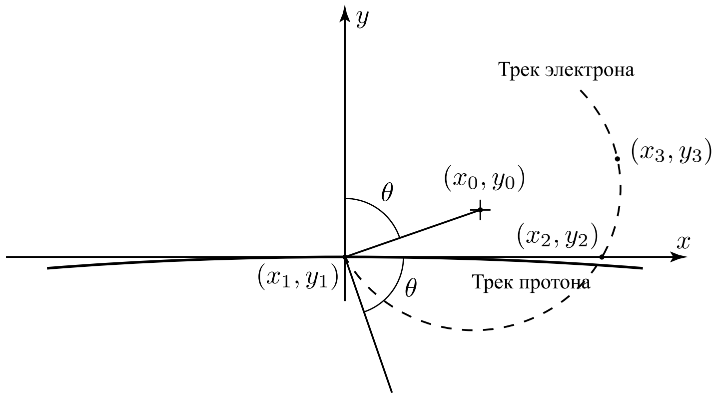

Fig. 4

equations:

$$\begin{aligned} &x_0^2 + y_0^2 = R^2, \\ &(x_2 - x_0)^2 + (y_2 - y_0)^2 = R^2, \\ &x_0^2 + (y_3 - y_0)^2 = R^2. \end{aligned} \tag{27}$$

Then

$$y_0 = \frac{y_3}{2}, \qquad x_0 = \frac{x_2^2 + y_2^2 - y_2 y_3}{2x_2}. \tag{28}$$

For the case shown in Fig. 4 two additional points are: the point $(x_2,$

y_2) of the trajectory intersection with the abscissa and an arbitrary point (x_3, y_3). This gives the following set of equations:

$$\begin{aligned} &x_0^2 + y_0^2 = R^2,\\ &(x_2 - x_0)^2 + y_0^2 = R^2,\\ &(x_3 - x_0)^2 + (y_3 - y_0)^2 = R^2. \end{aligned} \tag{29}$$

Therefore

$$x_0 = \frac{x_2}{2}, \qquad y_0 = \frac{x_3^2 + y_3^2 - x_2 x_3}{2y_3}. \tag{30}$$

It is convenient to choose the point of the trajectory intersection with the ordinate as the third point providing the trajectory does not deviate significantly from a circle. Then

$$x_0 = \frac{x_2}{2}, \qquad y_0 = \frac{y_3}{2}. \tag{31}$$

The radius of a circle is always found as

$$R = \sqrt{x_0^2 + y_0^2}. \tag{32}$$

For cross-checking it is advisable to measure directly the distance between the origin and the center of δ-electron trajectory on the screen using the coordinate grid.

In particle physics energy is usually measured in electron-volts (eV) or the derived units: kiloelectron-volt (1 $KeV = 10^3$ eV), megaelectron-volt (1 $MeV = 10^6$ eV), and gigaelectron-volt (1 $GeV = 10^9$ eV). Momentum and mass are conveniently replaced by pc and mc^2, respectively. These quantities have dimension of energy and expressed in electron-volts, which simplifies calculations. Using these units in the lab is mandatory. The masses of electron and proton are $mc^2 = 0{,}511$ MeV and $Mc^2 = 938$ MeV, respectively.

LABORATORY ASSIGNMENT

1. Make the table for recording the results of the measurements and calculations:

N	x_2, mm	y_2, mm	y_3, mm	R_{scr}, mm	R, mm	$\cos\theta$	$p_e c$, MeV	z

Here N is the track number and R_{scr} is the radius measured on the screen.
2. Using the magnifying glass project the image of the tracks on the screen.
3. Select an appropriate electron track (the scattering angle exceeds 2–3° and the diameter of the first curve revolution is 8–80 mm).
4. Place the origin of reference frame at the initial point (x_1, y_1) of the δ-electron trajectory. Choose the abscissa direction along the proton trajectory (see Fig. 3).
5. Measure and tabulate the coordinates x_2, y_2, y_3 of the corresponding points for a case shown in Fig. 3 and x_2, x_3, y_3 or x_2, y_3 for a case shown in Fig. 4.
6. Measure and tabulate the radius R of the first revolution of the track.
7. Repeat the measurements 3–6 for 40-50 tracks.
8. Calculate and tabulate the coordinates of the circle using Eqs. (28) and (30) or (31), the radius of the circle using Eq. (32), the cosine of the scattering angle using Eq. (26), the electron momentum multiplied by the speed of light using Eq. (24) and the relation $r = 0.427R$, and $z(\cos\theta)$ using Eq. (20).
9. Plot the points with coordinates ($p_e c$, $\cos\theta$). On the same graph plot the points $cp_e(\cos\theta)$ calculated using non-relativistic and relativistic Eqs. (7) and (19).
10. Plot the points with coordinates (z, $\cos\theta$). Draw a straight line through the points and the origin (using the method of least squares is preferable). Using the value of the slope and Eqs. (20), (9), (10), and (15) calculate: the momentum of the incoming proton, the proton energy, the proton velocity divided by the speed of light $\beta = v/c$, and the quantity $\gamma = 1/\sqrt{1-\beta^2}$.
11. Estimate the random error of the proton momentum and energy using the following graphic method. Draw two additional straight lines through the origin with the slopes $\beta \pm \Delta\beta$ (β is the slope of the line drawn previously) by choosing $\Delta\beta$ so that two thirds of the points are between the lines. Calculate the error of the momentum using

$$\Delta p \approx \frac{p(\beta + \Delta\beta) - p(\beta)}{\sqrt{n}}$$

and compare the obtained value with the error given by the method of least squares (1.40).

Questions

1. Derive equations relating electron scattering angle and its momentum in relativistic and non-relativistic mechanics.
2. Derive the formula relating velocity of a relativistic particle with its momentum and energy.

3. Derive the equation relating electron momentum and the radius of its trajectory in magnetic field. Show that this equation is valid both in relativistic and non-relativistic mechanics.

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Example of lab report 1.1.5

The laboratory equipment: a film with photographs of events in a hydrogen bubble chamber and a slide projector with coordinate grid for surveying the film.

The momentum and the scattering angle (the angle the recoiled electron makes with the direction of the incoming proton) of an electron are determined by its (spiral) trajectory in the magnetic field. The initial part of the spiral is approximated by a circular arc. The radius and the scattering angle are calculated from the coordinates of three points lying on the arc: x_2, y_2, and y_3 (see Fig. 3). The origin of the reference frame is at the collision point. The corresponding data are tabulated in Table 1. The coordinates are measured on the screen with an error of 1 mm.

The table also contains the results of the calculation. The radius and the cosine of the scattering angle are evaluated using Eqs. (32), (28) and (26).

Electron momentum is evaluated using Eq. (24) in which $r = 0.427R$ (R is in mm). The values of z are obtained from (20). The errors can be evaluated using (1.33).

The points with coordinates ($p_e c$, $\cos\theta$) are plotted in Fig. 5. The large scatter is due to a large measurement error.

It is evident that electron momentum increases together with $\cos\theta$ (the angle decreases).

In a non-relativistic case and for a constant energy of protons the electron momentum is determined by Eq. (7), so it is directly proportional to $\cos\theta$.

In a relativistic case the corresponding dependence is non-linear and it is given by Eq. (19). It is convenient to introduce the function

$$z = \frac{p_e c}{\sqrt{p_e^2 c^2 + m^2 c^4} + mc^2} = \frac{p_0 c}{E_0 + mc^2}\cos\theta \approx \frac{p_0 c}{E_0}\cos\theta = \beta\cos\theta.$$

The function depends linearly on $\cos\theta$, which allows one to determine the velocity of incoming protons using graphical methods.

The calculated values of z are presented in Table 1.

The final results are shown in Fig. 6, the straight line is drawn using the method of least squares (Eqs. 1.39)and (1.40)).

The line slope is $\beta = 0.936 \pm 0.014$.

The relative error of β found by the method of least squares is:

$$\frac{\Delta\beta}{\beta} = \frac{0.014}{0.936} = 0.015 = 1.5\%.$$

Now let us evaluate the random error of β graphically. To this end we draw two additional straight lines, so that approximately $40 \cdot 1/3 \cdot 1/2 \approx 7$ points lie outside the lines. The slopes of the lines differ from the slope of the central line by ± 0.08. The random error of β is

$$\Delta\beta = \frac{0.08}{\sqrt{40}} \approx 0.013; \qquad \frac{\Delta\beta}{\beta} = 0.14 = 1.4\%,$$

which agrees with the results of the method of least squares.

Calculate γ:

$$\gamma = \frac{1}{\sqrt{1 - 0.936^2}} = 2.84.$$

Equations (1.33) and (12) give the error of γ:

$$\frac{\Delta\gamma}{\gamma} = \gamma^2\beta^2\frac{\Delta\beta}{\beta} \approx \gamma^2\frac{\Delta\beta}{\beta} \approx 8 \cdot 1{,}5\% = 12\%.$$

Finally: $\gamma = 2.8 \pm 0.3$.

The initial proton momentum is found from Eq. (14):

$$p_0 c = \gamma\beta mc^2 = 2.8 \cdot 0.936 \cdot 938\ MeV = 2.5 \pm 0.3\ GeV.$$

The initial proton energy is

$$E_0 = \gamma mc^2 = 2.8 \cdot 938\ MeV = 2.6 \pm 0.3\ GeV.$$

The proton velocity is $v = \beta c = 0.936\,c$. The dashed line in Fig. 5 corresponds to $p(\cos\theta)$ calculated using the non-relativistic Eq. (7). The solid line on the same plot corresponds to the relativistic dependence (19).

It is obvious that the electron momentum should be determined from relativistic formulae.

Table 1

# track	x_2 mm	y_2 mm	y_3 mm	R mm	R_{scr} mm	$\cos\theta$	p_ec MeV	z
1	7.5	10	24	13.2	13	0.91	3.4	0.861
2	8	15	25	13.6	13	0.919	3.5	0.864
3	3	3	8.5	4.4	4	0.95	1.1	0.64
4	2	5	10.5	8	8	0.66	2.0	0.78
5	11.5	20	33.5	17.8	18	0.94	4.6	0.895
6	29	20	40.5	21.6	22	0.939	5.5	0.911
7	11.5	20	40	23	23	0.87	5.8	0.916
8	15	10	15.5	9.6	10	0.81	2.5	0.816
9	18	23	45	23.1	23	0.97	5.9	0.917
10	8	10	19.5	9.9	10	0.98	2.5	0.822
11	6	3	6	3.8	4	0.8	0.97	0.60
12	2.5	5	10.5	7	7	0.78	1.7	0.75
13	6.5	8	13.5	6.8	7	0.99	1.74	0.75
14	22.5	15	22	14.2	14	0.77	3.64	0.869
15	24	30	57	28.9	29	0.98	7.4	0.933
16	9.5	15	28.5	15.4	15	0.92	3.9	0.879
17	37.5	47	94	48.1	48	0.97	12.32	0.959
18	21.5	12.5	24.5	14.2	14	0.86	3.64	0.869
19	30	23	47	24.2	24	0.97	6.2	0.921
20	21	15	27	14.9	15	0.91	3.82	0.875
21	5	10	19	12	12	0.82	2.9	0.84
22	22	27	50.5	22.5	22	0.99	6.53	0.925
23	6	10	19	10.5	10	0.9	2.7	0.828
24	12.5	8	19	9.9	9	0.96	2.53	0.818
25	2.5	7.5	12	8	8	0.7	2.1	0.79
26	7.5	10	21	11.1	11	0.95	2.8	0.836
27	19.5	15	30.5	15.7	16	0.97	4.02	0.881
28	17	20	40.5	20.6	20	0.98	5.28	0.908
29	16	24	47.5	25.6	25	0.93	6.6	0.925
30	9	6	10.5	6.0	6	0.87	1.55	0.72
31	5.5	9	17	9.3	9	0.9	2.4	0.81
32	10	15	28.5	15.1	15	0.94	3.9	0.877
33	35.5	26	51.5	28.2	28	0.96	7.22	0.932
34	24.5	19	38	19.6	20	0.97	5.02	0.903
35	12.5	12.5	22	11.1	11	0.99	2.84	0.836
36	8	15	28.5	17	17	0.85	4.3	0.888
37	33	40	81	41.4	41	0.98	10.61	0.953
38	11	16	32.5	17.5	18	0.93	4.5	0.892
39	12.5	17	35	18.5	18	0.95	4.7	0.898
40	34.5	40	80	40.4	40	0.99	10.35	0.952

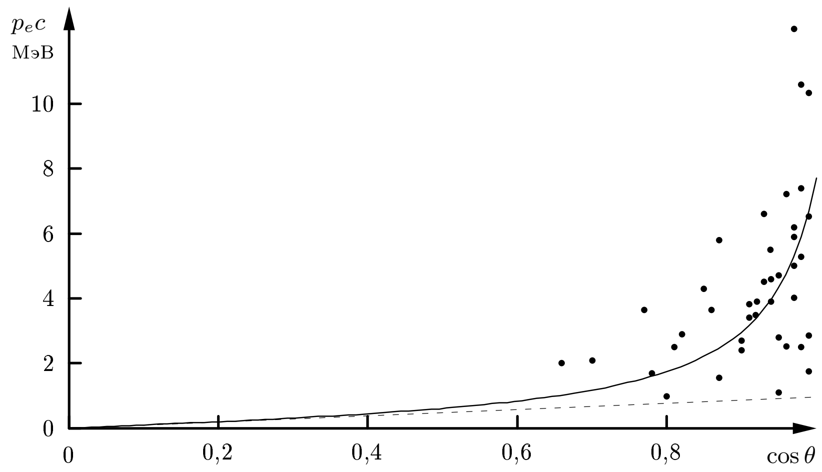

Fig. 5. Plot $p_ec(\cos\theta)$

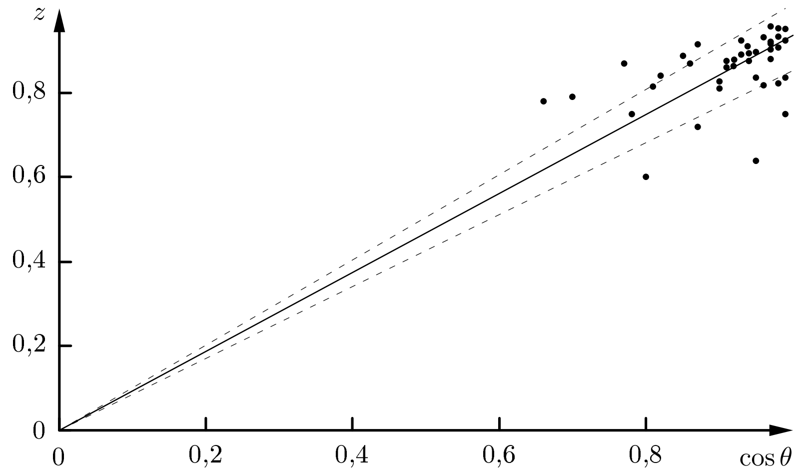

Fig. 6. Plot $z(\cos\theta)$

Lab 1.1.6

Study of electronic oscilloscope

Purpose of the lab: to study operation principles and design of electronic oscilloscope.

Tools and instruments: an oscilloscope, generators of electric signals, and cables.

Oscilloscope is an instrument which displays an electric signal as time-dependent curve. Oscilloscopes are widely used in experiments. Any time-dependent physical quantity which can be converted to electric signal can be studied with the aid of an oscilloscope.

The oscilloscope used in the lab is a modified version of the models C1-94 and C1-1.

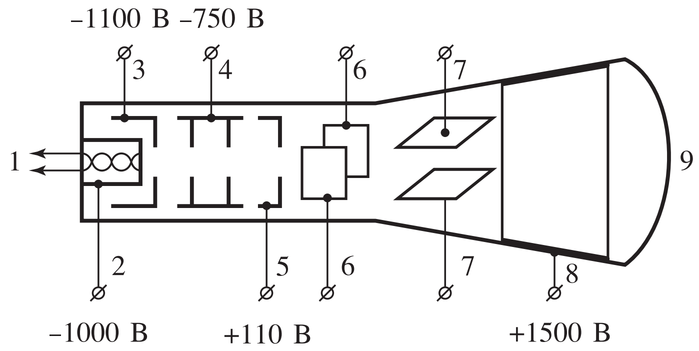

Fig. 1. Cathode-ray tube

Cathode-ray tube. The main part of oscilloscope that determines its most important specifications is a cathode-ray tube (CRT). It is a glass vacuum tube containing the following elements (see Fig. 1): cathode heater 1, cathode 2, modulator 3 (an electrode which controls image brightness), first (focusing) anode 4, second (accelerating) anode 5, deflecting plates 6 and 7, third (accelerating) anode 8, and screen 9.

An electron beam is formed by a set of electrodes called «electron gun»: the cathode and the heater, the modulator, and the focusing and accelerating anodes. The electrodes are arranged to accelerate electrons and to focus the beam on the screen. A voltage difference between the first (focusing) anode and the cathode can be adjusted by knob «FOCUS». The size of the screen bright spot is determined by the quality of the focusing

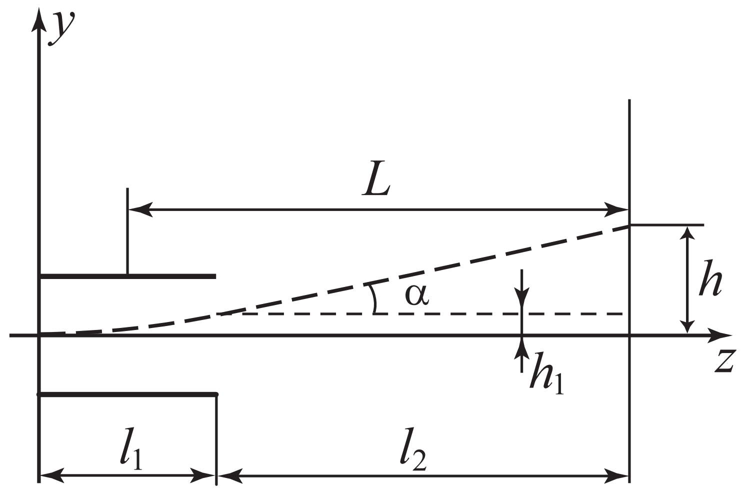

Fig. 2. Deflection of electron beam by electric field of the plates

system, the size does usually not exceed 1 mm. Spot brightness is proportional to the electron beam current which can be adjusted by varying the modulator voltage (knob «BRIGHTNESS»). The oscilloscope screen is the tube front surface covered with a phosphor layer.

On its way to screen the beam of electrons passes two pairs of deflecting plates. Two vertical plates are a capacitor which electric field deflects the beam in the horizontal direction. Two horizontal plates deflect the beam in the vertical direction. By applying the appropriate voltage on the plates it is possible to «draw» a figure on the screen using the beam as a «marker».

Consider the motion of an electron in a homogeneous electric field of deflecting plates (see Fig. 2). Let an electron enter the field at the speed v_0 and go along z-axis, i.e. perpendicular to the field lines. The motion is free along the z-axis and it is uniformly accelerated along the y-axis:

$$z = v_0 t, \qquad y = \frac{at^2}{2}. \tag{1}$$

The acceleration can be found by using the second law of Newton:

$$a = \frac{eE_y}{m}. \tag{2}$$

Using Eqs. (1) and (2) one finds:

$$y = \frac{eE_y}{2mv_0^2} z^2. \tag{3}$$

Therefore the electron path between the deflecting plates is a parabola. The electron is displaced by distance h_1 from the point of entry at the field exit and its velocity is deflected by the angle α from z-axis:

$$h_1 = \frac{eE_y}{2mv_0^2} l_1^2, \qquad \tan\alpha = \frac{eE_y}{mv_0^2} l_1. \tag{4}$$

Here l_1 is the length of the plates. After leaving the field the electron goes along a straight line. The displacement h from the center of oscilloscope screen can be obtained from Fig. 2:

$$h = h_1 + l_2 \tan\alpha = \frac{eE_y l_1}{mv_0^2}\left(\frac{l_1}{2} + l_2\right). \tag{5}$$

Let the distance between the center of a plate and the screen be L. Then

$$h = \frac{eE_y l_1 L}{mv_0^2}. \tag{6}$$

The speed v_0 is determined by accelerating voltage U_a on the second anode:

$$\frac{mv_0^2}{2} = eU_a. \tag{7}$$

The electric field E_y between the deflecting plates is

$$E_y = \frac{U_y}{d}, \tag{8}$$

where U_y is the voltage between the plates and d is the distance between them. Using Eqs. (6) –(8) one obtains:

$$h = \frac{l_1 L}{2dU_a} U_y. \tag{9}$$

Therefore beam displacement is directly proportional to the deflecting voltage U_y. The proportionality coefficient k in Eq. (9) is called tube voltage sensitivity:

$$k = \frac{h}{U_y} = \frac{l_1 L}{2dU_a}\left[\frac{cm}{V}\right]. \tag{10}$$

The tube sensitivity to voltage on the second pair of plates is calculated in the same way.

Equation (9) also applies when deflecting voltage is time-dependent providing the corresponding variation of time τ of electron passage between the plates is small. Typical time interval T, which defines signal variation

rate, can be the signal period, duration, build-up time, etc. Let us estimate the minimum value T_{min} which satisfies $T_{min} \gg \tau$. The speed of electron leaving the «electron gun» is approximately $2 \cdot 10^7$ m/s (for $U_a \approx 10^3$ V). For $l = 3$ cm this gives $\tau = 1.5 \cdot 10^{-9}$ s. Assuming that Eq. (9) applies if $T_{min}/\tau \geqslant 10$ one obtains $T_{min} = 15 \cdot 10^{-9}$ s. Therefore Eq. (9) correctly determines the electron coordinates on the screen if the frequency of sinusoidal voltage on the deflecting plates is less than $\sim 10^8$ $Hz = 0.1$ GHz.

However, the actual maximum frequency is sufficiently less. Voltage sensitivity of the tube is a fraction of mm/V, so the input signal must be amplified before it is applied to oscilloscope. Any amplifier has a working frequency range in which its coefficient of amplification is constant, outside the range the coefficient falls sharply. The upper frequency is determined by the time constant of oscilloscope circuit. Usually the working frequency range of oscilloscope is limited by that of the amplifier.

For the oscilloscope used in this lab the working range is 0 – –1 MHz. In this range, a beam displacement on the screen in horizontal and vertical directions can be considered directly proportional to the voltage on the corresponding deflecting plates.

Sweeps. According to Eq. (9) x and y coordinates of the point where the beam strikes the screen are proportional to instantaneous voltages $U_x(t)$ and $U_y(t)$.

The signal amplitude varies between tens of microvolts and several hundred volts whereas the sensitivity of the deflection plates is a fraction of mm/V. Therefore before the signal is applied to oscilloscope it must be either amplified or diminished.

Amplifiers «Y» and «X» serve to amplify the signal applied to the horizontal and vertical plates, respectively. The attenuator (divider) at the «Y» input allows one to reduce the input signal by a required factor.

Two requirements must be met in order to obtain an «image» of periodic electric signal $U_c(t)$ on the screen.

1. The voltage U_y applied to the vertically deflecting plates must be related to U_c as:

$$U_y(t) = U_{0y} + k_{yu} U_c(t). \tag{11}$$

Here U_{0y} is a constant voltage which determines image location on the Y axis of the screen and k_{yu} is the amplification coefficient of the input signal in the vertical channel.

2. The voltage U_x applied to horizontally deflecting plates must be linearly proportional to time t:

$$U_x = U_{0x} + k_{xu} t. \tag{12}$$

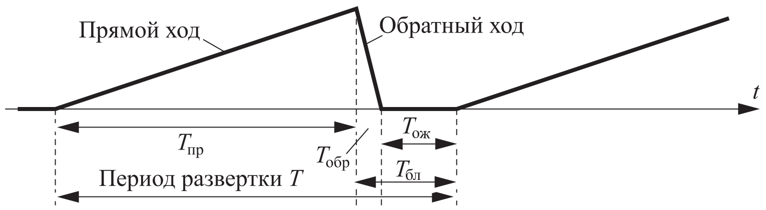

Fig. 3. Sweeps voltage

Here U_{0x} is a constant voltage which determines the image location on the X axis of the screen and k_{xu} is a coefficient which depends on working parameters of the sweep oscillator and the «X» channel amplifier.

A sawtooth voltage generated by the sweep oscillator is also called sweep voltage (see Fig. 3). During the forward sweep (T_{fs}) the voltage increases to maximum, so the beam crosses the screen from left to right at a constant rate. When the forward sweep is completed, the voltage returns to its initial value (T_{bs}), so the beam returns to its initial position on the left side of the screen. The rate of forward sweep, i.e. the scale of X-axis, is controlled by knob «TIME/DIV» which graduation corresponds to the time of beam crossing a cell of the graticule. Waiting interval T_w allows one to vary the scale of X axis regardless of the sweep period.

A potential difference between the modulator of «electron gun» and the cathode is positive during the forward sweep, so the bright trace on the screen is visible. During the backward sweep (T_{bs}) the modulator voltage «blocks» the beam, so there is no trace on the screen during the blocking interval.

Triggering. Observation of periodic and especially fast processes requires the period of sweeps be a multiple of the signal period. However either the sweep oscillator or the signal is not stable. In practice sweeps are controlled by the studied periodic signal: the beginning of a forward sweep must coincide with a selected point of the signal. Process of synchronizing sweeps by means of a selected point of the signal is called triggering. This method of synchronization is illustrated in Fig. 4.

Signal U_y of an arbitrary shape (trapezoid in the figure) reaches the threshold voltage U_l (triggering level) from below that is controlled by knob «LEVEL» on the oscilloscope front panel. At this moment the forward sweep of the «saw» starts provided the threshold is crossed during the waiting interval T_w (Figs. 3 and 4). The «saw» can start when the signal U_y crosses threshold U_l either from below (like in Fig. 4) or from above according to the chosen triggering mode (the switch «TRIGGER

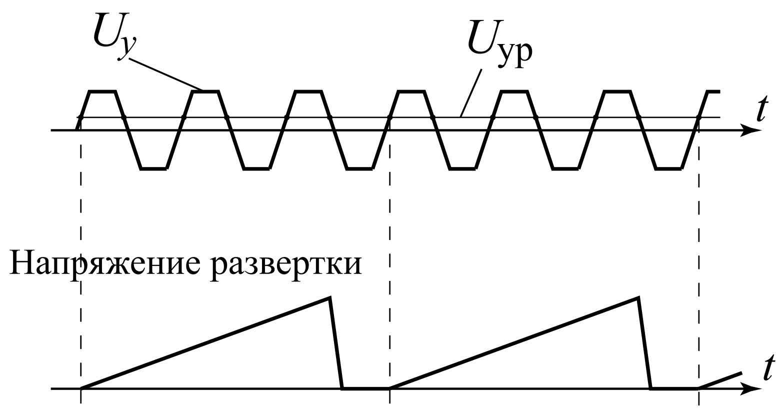

Fig. 4. Triggering of sweeps

+» or «TRIGGER −» on the front panel of oscilloscope). Adjusting the knobs «TRIGGER» and «LEVEL» one controls a signal phase at the beginning of the sweep and achieve a desired image stability and observation convenience. Synchronization is impossible unless U_y crosses U_l.

Sweep oscillator can work in automatic or trigger mode that is controlled by switch «AUTO/TRIG». In automatic mode the waiting time T_w can not exceed some maximum $T_{w,\max}$. If the signal U_y does not cross U_l during $T_{w,\max}$ the forward sweep starts automatically at the moment which is not related to signal phase; the period T_{auto} of sawtooth voltage is determined by internal parameters of oscilloscope. In this case the image on the screen is «running»; if there is no signal the horizontal line is displayed.

If the signal U_y crosses U_l during the waiting interval, a forward sweep is triggered at the moment corresponding to a certain phase of the signal. A stable image is then displayed.

Synchronization in automatic mode is possible only if the internal period of sweep oscillator is greater than the period of studied signal, $T_{auto} > T_s$. Otherwise the first sweep cycle will be followed by another one or more forward sweeps of the «saw» triggered at the moments not related to a certain phase of the signal, which will result in several superimposed images.

In the waiting mode a forward sweep is triggered only if U_y crosses U_l during the waiting time T_w. The time can be as long as necessary, so synchronization is realized for any period of the signal $U_s(t)$. A short interval of signal (e.g. the signal front or a short pulse which duration is much less than the time interval between pulses) can be observed only in the waiting mode.

Sweeps can also be synchronized by an external signal (instead of U_y) which is synchronous to the signal under study. The external signal is applied to the input connector «EXT.TRIG.» on the oscilloscope front panel. The switch «TRIGGER» must be in position «EXT.». Operation of the triggering circuit is similar to the one described above. The sweep range (scale) is controlled by switch «TIME/DIV».

The vertical image dimension is controlled by switch «V/DIV» which graduation in volts corresponds to beam displacement by one cell of the graticule (this quantity is called deflection coefficient). Knob «↕» is used to shift the image up and down by varying the constant U_{0y} (see Eq. (11)).

Now consider frequency response of vertical and horizontal deflection channels of oscilloscope. Suppose that the sinusoidal signal $U_y = U_0 \sin(2\pi ft)$ is applied to the «Y» channel. Beam position on the oscilloscope screen is then $y = y_0(f)\sin(2\pi ft + \Delta\Phi_y(f))$, where $y_0(f)$ is the position amplitude as a function of frequency f and $\Delta\Phi_y(f)$ is the difference between the phase of y and the phase of the signal U_y (phase shift) at the frequency f.

Then the frequency response of the vertical channel is given by

$$K_y(f) = \frac{y_0(f)}{U_0},$$

and the phase response is the function $\Delta\Phi_y(f)$. Frequency and phase responses of the horizontal deflection channel are defined in the same way.

Usually the frequency response $K_y(f)$ remains constant, $K_y = K_{y,\max}$, in the range from $f_{\min}$ to $f_{\max}$ and decreases for $f < f_{\min}$ and $f > f_{\max}$. The frequency range between $f_{\min}$ and $f_{\max}$ is called bandwidth. The values $f_{\min}$ and $f_{\max}$ are determined according to

$$\frac{K_y(f_{\min})}{K_{y,\max}} = \frac{K_y(f_{\max})}{K_{y,\max}} = \frac{1}{\sqrt{2}} \simeq 0{,}7.$$

Since $K_y(f)$ and $\Delta\Phi(f)$ are not constant in the whole frequency range, the shape of a high frequency pulse is distorted in the vertical deflection channel.

The «Y» channel can be used with an open and closed input. In the first case both the variable $U_\sim$ and constant $U_=$ components of a signal are transmitted, while in the second case it is only the variable one. In the closed input mode the constant component is blocked by a dividing capacitor connected to the input. By switching «$\sim/\simeq$» on the front panel one can choose a required input of the «Y» amplifier. The horizontal deflection channel has the similar «X» amplifier.

To observe the dependence $U_y = F(U_x)$ one applies signal U_x to the closed input «→⊃X». The horizontal image size can not be adjusted in the lab oscilloscope. To shift the image horizontally one uses the potentiometer «↔» which changes the constant U_{0x} (see Eq. (12)).

Lissajous curves. Two oscillations with equal or multiple frequencies applied to the oscilloscope inputs make the beam draw a stationary closed curve called Lissajous curve. The curve slowly rotates if the frequencies are not exact multiples; for arbitrary frequencies the pattern is smeared.

Let us apply signal $U_x = U_a\cos(2\pi ft + \varphi_1)$ to the horizontally deflecting plates (the internal sweep oscillator must be switched off) and apply the signal of the same frequency but with the phase shifted, $U_y = U_b\cos(2\pi ft + \varphi_2)$, $\varphi_1 \neq \varphi_2$, to the vertically deflecting plates.

For sensitivities k_x and k_y the beam coordinates x, y on the screen are:

$$x = A\cos(2\pi ft + \varphi_1), \quad y = B\cos(2\pi ft + \varphi_2), \qquad A = k_x U_a, \quad B = k_y U_b.$$

Excluding time t from these equations one readily obtains beam trajectory:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{xy}{AB}\cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1).$$

Thus the curve obtained by superimposing two oscillations of the same frequency is ellipse. The ellipse orientation depends on the phase shift between the oscillations $(\varphi_2 - \varphi_1)$.

The particular Lissajous curve depends on the relation between periods, phases, and amplitudes of the oscillations. Some Lissajous curves for different periods and phases are shown in Fig. 5. Parameters of an oscillation, e.g. f_x, can be determined from the Lissajous curve provided the parameters of the other oscillation, e.g. f_y, are known. For example, one should imagine two straight lines, a vertical and horizontal one, which cross the curve without crossing its nodes. The number of crossings with the horizontal line n_x and the vertical line n_y determines the ratio of the frequencies according to $f_y/f_x = n_x/n_y$.

If either or both oscillations are not harmonic the curves are more complicated.

Calibration signal. The oscilloscope has internal generator of rectangular pulses of a fixed amplitude and the frequency of 50 Hz. The signal is used to check deflection and sweep coefficients. When the signal is applied to «Y» input (the switch «V/DIV» is in K position), a deflection on

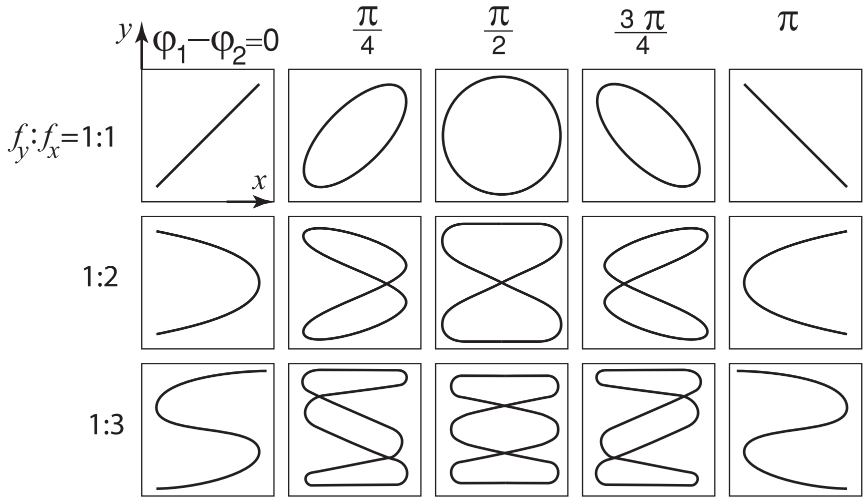

Fig. 5. Lissajous curves for oscillations of the same amplitude

Y-axis must be 4.5–5.0 divisions and the oscillation period on X-axis must be 20 ms.

Preparation of equipment

1. Check that the device casings are properly grounded. Switch on power supplies of the oscilloscope and the generators and let them warm up for 3–5 min.
2. Set the following knobs in intermediate positions (see Fig. 6): «BRIGHTNESS», «FOCUS», «↕», «↔», and «LEVEL».
3. Set the switch «TRIGGER» in position «INT +» and the switch «AUTO/WAIT» in position «AUTO». Switch «V/DIV» should be set to a low sensitivity, e.g. 5 V/div.
4. Set the switch «TIME/DIV» in position 2 ms.
5. A horizontal line appears on the screen in 1–2 min after the oscilloscope is switched on. If the line does not appear adjust the line position by knobs «↕» and «↔». Use the knobs «BRIGHTNESS» and «FOCUS» to obtain a clear sharp image.

CAUTION!

1. Do not increase brightness beyond the level at which the image starts to grow.
2. The beam on the oscilloscope screen is visible only during the forward sweep. In the waiting mode there is no image unless U_l and U_y cross (see

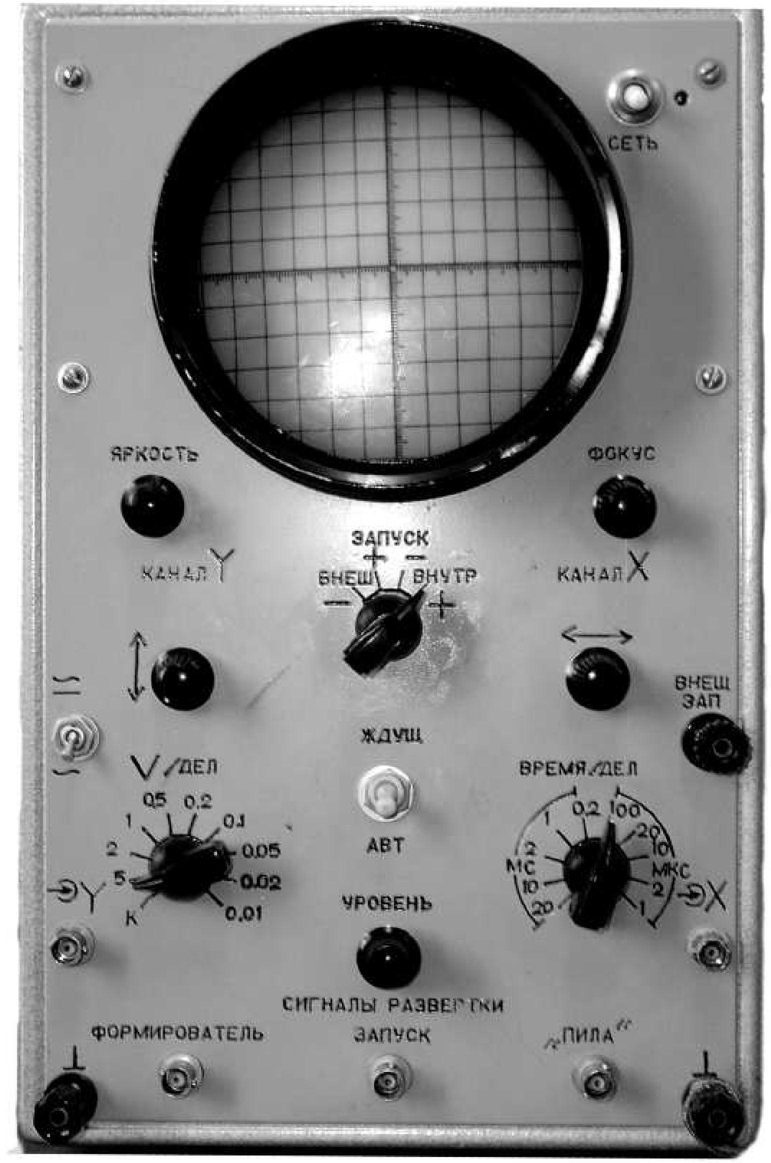

Fig. 6. Front panel of oscilloscope

Fig. 4). Therefore an experiment should begin in «AUTO» mode at the lowest sensitivity of the input «Y». In so doing the horizontal line is visible even if a signal is absent. By increasing the sensitivity with «V/DIV» set the image amplitude of 2–6 divisions. Use the knob «LEVEL» to stabilize image. A convenient horizontal dimension is adjusted by knob «TIME/DIV». If this fails try again the synchronization attempt in the «WAIT» mode.

3. To apply a signal to the «X» input (to observe $U_y = F(U_x)$ dependence) one should switch off the internal sweep oscillator as follows:
   — set the beginning of sweep at the screen center in «AUTO» mode;
   — switch the trigger to waiting mode («WAIT»);
   — turn the knob «LEVEL» to the minimum, the image must vanish;
   — increase the screen brightness if necessary («BRIGHTNESS»).

LABORATORY ASSIGNMENT

I. Observation of periodic signal of acoustic frequency generator (AFG)

1. Figure out how the signal image depends on synchronization modes. To this end connect the input «Y» to output of AFG. Set the following switches as: «TRIGGER» to «INT +», «WAIT–AUTO» to «AUTO», «V/DIV» to 5, and «TIME/DIV» to 2 ms. Apply a signal of frequency 100 Hz and arbitrary amplitude (e.g. set the attenuator of AFG at 0 dB) to the input «Y». The oscilloscope must display a sinusoid. If the sinusoid is «running», stabilize it by turning knob «LEVEL». Shift the image horizontally until the initial point of the sinusoid appears.
2. Turn the knob «LEVEL» and observe how the curve changes. Perform the same observations at the modes «AUTO», «WAIT», and internal trigger modes «INT +» and «INT −». Figure out how the curve appearance depends on triggering mode.
3. Obtain a stable image for three arbitrary sets of AFG controls (e.g. 100 Hz, 0 dB; 1000 Hz, 10 dB; and $3 \cdot 10^5$ Hz, 30 dB). Adjust the image size using knobs «TIME/DIV», «V/DIV».

II. Measurement of amplitude of sinusoidal signal. Correspondence between step-wise attenuator of AFG (the switch «◁ dB») and the control switch of vertical image scale («V/DIV» on the oscilloscope front panel).

1. Set the switch «V/DIV» in position «5», the frequency of AFG at $f_{afg} = 1000$ Hz, and the attenuator at «0 dB»; by adjusting the AFG output set the sinusoid amplitude at $2A = 4$ divisions. Obtain a stable sinusoid on the screen. After that the output voltage of the generator should not be altered.

2. Set the step-wise attenuator of AFG in position «10 dB». Adjust the vertical image size by the knob «V/DIV» and measure the signal amplitude. Perform the measurement for all positions of the attenuator (they correspond to different values of attenuation α, [dB]) and tabulate the results in Table 1.

Table 1

Settings of AFG attenuator and oscilloscope divider

α, dB	V/DIV	$2A$, DIV	$2A$, B	β, dB	$\lvert\alpha-\beta\rvert$, dB
0					
10					
...	...	...	...	...	...
70					

Parameter β is defined as $\beta = -20\lg(2A[V]/20[V])$, the measurement unit is 1 dB (1 decibel). Plot a graph in coordinates β, α. Find the maximum discrepancy between β and α.

III. Measurement of frequency of sinusoidal signal

Set the amplitude of sinusoidal signal at 6 divisions and the AFG frequency in accordance with Table 2. Obtain a stable image. Set a convenient horizontal size of the image by using the switch «TIME/DIV». Measure the signal period, calculate the frequency and tabulate the results in Table 2.

Table 2

Period and frequency of sinusoidal signal

f_{afg}, Hz	$\frac{TIME}{DIV}$	T, DIV	T, s	f_{mes}, Hz	$\lvert f_{afg}-f_{mes}\rvert$, Hz	$\frac{\lvert f_{afg}-f_{mes}\rvert}{f_{mes}}$
$2 \cdot 10$						
$2 \cdot 10^2$						
...	...	...	...	...	...	...
$2 \cdot 10^6$						

IV. Measurement of frequency response of the amplifiers of «X» and «Y» channels

1. Connect the output of AFG to the «Y» input of oscilloscope. Set the switch «V/DIV» in position «1». Set the amplitude of sinusoidal signal at 6 divisions at AFG frequency $f_{afg} = 10^3$ Hz. Obtain a stable image.

Measure the signal amplitude $2A_y$ (or $2A_x$) in the whole working frequency range of AFG according to Table 3 both for open ($\simeq$) and closed ($\sim$) input. Calculate the values of parameter K using Eq. (13):

$$K(f_{afg}) = \frac{2A(f_{afg})[V]}{6[V]}. \tag{13}$$

Tabulate the results in Table 3.

Table 3

Frequency response of channel amplifiers

f_{afg}, Hz	10	...	10^2	10^3	10^4	10^5	10^6	...	10^7
$2A_y$, div									
K_y, $\simeq$									
$2A_y$, div									
K_y, $\sim$									
$2A_x$, div									
K_x									

One should determine the frequencies at which coefficients K_x and K_y are equal to approximately 0.7 of their maximum values. These frequencies define the amplifier bandwidth.

2. Turn off the internal sweep oscillator of X. To do this set the switches in the following positions: «TRIGGER» to «EXT», «WAIT–AUTO» to «WAIT», turn «LEVEL» clockwise to halt, and «BRIGHTNESS» to maximum. Connect the output of AFG to the «X» input of oscilloscope and set the signal amplitude at 6 divisions at AFG frequency $f_{afg} = 10^3$ Hz. The image should be a segment of straight horizontal line at the screen center.
3. Measure the signal amplitude $2A_x$, calculate $K_x(f)$ in the same way as $K_y(f)$, and tabulate the results in Table 3.
4. Plot $K_{y,\simeq}(f)$, $K_{y,\sim}(f)$, and $K_x(f)$ on the same graph using logarithmic scale for frequency f.
5. Turn on the internal sweep oscillator. Consider (qualitatively) how the frequency response of «Y» channel affects a pulse signal. Set the switch of signal shape of AFG in position «⊓». Set the signal amplitude at 4 divisions on the oscilloscope screen. Observe the signal at open ($\simeq$) and closed ($\sim$) inputs at frequencies of 10 Hz, 10^3 Hz, $2 \cdot 10^5$ Hz, and 10^6 Hz. Sketch the curves.

V. Measurement of phase shift between output signals of «Y» and «X» channels when input signal is the same for both channels, i.e. measurement of the difference between phase responses of «X» and «Y» channels.

1. Turn off the internal sweep oscillator as in IV.2. Apply a signal of frequency 10^4 Hz from the AFG output simultaneously to inputs of «X» and <Y» channels using a tee connector. By adjusting the AFG output set the amplitude of X at 6 divisions. The scale on Y must be 0.5 V/DIV. The image must be a segment of straight line at the angle of 30–60° to the vertical (a degenerate ellipse). By varying frequency observe transformation of the segment to ellipse.

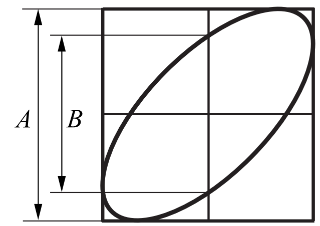

Fig. 7. Lissajous curve for $f_x = f_y$ and arbitrary phase shift $\Delta\Phi_{xy}$

2. Using the graticule measure the parameters A and B (see Fig. 7) in the whole range of AFG frequencies and calculate the phase shift $\Delta\Phi_{xy}$ as

$$\Delta\Phi_{xy} = \begin{cases} \pm \arcsin \frac{B}{A}, & \text{if the ellipse is tilted to the right, as in Fig. 7;} \\ \pm\pi \mp \arcsin \frac{B}{A}, & \text{if the ellipse is tilted to the left.} \end{cases}$$

The sign «+» or «−» corresponds to clockwise or counterclockwise motion of the point tracing the ellipse. By increasing or decreasing frequency one transforms the ellipse to a straight line and the motion reverses.

Tabulate the data in Table 4. Plot $\Delta\Phi_{xy}(f)$ using a logarithmic scale for f_{afg}.

Table 4

Phase shift $\Delta\Phi$ versus frequency

f_{afg}, Hz	10	...	50	10^2	10^3	10^4	10^5	10^6	...	10^7
A, div										
B, div										
$\Delta\Phi_{xy}$										

VI. Observation of Lissajous curves obtained by superimposing orthogonal oscillations

Turn off the internal sweep oscillator. Apply a sinusoidal signal of frequency f_x to the input «X» from the first AFG, and a sinusoidal signal of frequency f_y to the input «Y» from the second AFG. Adjust the amplitudes of the signals and the switch «V/DIV» so that the Lissajous curve occupies the major part of oscilloscope screen. Set f_y at 1 kHz. By varying f_x obtain a stable curve for the following values of the ratio f_y/f_x: 1:1, 2:1, 3:1, and 3:2. Sketch the curves and compare them with the curves shown in Fig. 5.

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Chapter II

DYNAMICS

Dynamics of many particles. In classical mechanics the state of motion of a particle is defined by the particle position vector $\vec{r}$ and momentum $\vec{p} = m\vec{v}$. A state evolves in time according to equation of motion, (Newton's second law):

$$\frac{d\vec{p}}{dt} = \vec{F}(\vec{r}, \vec{p}, t). \tag{2.1}$$

It is important that the right-hand side (the force) depends only on the particle state. Solution of Eq. (2.1) for some boundary conditions gives a law of particle motion:

$$\vec{r} = \vec{r}(t).$$

Since the equation of motion of a particle is linear, it becomes for a set of particles

$$\frac{d\left(\sum\limits_{i=1}^{n} \vec{p}_i\right)}{dt} = \sum_{i=1}^{n} \vec{F}_i. \tag{2.2}$$

Here only the external forces are counted because the internal forces acting between the particles cancel out.

Any set of particles has a remarkable geometric point called the center of mass. The position vector of the center of mass is defined as

$$\vec{R} = \frac{\sum\limits_{i=1}^{n} m_i \vec{r}_i}{\sum\limits_{i=1}^{n} m_i}.$$

Obviously, the velocity of the center of mass is

$$\vec{v} = \frac{\vec{P}}{m},$$

where $\vec{P} = \sum\limits_{i=1}^{n} \vec{p}_i$ is the net momentum of the particles, and $m = \sum\limits_{i=1}^{n} m_i$ is its mass. According to Eq. (2.2)

$$m\frac{d\vec{v}}{dt} = \sum_{i=1}^{n} \vec{F}_i.$$

Therefore the center of mass behaves as the single particle which mass is equal to the total mass of particles and the force exerted on this particle equals the sum of all the external forces. The center of mass velocity can be regarded as the velocity of the set as a whole.

If there are no forces exerted on the set of particles, the set is called isolated or closed. In this case Eq. (2.2) predicts conservation of the net momentum:

$$\sum_{i=1}^{n} \vec{p}_i = \text{const}, \tag{2.3}$$

i. e.

$$\vec{v} = \text{const}.$$

The center of mass of an isolated set of particles serves as the origin of a special inertial frame of reference called the center of mass frame.

The net momentum in the center of mass frame is zero.

The sum of momenta of two particles before and after an interaction, e.g. a collision, is the same:

$$\vec{p}_{10} + \vec{p}_{20} = \vec{p}_1 + \vec{p}_2 \tag{2.4}$$

or

$$m_1\vec{v}_{10} + m_2\vec{v}_{20} = m_1\vec{v}_1 + m_2\vec{v}_2. \tag{2.5}$$

Here the subscript «0» refers to the quantities before the interaction.

Let $\vec{p}$ be the particle momentum and $\vec{r}$ be its position vector with respect to some point of origin O. Then the angular momentum $\vec{L}$ of the particle with respect to O is defined as the cross product:

$$\vec{L} = \vec{r} \times \vec{p}. \tag{2.6}$$

Similarly, if there is a force $\vec{F}$ exerted on the point, the torque due to the force with respect to O is defined as the vector product

$$\vec{M} = \vec{r} \times \vec{F}. \tag{2.7}$$

Multiplying Eq. (2.1) by $\vec{r}$ on the left and using $\vec{p} = m\frac{d\vec{r}}{dt}$ one finds:

$$\frac{d\vec{L}}{dt} = \vec{M}. \tag{2.8}$$

Equation (2.7) can be written in a more transparent form as

$$M = rF\sin\theta = Fh,$$

where θ is the angle between the vectors $\vec{r}$ and $\vec{F}$ and $h = r\sin\theta$ is the length of the perpendicular drawn from the point O to the direction of the force, this distance is called the lever arm with respect to O.

On the other hand,

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} =$$

$$= \vec{i}(yF_z - zF_y) + \vec{j}(zF_x - xF_z) + \vec{k}(xF_y - yF_x).$$

Here $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit basis vectors corresponding to the axes Ox, Oy, and Oz. Let us choose the reference frame so that vectors $\vec{r}$ and $\vec{F}$ lie in the same plane. In addition, let the axis Ox be directed along $\vec{r}$. Then

$$\vec{r} = (x, 0, 0), \qquad \vec{F} = (F_x, F_y, 0),$$

i.e.

$$M_x = 0, \qquad M_y = 0, \qquad M_z = xF_y = xF\sin\theta = Fh.$$

Since the perpendicular drawn from the point O to the direction of the force $\vec{F}$ is perpendicular to Oz, its length h can be called the lever arm with respect to Oz. For this reason the projections of $\vec{M}$ on the coordinate axes are called moments of force with respect to these axes. Similar consideration applies to angular momentum $\vec{L}$.

In an arbitrary frame vectors $\vec{r}$ and $\vec{F}$ can be written as follows:

$$\vec{r} = \vec{r}_\perp + \vec{r}_\parallel, \qquad \vec{F} = \vec{F}_\perp + \vec{F}_\parallel.$$

Here $\vec{r}_\perp$ is the component of $\vec{r}$ perpendicular to Oz and $\vec{r}_\parallel$ is the parallel component. The vectors $\vec{F}_\perp$ and $\vec{F}_\parallel$ are similarly defined. One can show that

$$\vec{M}_\parallel = \vec{r}_\perp \times \vec{F}_\perp,$$

i.e.

$$M_z = r_\perp \cdot F_\perp \sin\varphi,$$

where φ is the angle between $\vec{r}_\perp$ and $\vec{F}_\perp$. For the component L_z of angular momentum one has

$$L_z = r_\perp \cdot p_\perp \sin\alpha,$$

where α is the angle between $\vec{r}_\perp$ and $\vec{p}_\perp$.

The net angular momentum of a set of particles is the sum of angular momenta of all particles and the net torque is due to external forces only because the torques due to inter-particle forces cancel out. Therefore

$$\frac{d\left(\sum\limits_{i=1}^{n} \vec{L}_i\right)}{dt} = \sum_{i=1}^{n} \vec{M}_i. \tag{2.9}$$

If the set of particles is isolated, i.e. no external force acts on it, the net torque is zero and the net angular momentum of the particles is conserved:

$$\sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i = \text{const.} \tag{2.10}$$

Sometimes vector quantities like momentum and angular momentum are not conserved but a certain component is. For instance, the component of momentum perpendicular to the lines of force of uniform gravitational field and angular momentum with respect to an axis parallel to the field are conserved. In a central field the angular momentum with respect to the field center is conserved.

The work done by force $\vec{F}$ is defined as the dot product

$$dA = \vec{F} \cdot d\vec{r}, \tag{2.11}$$

where $d\vec{r}$ is the particle displacement due to the force.

Using the second Newton' law (2.1) one obtains

$$dA = \frac{d\vec{p}}{dt} d\vec{r} = \vec{v} d\vec{p} = \frac{1}{m}\vec{p}\, d\vec{p} = \frac{1}{m} p\, dp = d\left(\frac{p^2}{2m}\right).$$

Therefore the work changes the quantity called kinetic energy of the particle:

$$K = \frac{p^2}{2m} = \frac{mv^2}{2}. \tag{2.12}$$

If the work done by a force on a particle which travels in a closed path is zero, the force is called conservative. An equivalent definition is that the work done by a conservative force is path independent. Gravitational field is an example of conservative force. The field of a conservative force

can be specified by potential energy U. By definition the work done by a conservative force equals the loss of the potential energy:

$$dU = -\vec{F}\, d\vec{r}. \tag{2.13}$$

Using the second Newton's law one obtains:

$$dU = -\frac{d\vec{p}}{dt} d\vec{r} = -\vec{v} d\vec{p} = -\frac{1}{m}\vec{p}\, d\vec{p} = -\frac{1}{m} p dp = -d\left(\frac{p^2}{2m}\right),$$

i.e.

$$d\left(U + \frac{p^2}{2m}\right) = 0. \tag{2.14}$$

This is the law of conservation of mechanical energy.

The net kinetic energy of a set of particles is equal to the sum of kinetic energies of the particles. In an isolated set of particles, i.e. no external force acts on the set, the net kinetic energy can change (unlike the net momentum and angular momentum) due to the work done by internal forces. The net kinetic energy is conserved provided the interactions between particles are elastic, i.e. energy transforms only from kinetic to potential and back. For two particles with an elastic interaction between them the law of conservation of kinetic energy is

$$\frac{p_{10}^2}{2m_1} + \frac{p_{20}^2}{2m_2} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$$

or

$$\frac{m_1 v_{10}^2}{2} + \frac{m_2 v_{20}^2}{2} = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}. \tag{2.15}$$

Here the subscript «0» stands for a quantity before the interaction.

Using these relations in Eqs. (2.4) and (2.5) one can prove that in the center of mass frame the momentum of a particle changes only its direction while its magnitude remains the same.

The net kinetic energy of a set of particles in an arbitrary frame is the sum of the kinetic energies of particles in the center of mass frame and the kinetic energy of the set which speed equals that of the center of mass.

The laws of conservation of momentum, angular momentum, and energy derived from equations of motion are, in fact, fundamental properties of an isolated system, which follow from homogeneity and isotropy of space and homogeneity of time.

A particle which speed is close to the speed of light ($v \sim c$, $c = 3 \cdot 10^{10}$ cm/s) is called relativistic. High energy physics experimentally confirms the relation between the momentum of a relativistic particle and its velocity:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{2.16}$$

where m is the particle mass. Equation (2.1) remains the same although the relation between momentum and velocity is different. Using Eq. (2.1) one can show that

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{dA}{dt},$$

where $dA = \vec{F}\cdot d\vec{r}$ is infinitesimal work. The kinetic energy K of a particle can be defined as the work done by a force accelerating the particle from zero speed to v. Then

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2. \tag{2.17}$$

Since

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots,$$

Eq. (2.17) becomes for $v \ll c$

$$K = \frac{mv^2}{2},$$

which is to be expected.

According to collider experiments the energy of a free particle does not vanish at $v = 0$ but tends to a constant value of mc^2. Therefore the particle energy is actually the quantity

$$\mathcal{E} = K + mc^2,$$

i.e.

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.18}$$

The constant mc^2 is called the particle rest energy. By comparing Eqs. (2.16) and (2.18) one can see that the particle momentum is

$$\vec{p} = \frac{\mathcal{E}}{c^2}\vec{v}. \tag{2.19}$$

At $v = c$ both momentum and energy of a massive particle tend to infinity. Therefore a massive particle cannot move faster than light. However relativistic mechanics admits existence of massless particles which travel at the speed of light (e.g. photons and neutrinos). Equation (2.19) for these particles becomes

$$p = \frac{\mathcal{E}}{c}. \tag{2.20}$$

We use the term «particle» although its «elementariness» is never used. Therefore Eqs. (2.16), (2.18), and (2.19) can be equally applied to any body comprised of many particles. The mass m is then the total body mass and v should be understood as the body velocity as a whole.

The energy of a body at rest consists of the rest energy of the constituent particles, their kinetic energies and the interaction energy of the particles. Therefore

$$mc^2 \neq \sum_i m_i \cdot c^2,$$

where m_i is the mass of i-th particle.

Thus mass is not conserved in relativistic mechanics, there is only the law of conservation of energy which also includes rest energy.

By taking the squares of Eqs. (2.16) and (2.18) one can see that

$$\mathcal{E}^2 - (pc)^2 = m^2c^4, \tag{2.21}$$

i.e.

$$\mathcal{E} = \sqrt{(mc^2)^2 + (pc)^2}. \tag{2.22}$$

Equation (2.21) is often called the main kinematic identity of relativistic mechanics.

Notice that a particle for which

$$p \gg mc,$$

is called ultrarelativistic. For such a particle Eq. (2.20) holds approximately.

When describing collisions of relativistic particles it is convenient to write the main kinematic identity as

$$\left(\sum_i \mathcal{E}_i\right)^2 - \left(\sum_i \vec{p}_i c\right)^2 = \text{invariant}. \tag{2.23}$$

The term «invariant» means that the right-hand side of Eq. (2.23) remains the same in another inertial frame of reference.

Rigid body dynamics. One of the most important mechanical concepts is that of absolutely rigid body. Absolutely rigid body is a set of particles in which the distance between any pair of particles remains the same during the body motion.

Consider rotation of a rigid body around some axis. In this case all the body particles move around the circles which centers belong to the same straight line called rotation axis. The axis can be either inside or outside the body. Let us choose a point O on the rotation axis. The position of the particle A of the rigid body can be specified by the radius-vector $\overrightarrow{OA} = \vec{r}$. If the body rotates by the angle $d\varphi$ for the time interval dt, the displacement of A is then

$$|d\vec{r}| = r_\perp \cdot d\varphi, \tag{2.24}$$

where $r_\perp$ is the distance between A and the rotation axis.

The rate of change of the angular displacement is called angular velocity ω. Since the linear displacement $dl = rd\varphi$ for the same time interval equals vdt,

$$v = \omega \cdot r. \tag{2.25}$$

This relation can be also written in vector form by introducing the vector of rotation angle $\vec{\varphi}$ and the vector of angular velocity $\vec{\omega}$. These quantities together with torque and angular momentum are vectors albeit unusual. Unlike ordinary vectors (e.g. position vector, velocity, and force) which are called polar vectors, these vectors have opposite directions in the right-handed (the z-axis is along the motion of a right screw when turning the screw from x to y) and left-handed coordinate frames. Vectors which possess this property are called axial vectors. As long as one employs the same coordinate frame (usually it is the right-handed) the axial and polar vectors can be treated on the same footing. In vector form Eq. (2.25) becomes:

$$\vec{v} = \vec{\omega} \times \vec{r}. \tag{2.26}$$

The rotation angle is related to the angular velocity as

$$\vec{\omega} = \frac{d\vec{\varphi}}{dt}. \tag{2.27}$$

Any body can be treated as a set of n particles (including $n \to \infty$). In this case the torque and the angular momentum are defined as the sums:

$$\vec{M} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i, \tag{2.28}$$

$$\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i. \tag{2.29}$$

As it was already mentioned a set of particles in which the distance between any two particles remains constant during a motion is called rigid body. Consider rotation of a rigid body around immobile axis Oz. The angular velocity vector $\vec{\omega}$ is the same for all particles of the body and it is parallel to the axis. The particle velocity is

$$\vec{v_i} = \vec{\omega} \times \vec{r}_i,$$

where $\vec{r}_i$ is the position vector of the particle drawn from the origin O. Any particle moves around the circle which radius is $r_{i\perp}$. The vectors $\vec{r}_{i\perp}$ and $\vec{v}_{i\perp}$ are perpendicular, i.e. the angular momentum of i-th particle is

$$L_{i\perp} = m_i r_{i\perp} v_{i\perp} = m_i r_{i\perp}^2 \omega.$$

The net angular momentum of the body is

$$L_z = \sum_{i=1}^{n} m_i r_{i\perp}^2 \omega = I_z \omega.$$

The quantity I_z introduced here specifies the body's rotational inertia, it is called the moment of inertia around z axis. It is determined not only by the body mass but also the mass distribution with respect to the axis of rotation:

$$I_z = \sum_{i=1}^{n} m_i r_{i\perp}^2. \tag{2.30}$$

The moment of inertia I around an axis of rotation can be expressed via the moment of inertia I_0 around the parallel axis which passes through the center of mass of the body, the mass of body m, and the distance between the axes a_0:

$$I = I_0 + m a_0^2. \tag{2.31}$$

This relation is called Huygens-Steiner theorem.

The distance of mass m_i from the axis of rotation in Eq. (2.30) can be expressed via its coordinates as $r_{i\perp}^2 = x_i^2 + y_i^2$. Similar equations can be written for moments of inertia around x and y axes:

$$I_x = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2), \quad I_y = \sum_{i=1}^{n} m_i(z_i^2 + x_i^2), \quad I_z = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2). \tag{2.32}$$

Adding the moments of inertia and taking into account that $r_i^2 = x_i^2 + y_i^2 + z_i^2$ one obtains the relation:

$$I_x + I_y + I_z = 2\sum_{i=1}^{n} m_i r_i^2 = 2I_{\odot}. \tag{2.33}$$

Here the moment of inertia around the point $I_{\odot}$ is introduced.

Equation (2.33) turns out to be very useful in calculating the moments of inertia. For example, by placing the origin at the center of a thin spherical shell of radius R one obtains:

$$I_x = I_y = I_z = \frac{2}{3}I_{\odot} = \frac{2}{3}mR^2. \tag{2.34}$$

Equation of motion of a rigid body rotating around a fixed axis Oz is

$$I_z\frac{d\omega}{dt} = M_z. \tag{2.35}$$

Comparing this equation with Newton's second law (2.1) one can see that two equations are identical up to replacement of the force with the torque, the acceleration with the angular acceleration, and the mass with the moment of inertia (the latter depends on the mass and its distribution relative to the axis). Similar correspondence exists in the expression of kinetic energy K:

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n} m_i r_{i\perp}^2\omega_z^2 = \frac{1}{2}I_z\omega_z^2 = \frac{L_z^2}{2I_z}. \tag{2.36}$$

Notice that the linear displacement in the expression for work is similar to the rotation angle. In the simplest case of the force tangential to the circular path of a particle one obtains:

$$dA = F\,dr = Fr\,d\varphi = M\,d\varphi. \tag{2.37}$$

Equation (2.30) also applies to continuous mass distribution if the sum over the particles is replaced by the integral over infinitesimal body elements:

$$I_z = \int r_\perp^2\,dm. \tag{2.38}$$

Vectors and tensors. Many problems of physics require the concept of tensor to be properly formulated. Often a vector is defined as an ordered triplet of numbers. However one can see that not any ordered triplet forms

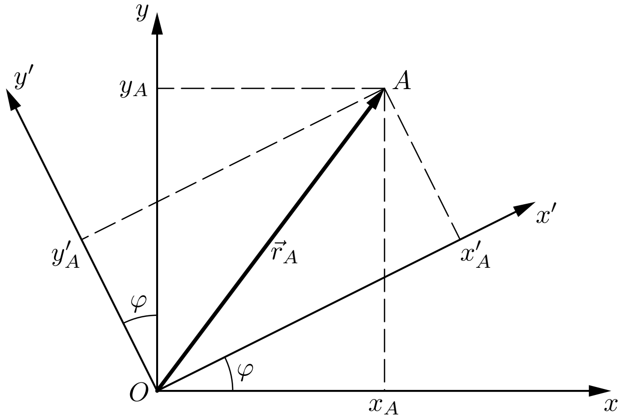

Fig. 2.2. Rotation of coordinate frame

a vector. For instance, pressure, volume, and temperature (P, V, T) of any mass of gas form the ordered triplet which is not vector. On the other hand, the triplet (x, y, z), where x, y, and z are coordinates of some point in a Cartesian frame form the vector called radius vector. What is the difference?

Vector is a concept originated from experience. The latter teaches that particle displacements (arrows) are added according to the parallelogram rule (see Fig. 2.1):

$$\vec r_{13} = \vec r_{12} + \vec r_{23}.$$

This is one of the defining properties of vector that is independent of the coordinate frame. However the definition of the radius vector as a triplet of numbers (x, y, z) depends on the coordinate frame. This definition can be made invariant by specifying the rule relating the coordinates of a point in different frames.

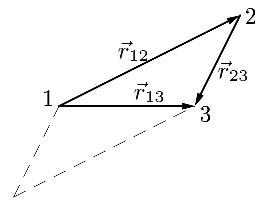

Fig. 2.1. Vector addition: $\vec r_{13} = \vec r_{12} + \vec r_{23}$

Let a coordinate frame be rotated around z axis by the angle φ (see Fig. 2.2).

Coordinates of the point A transform as:

$$\begin{aligned} x'_A &= x_A\cos\varphi + y_A\sin\varphi,\\ y'_A &= -x_A\sin\varphi + y_A\cos\varphi,\\ z'_A &= z_A. \end{aligned}$$

These equations define the law of transformation of the components of radius vector because $\vec{r}_A \equiv (x_A, y_A, z_A)$. The same transformation law holds for components of any vector. For instance, the vector of force $\vec{F} \equiv (F_x, F_y, F_z)$ transforms as:

$$\begin{aligned} F'_x &= F_x \cos\varphi + F_y \sin\varphi, \\ F'_y &= -F_x \sin\varphi + F_y \cos\varphi, \\ F'_z &= F_z. \end{aligned}$$

Thus the transformation law of the components of radius vector defines vector of any kind. The triplet of numbers (P, V, T) does not satisfy this law since it is independent of the coordinate frame.

Let us give a general definition of a vector. Suppose there are two coordinate frames, $Ox_1x_2x_3$ and $Ox'_1x'_2x'_3$, with the common origin O. Vector $\vec{A}$ is an ordered triplet of numbers (A_1, A_2, A_3) which transforms under rotation of the coordinate frame as the triplet of coordinates (x_1, x_2, x_3) of a radius-vector:

$$A'_i = \sum_{k=1}^{3} \alpha_{ik} A_k, \qquad i = 1, 2, 3. \tag{2.39}$$

Here α_{ik} is the cosine of the angle between the axes Ox'_i and Ox_k.

This definition can be generalized. A second-rank tensor is a triplet of vectors ($\vec{T}_1$, $\vec{T}_2$, $\vec{T}_3$) which under rotation of the frame transforms according to the same law:

$$\vec{T}'_i = \sum_{k=1}^{3} \alpha_{ik} \vec{T}_k, \qquad i = 1, 2, 3. \tag{2.40}$$

Vectors $\vec{T}_1$, $\vec{T}_2$, and $\vec{T}_3$ can be called components of tensor T on the axes Ox_1, Ox_2, and Ox_3, respectively, and vectors $\vec{T}'_1$, $\vec{T}'_2$, and $\vec{T}'_3$ are the components on the axes Ox'_1, Ox'_2, and Ox'_3. Obviously,

$$\begin{aligned} \vec{T}_1 &= \vec{i}T_{11} + \vec{j}T_{12} + \vec{k}T_{13}, \\ \vec{T}_2 &= \vec{i}T_{21} + \vec{j}T_{22} + \vec{k}T_{23}, \\ \vec{T}_3 &= \vec{i}T_{31} + \vec{j}T_{32} + \vec{k}T_{33}, \end{aligned} \tag{2.41}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors of the coordinate frame. Thus tensor T can be specified by the matrix T_{ik} which elements are called tensor components.

The set of Eqs. (2.41) can be written in a compact form:

$$\vec{T}_k = \sum_l \vec{e}_l T_{kl}, \qquad k = 1, 2, 3, \tag{2.42}$$

where $\vec{e}_1 = \vec{i}$, $\vec{e}_2 = \vec{j}$, and $\vec{e}_3 = \vec{k}$. Similarly,

$$\vec{T}'_i = \sum_m \vec{e}'_m T'_{im}, \qquad i = 1, 2, 3, \tag{2.43}$$

where $\vec{e}'_1 = \vec{i}'$, $\vec{e}'_2 = \vec{j}'$, and $\vec{e}'_3 = \vec{k}'$. Substituting Eqs. (2.42) and (2.43) in Eq. (2.40) one finds

$$\sum_m \vec{e}'_m T'_{im} = \sum_{k,l} \alpha_{ik}\, \vec{e}_l\, T_{kl}. \tag{2.44}$$

Scalar multiplication of Eq. (2.44) by $\vec{e}'_n$ gives the transformation law for the tensor components:

$$T'_{in} = \sum_{k,l} \alpha_{ik}\, \alpha_{nl}\, T_{kl}. \tag{2.45}$$

Here one uses the relation $(\vec{e}'_n, \vec{e}_l) = \alpha_{nl}$, $(\vec{e}'_m, \vec{e}'_n) = \delta_{mn}$, where δ_{mn} is identity matrix, i.e.

$$\delta_{mn} = \begin{cases} 1, & \text{если} \quad m = n, \\ 0, & \text{если} \quad m \neq n. \end{cases}$$

Equation (2.45) defines the transformation law of a second-rank tensor under rotations of coordinate frame. One can see that it is reasonable to classify vectors and scalars as tensors of the first-rank and zero-rank, respectively. Components of a third-rank tensor transform as

$$T'_{ikl} = \sum_{m,n,p} \alpha_{im}\, \alpha_{kn}\, \alpha_{lp}\, T_{mnp}.$$

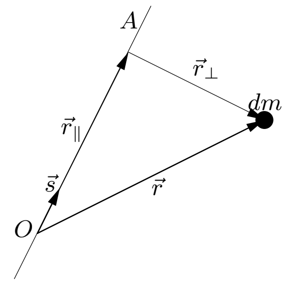

Fig. 2.3. Calculation of moment of inertia around arbitrary axis

As an example, consider tensor of inertia of a rigid body. Let us calculate the moment of inertia I of the body around arbitrary axis OA passing through the origin O (see Fig. 2.3).

Let us write the radius vector $\vec{r}$ of a body element of mass dm as the sum of the vector components along OA and perpendicular to it:

$$\vec{r} = \vec{r}_\parallel + \vec{r}_\perp.$$

By definition the moment of inertia is

$$I = \int r_\perp^2\, dm = \int (r^2 - r_\parallel^2) dm.$$

If $\vec{s}$ is a unit vector along the axis OA, then

$$r_{\parallel} = (\vec{r},\vec{s}) = x_1 s_1 + x_2 s_2 + x_3 s_3.$$

Also

$$r^2 = x_1^2 + x_2^2 + x_3^2, \qquad s_1^2 + s_2^2 + s_3^2 = 1.$$

Combining the above relations one obtains:

$$I = I_{11}s_1^2 + I_{22}s_2^2 + I_{33}s_3^2 + 2I_{12}s_1 s_2 + 2I_{23}s_2 s_3 + 2I_{31}s_3 s_1, \qquad (2.46)$$

where

$$\begin{array}{ll} I_{11} = \int(x_2^2 + x_3^2)dm, & I_{12} = I_{21} = -\int x_1 x_2\, dm, \\ I_{22} = \int(x_3^2 + x_1^2)dm, & I_{23} = I_{32} = -\int x_2 x_3\, dm, \\ I_{33} = \int(x_1^2 + x_2^2)dm, & I_{31} = I_{13} = -\int x_3 x_1\, dm. \end{array} \qquad (2.47)$$

Equation (2.46) shows how the moment of inertia around axis OA depends on the cosines of the axis. The equation has a geometric interpretation. Let us draw straight lines through the origin O in various directions and plot the points on them at the distance $1/\sqrt{I}$ from the origin. The points form a surface. Let us find the equation of the surface. Radius-vector of a point on the surface is

$$\vec{r} = \frac{\vec{s}}{\sqrt{I}},$$

i.e.

$$s_i = x_i\sqrt{I}, \qquad i = 1, 2, 3. \qquad (2.48)$$

Substitution of Eq. (2.48) in Eq. (2.46) gives

$$I_{11}x_1^2 + I_{22}x_2^2 + I_{33}x_3^2 + 2I_{12}x_1 x_2 + 2I_{23}x_2 x_3 + 2I_{31}x_3 x_1 = 1. \qquad (2.49)$$

This surface of the second order is an ellipsoid since it does not have points at infinity ($I \neq 0$). The ellipsoid is called inertia ellipsoid of the body constructed around the point O. Inertia ellipsoid depends on the point of construction. The central inertia ellipsoid is the ellipsoid constructed around the center-of-mass. One can show that the moment of inertia of a rigid body has all the features of a second-rank tensor: it is in one-to-one correspondence with matrix I_{ik} and its vector components are

$$\begin{array}{l} \vec{I}_1 = \vec{e}_1 I_{11} + \vec{e}_2 I_{12} + \vec{e}_3 I_{13}, \\ \vec{I}_2 = \vec{e}_1 I_{21} + \vec{e}_2 I_{22} + \vec{e}_3 I_{23}, \\ \vec{I}_3 = \vec{e}_1 I_{31} + \vec{e}_2 I_{32} + \vec{e}_3 I_{33}. \end{array} \qquad (2.50)$$

There is a theorem in algebra that Eq. (2.49) can be reduced to the main axes Ox, Oy, and Oz:

$$I_x x^2 + I_y y^2 + I_z z^2 = 1. \qquad (2.51)$$

The origin O of coordinate frame is usually placed at the center-of-mass. The quantities I_x, I_y, and I_z are called the main moments of inertia of the body. Vector components of the tensor on the main axes Ox, Oy, and Oz are

$$\vec{I}_x = \vec{i} I_x, \qquad \vec{I}_y = \vec{j} I_y, \qquad \vec{I}_z = \vec{k} I_z. \qquad (2.52)$$

If the cosines of a given axis with respect to the main axes are known,

$$s_x = \cos\alpha, \qquad s_y = \cos\beta, \qquad s_z = \cos\gamma,$$

then taking into account that

$$I_{xy} = 0, \qquad I_{yz} = 0, \qquad I_{zx} = 0,$$

and using Eq. (2.46) one obtains:

$$I = I_x \cos^2\alpha + I_y \cos^2\beta + I_z \cos^2\gamma. \qquad (2.53)$$

Otherwise, if the moments of inertia I_1, I_2, and I_3 around three arbitrary axes are known, one can solve the set of linear equations

$$\begin{array}{l} I_1 = I_x \cos^2\alpha_1 + I_y \cos^2\beta_1 + I_z \cos^2\gamma_1, \\ I_2 = I_x \cos^2\alpha_2 + I_y \cos^2\beta_2 + I_z \cos^2\gamma_2, \\ I_3 = I_x \cos^2\alpha_3 + I_y \cos^2\beta_3 + I_z \cos^2\gamma_3, \end{array} \qquad (2.54)$$

and determine the main moments of inertia: I_x, I_y, and I_z.

The main axes of a body can be found from its symmetry. The main axes of a homogeneous rectangular parallelepiped are parallel to its edges. If a body is rotationally symmetric its inertia ellipsoid has the same symmetry. A cylinder is an example. In this case the moments of inertia around the axes perpendicular to the symmetry axis are the same. The symmetry axis is one of the main axes. Any axis which is perpendicular to it is also the main one. For a spherical body any axis passing through its center is the main axis.

For example, consider a homogeneous rectangular parallelepiped which edges are a, b and c (see Fig. 3 on p. 139).

Let us place the origin O of the coordinate frame $Oxyz$ at the center of mass of the parallelepiped. It is not difficult to calculate the main moments of inertia:

$$I_x = \frac{m}{12}(b^2 + c^2), \qquad I_y = \frac{m}{12}(a^2 + c^2), \qquad I_z = \frac{m}{12}(a^2 + b^2).$$

Now let us find the moment of inertia with respect to the diagonal OO'. For this purpose we use Eq. (2.53). One can see that the cosines of the axis OO' are

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}},\ \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}},\ \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Thus the desired moment of inertia is

$$I_d = \frac{m}{6}\cdot\frac{a^2b^2+a^2c^2+b^2c^2}{a^2+b^2+c^2}. \qquad (2.55)$$

For a cube

$$I_x = \frac{ma^2}{6}, \qquad I_y = \frac{ma^2}{6}, \qquad I_z = \frac{ma^2}{6}, \qquad I_d = \frac{ma^2}{6}.$$

The latter is clear because the inertia ellipsoid of a cube is sphere.

Notice that the angular momentum $\vec{L}$ of a rigid body can be written as the dot product of inertia tensor I and angular velocity vector $\vec{\omega}$:

$$\begin{aligned} L_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3,\\ L_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3,\\ L_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3. \end{aligned} \qquad (2.56)$$

These equations are simplified when written in the coordinate frame of the main axes:

$$L_x = I_x\omega_x, \qquad L_y = I_y\omega_y, \qquad L_z = I_z\omega_z. \qquad (2.57)$$

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Lab 1.2.1

Determination of pellet velocity by means of ballistic pendulum

Purpose of the lab: determination of pellet velocity using conservation laws and employing ballistic pendulums.

Tools and instruments: an air-riffle on a support, a spotlight, an optical system to measure pendulum displacement, a ruler, pellets and a balance to weigh them, and ballistic pendulums.

Muzzle velocity of an air rifle is in the range from 150 to 200 m/s, and that of a rifle is ~1000 m/s.

These velocities are large in comparison with a pedestrian speed (~2 m/s) or even with the speed of an automobile (~20 m/s). A laboratory bench is usually about a few meters long, so the time of pellet flight is about 10^{-2}–10^{-3} s. Measurement of such time interval requires an expensive equipment capable of registering fast processes. It is cheaper to determine pellet velocity by measuring the momentum transferred by the pellet to some body in an inelastic collision. Net pellet-body momentum is conserved providing external forces are negligible or the collision time is small. If the body mass exceeds considerably the pellet mass the speed of the body (with the pellet stuck in it) is significantly less than the initial pellet velocity and can be easily measured. Duration of the inelastic collision, which lasts from the initial contact between the pellet and the body until the pellet gets stuck, depends on resistance of the body material. The time can be estimated using the pellet penetration depth and assuming that the resistance force is constant. A velocity of 200 m/s and a penetration depth of ~1 cm allows one to estimate the collision time as $\sim 10^{-4}$ s. Within that period even a body one hundred times heavier than a pellet will change its position by 0.1 mm only. For small collision times a momentum transferred by external forces is far smaller than the pellet momentum.

The momentum transferred by the pellet and therefore its velocity can be measured by a ballistic pendulum. The latter is a pendulum which is set in motion by a short initial impact. The impact can be considered short if the collision time is much shorter than the pendulum period. In this case the pendulum displacement during the collision time is much smaller than the amplitude of the pendulum swing. For harmonic oscillations collision time τ, pendulum period T, angular deviation $\Delta\varphi$ developed for the collision time, and the maximum swing φ_m (amplitude) are related by a simple equation:

$$\frac{\Delta\varphi}{\varphi_m} \approx \frac{2\pi\tau}{T}.$$

Thus, if the collision time equals 0.01 of the period, the deviation is 0.06 of the amplitude.

Maximum swing of pendulum and initial velocity resulting from pulse impact can be determined from the law of of conservation of mechanical energy providing energy loss for oscillation period is much smaller than energy of oscillation. We consider an attenuation as small if the amplitude decreases less than by half after ten swings. Pellet momentum and velocity can be found from the initial maximum swing.

While carrying out the experiment one should ensure that the pendulum

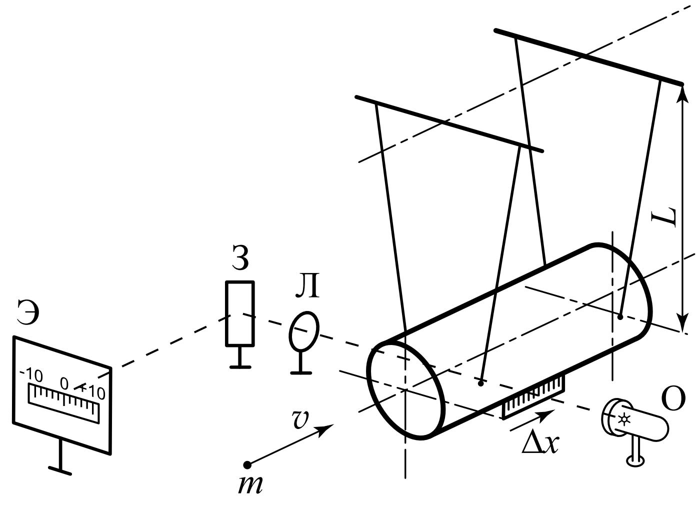

Fig. 1. Pellet-velocity measurement setup

swings in a plane and do not allow a transverse motion after the pellet strikes. This can be achieved by installing the rifle carefully. Also one should be aware that the pellet is followed by air jet which may affect pendulum motion thereby deteriorating the results. Therefore the rifle must be positioned at a distance sufficient for jet dispersion. The influence of the gas jet on the pendulum can be estimated by means of a blank shot.

The rifle is mounted on a special support. To load the rifle one should loosen the lock screw of the support and tilt the rifle to one side in the holder then bend the barrel in the trigger direction as far as it can go. The initial rifle position should be restored after it is loaded.

I. Pellet-velocity measurement setup

The ballistic pendulum used in this part of the lab is a heavy cylinder suspended on four threads of the same length. It is shown in Fig. 1 as a part of the measurement setup. When the pendulum is swinging any point of the cylinder executes circular motion with the radius equal to the suspension length. The motion is illustrated in Fig. 2 (side view of the swing plane). All the points of the cylinder move round circular arcs of the same radius L. In particular the center of mass M_0 moves to M_1 along the arc which center is at the point O.

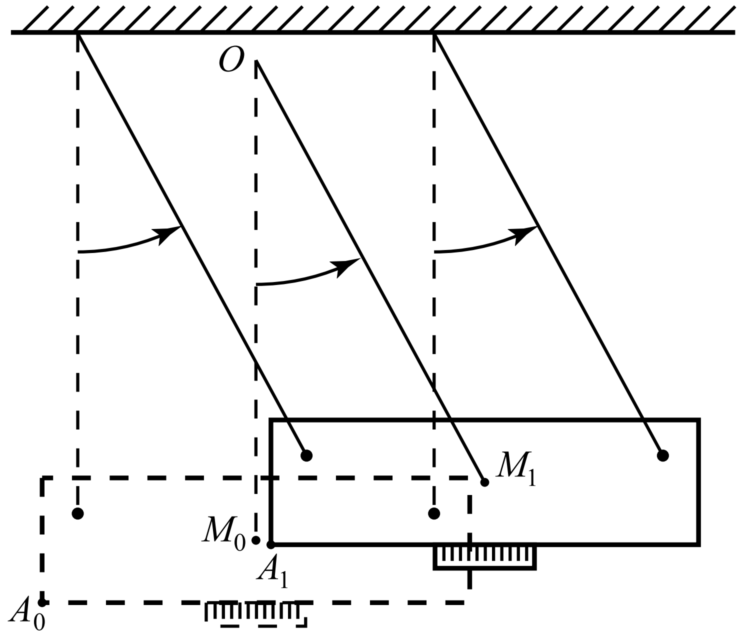

Fig. 2. Pellet-velocity measurement setup

We have already mentioned that the rifle must be appropriately installed. The rifle should be mounted so that the pellet velocity before collision would be directed along the cylinder axis (at least close enough). The external forces for the pellet-cylinder system are gravity force which has no horizontal component and the thread tension forces which develop horizontal components when the pendulum swings. However if the deviation is small these components are also small and their momentum transferred during the collision is negligible compared to the momentum of the pellet. Thus the law of conservation of momentum applied to the collision looks as follows:

$$mu = (M + m)V. \tag{1}$$

Here m is the pellet mass, M is the cylinder mass, u is the pellet velocity before collision, and V is the cylinder velocity after collision.

Taking into account that the pendulum mass exceeds considerably that of a pellet we can write

$$u = \frac{M}{m}V. \tag{2}$$

Having gained some kinetic energy during collision the pendulum will rise until its kinetic energy is converted into potential energy in the gravitational field (losses neglected). According to the law of conservation of mechanical energy the pendulum elevation h above its equilibrium position is related to the initial pendulum velocity V as

$$V^2 = 2gh. \tag{3}$$

Here g is the gravitational acceleration.

Pendulum elevation can be expressed via the angle φ of pendulum deviation from the vertical:

$$h = L(1 - \cos\varphi) = 2L\sin^2\frac{\varphi}{2}, \qquad \text{пïSпïSпïS} \quad \varphi \approx \frac{\Delta x}{L}. \tag{4}$$

From Eqs. (2), (3) and (4) we obtain the final formula for pellet velocity:

$$u = \frac{M}{m}\sqrt{\frac{g}{L}}\,\Delta x. \tag{5}$$

The pendulum deviation Δx is measured by means of an optical system shown in Fig. 1. Enlarged image of the scale attached to the cylinder makes it possible to determine its horizontal displacement. This allows one to measure successive amplitudes of pendulum swing and determine the attenuation.

Equation (3) and therefore the final formula (5) are valid as long as an energy loss during pendulum motion can be neglected.

The most important sources of swing attenuation are air drug and a loose pivot.

The energy lost during a swing quarter-period could be omitted from the conservation law (3) if it is small compared to the maximum potential energy. As it was already mentioned the attenuation can be neglected if the swing magnitude decreases less than by half for ten periods.

LABORATORY ASSIGNMENT

1. Examine the ballistic pendulum and the measurement setup, learn how to handle the air-rifle.
2. Using the precision balance weigh the pellets and place them into box compartments with appropriate numbers so that not to mix them up. Do not forget to reset the balance before changing pellets.
3. Measure the distance L (see Fig. 1) with a two-meter ruler.
4. Assemble the optical system designed for measuring pendulum displacement. Switch on the spotlight and obtain a clear image of the scale on the screen.

5. Fire a few blank shots at the pendulum to make sure that it does not respond to the impact of the air jet from the rifle.
6. Make sure that the swing attenuation is small: the amplitude decreases less than by half after ten swings.
7. Fire a few shots and determine pellet velocity for each shot using Eq. (5).
8. For each shot estimate an accuracy of determination of pellet velocity.
9. Find the average pellet velocity and a scatter near the average. What is the reason for the observed scatter? Is it due to the measurement inaccuracy or to different shot velocities?

Questions

1. Give a definition of ballistic pendulum and describe where it can be used.
2. When is initial momentum of ballistic pendulum equal to pellet momentum?
3. Why is it necessary to use inelastic collision between pellet and pendulum?
4. Estimate the time of pellet-pendulum collision in the experiment.
5. What factors are responsible for non-conservation of momentum during the collision?
6. What are the specific requirements for rifle installation?
7. What factors contribute to swing attenuation?
8. Which assumptions made in derivation of eq. (5) can be checked experimentally?
9. Why are the suspension threads not parallel (see Fig. 1)?

II. Method of torsion ballistic pendulum

The measurement setup is shown in Fig. 3. A pellet of mass m hits a target fixed on the rod aa which together with weights M and the wire Π is a torsion ballistic pendulum. To determine the pellet velocity we assume a pellet-target collision to be inelastic and use the law of angular momentum conservation

$$mur = I\Omega. \tag{6}$$

Here r is the distance between the pellet path and the pendulum axis of rotation (the wire Π), I is the pendulum moment of inertia, and Ω is its angular velocity right after the collision.

The law of angular momentum conservation can be used if the time of pellet-target collision is much less than the period of small oscillations of the pendulum. An angle of pendulum rotation during the collision is small compared to the amplitude of pendulum swing. Consequently the torque in the wire right after the collision is small compared to the torque at the maximum swing which is always finite. What matters is that the product of the torque and the collision time is small compared to the angular momentum of the pellet before the collision.

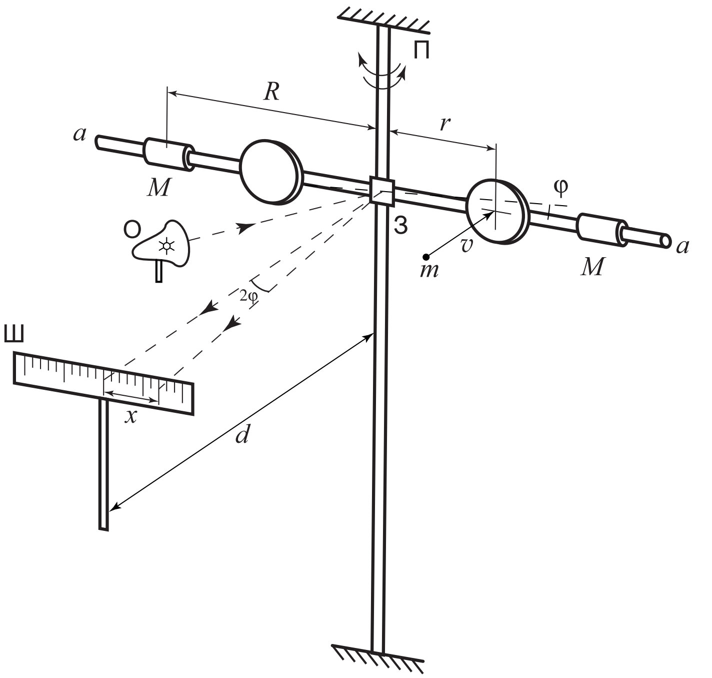

Fig. 3. Measurement of pellet velocity using a torsion ballistic pendulum

Initial kinetic energy of the pendulum converts to potential energy, i.e. elastic energy of the wire torsion, and a part of it is irreversibly lost, first of all, due to air friction. The loss can be estimated by measuring the decrement of swing amplitude in 10 periods. The swing attenuation is considered small if the amplitude decreases by half or less. This means that the energy loss during oscillation period is considerably less than the swing energy. Neglecting the losses we can write the energy balance as

$$k\frac{\varphi^2}{2} = I\frac{\Omega^2}{2}. \tag{7}$$

Here k is the torsion modulus of the wire Π and φ is the maximum swing angle.

From Eqs. (6) and (7) we obtain

$$u = \varphi\frac{\sqrt{kI}}{mr}. \tag{8}$$

The maximum angle in the experiment is always small. It can be easily determined from a displacement x of the image of the filament spotlight on the measurement scale. It follows from Fig. 3 that

$$\varphi \approx \frac{x}{2d}. \tag{9}$$

Here d is the distance from the scale III to the pendulum rotation axis.

Equation (8) includes the product kI which can be found by measuring the period of the pendulum with the weights M and without them. In the former case the pendulum period is equal to

$$T_1 = 2\pi\sqrt{\frac{I}{k}}. \tag{10}$$

In the latter case

$$T_2 = 2\pi\sqrt{\frac{I - 2MR^2}{k}}. \tag{11}$$

It follows from Eqs. (10) and (11) that

$$\sqrt{kI} = \frac{4\pi MR^2 T_1}{T_1^2 - T_2^2}. \tag{12}$$

Here R is the distance from the centers of mass of the weights M to the wire.

LABORATORY ASSIGNMENT

1. Examine the experimental setup and learn how to handle the air-rifle.
2. Using the precision balance weigh the pellets and place them into box compartments with the appropriate numbers so that not to mix them up. Do not forget to reset the balance before changing pellets.
3. Measure the distances r, R and d (see Fig. 3) with a ruler.
4. Adjust the optical system designed for measuring pendulum rotation angle. Switch on the spotlight, direct the light to the mirror and obtain a clear image of the spotlight filament on the scale.
5. Fire a few blank shots at the pendulum to make sure that it does not respond to the impact of the air jet from the rifle.
6. Make sure that the swing attenuation is small: the amplitude must decrease by half or less after ten swings.
7. By measuring the time of 10–15 full swings of the pendulum determine T_1 and T_2. Using Eq. (12) find the value of $\sqrt{kI}$ and estimate its error.
8. Fire a few shots and determine the pellet velocity for each shot using Eqs. (9) and (8).

9. Estimate the pellet velocity error for each shot.
10. Find the average pellet velocity and a scatter near the average. What is the reason for the observed scatter? Is it due to the measurement inaccuracy or to different shot velocities?

Questions

1. How does a deviation of the pellet-target impact angle from 90 degrees affect the validity of the method employed in the experiment?
2. At which amplitudes of pendulum swing should the periods be measured?
3. How does pellet momentum affect pendulum swing?

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Lab 1.2.2

Experimental verification of the dynamical law of rotational motion using the Oberbeck pendulum

Purpose of the lab: 1) to verify that angular acceleration of the pendulum is directly proportional to the torque exerted on the pendulum, to determine the moment of inertia of the pendulum; 2) to access friction forces applied to the axis of rotation.

Tools and instruments: the Oberbeck pendulum, weights, a stopwatch, a ruler, and a caliper.

The purpose of the lab is to verify experimentally the dynamical law of rotational motion:

$$I\frac{d\omega}{dt} = M. \tag{1}$$

To this end the Oberbeck pendulum is used, its design is shown in Fig. 1.

The pendulum consists of four thin rods which are rigidly attached to the hub at right angles. The hub and two wheels of different radii (r_1 and r_2) are attached to the same horizontal shaft which is fixed between two spindle bearings. The moment of inertia of the pendulum can be varied by placing the weights m_1 along the rods. A thin thread is winded around one of the pendulum wheels. The light platform of a known mass is attached

to the thread, it is used for placing the weights. The torque exerted on the pendulum is due to the thread tension T:

$$M_{\text{н}} = rT, \tag{2}$$

where r is the wheel radius (r_1 or r_2). The force T can be easily found from the equation of motion of the platform with a weight on it:

$$mg - T = ma. \tag{3}$$

Here m is the mass of the platform and the weight.

If the torque M_{fr} due to the friction in the bearings is small compared to the torque M_T due to the tension in the thread, then the acceleration a is constant according to Eqs. (1), (2), and (3). The acceleration can be found by measuring the time t that takes the platform to descend through the distance h:

$$a = \frac{2h}{t^2}. \tag{4}$$

This acceleration is related to the angular acceleration $\beta = d\omega/dt$ by:

$$a = r\frac{d\omega}{dt} = r\beta. \tag{5}$$

Equations (2) – (5) specify the pendulum motion.

In real experiment the torque M_{fr} is often large, which significantly affects the results. At first sight, the effect due to friction could be mitigated by increasing the mass m. However this is not so because:
1) greater mass m increases the pressure exerted on the shaft by the pendulum thereby enhancing the friction;
2) large m reduces the time t and therefore deteriorates the accuracy of time measurement.

In our installation the friction in the spindle bearings (see Fig. 1) is small, so the friction torque is not large. However it is not negligible and should be taken into account in data treatment.

It is convenient to separate the friction torque in Eq. (1) explicitly:

$$M_T - M_{fr} = I\frac{d\omega}{dt}. \tag{6}$$

Before proceeding to the measurement the weights m_1 should be installed at some distance R from the rotation axis, so that the pendulum be in neutral equilibrium. To check the latter set the pendulum in motion and let it stop several times. (What is the use of the procedure? How can one

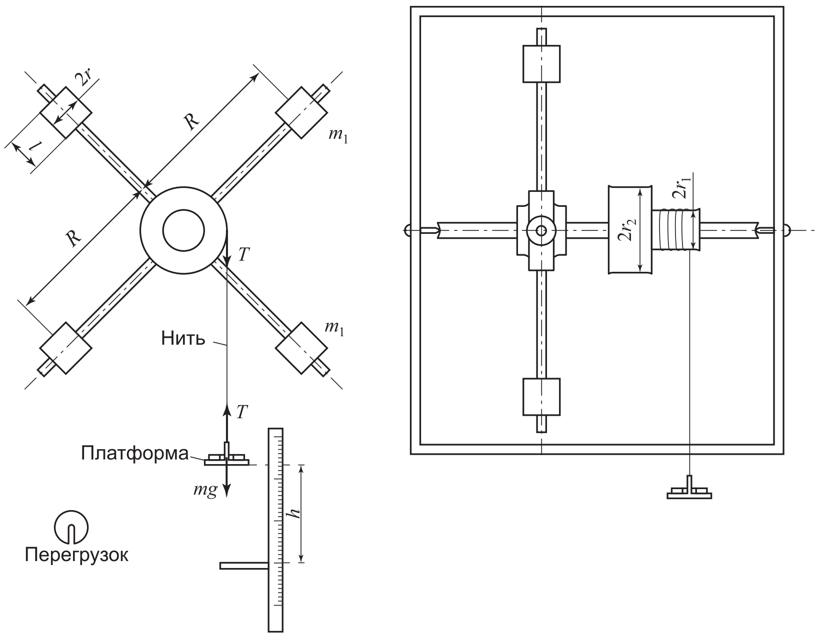

Fig. 1. Oberbeck pendulum

infer from the observations that the pendulum is well balanced?) Then wind one layer of the thread around a wheel and set the height h of the platform descent. The recommended height is 70-100 cm. It is convenient to perform measurements for the same height h using 3-5 different weights on the platform.

The experiment consists of two parts. In the first part the pendulum rotation is studied for different weights and the same moment of inertia (the positions of the weights m_1 are fixed). The results are used to calculate the moment of inertia I and the torque M_{fr} due to friction in the bearings.

In the second part the rotational motion is studied for different (5–6) values of the moment of inertia. The latter is varied by changing the distance R of the weights from the shaft. The measured value of the moment of inertia is compared to the calculated one. The weights m_1 are cylinders of radius r and height l. The moment of inertia of the pendulum is evaluated as

$$I = I_0 + 4m_1R^2 + 4\frac{m_1l^2}{12} + 4\frac{m_1r^2}{4}, \tag{7}$$

where I_0 is the moment of inertia without the weights m_1. The derivation of the formula is left to the reader.

LABORATORY ASSIGNMENT

1. Achieve the neutral equilibrium by varying the distance R between the weights m_1 and the shaft. The distance R should be measured and recorded.
2. Increase the tension T by loading the platform. Find the minimum mass m_0 of the weight for which the pendulum starts spinning. Perform the experiment for each wheel. Estimate the torque due to friction.
3. Put an additional weight on the platform and measure the time of the platform descent. Repeat the measurement 4-5 times and find the average t. Using Eqs. (2) – (5) determine the angular acceleration $\beta = \frac{2h}{rt^2}$ and the torque M_T. Tabulate the results using the table below.

Wheel diameter	Weight	Mass of platform with weights	Descent time					$\bar{t} \pm \sigma_{\bar{t}}$	$\beta \pm \sigma_\beta$	$M_T \pm \sigma_M$
cm	g	g	t_1	t_2	t_3	t_4	t_5	sec	sec^{-2}	N·m

4. Repeat the experiment for 3–4 different values of m for each wheel (6–8 measurements overall). Tabulate the results.
5. Plot the experimental results for two wheels. Plot the values of M_T on the abscissa and the angular acceleration β on the ordinate. Determine graphically the moment of inertia l and the friction torque M_{fr} (the x-intercept of the function $\beta(M_T)$). Estimate the errors.
6. Repeat the measurements of 3–5 for two different values of the moments of inertia corresponding to maximum and minimum distance of the weights m_1 from the shaft.
7. Compare the values of M_{fr} obtained in the experiments. Does the value of M_{fr} depend on the moment of inertia of the pendulum?
8. Repeat the experiments described in 3 for three different moments of inertia of the pendulum using only one weight and the large wheel. In each case determine I using Eq. (6). Take the value M_{fr} from 5.
9. Plot the values of I obtained for different R's as a function $I = f(R^2)$. Using the plot determine the moment inertia of the pendulum I_0 without the weights.

Do the experimental results agree with Eq. (7)? How does the relative contribution of two last terms in Eq. (7) depend on R? Is the corresponding correction comparable to the measurement errors? To answer these questions plot the value $\Delta I/I$ versus R^2, where

$$\Delta I = 4\frac{ml^2}{12} + 4\frac{mr^2}{4}.$$

10. What are possible sources of the experimental error?

Questions

1. Why must the torque due to friction in the shaft bearings be reduced as much as possible? It appears that Eq. (6) is valid for any value of M_{fr}.
2. What is the role of the thread thickness and elasticity?
3. Which quantity has to be measured with the greatest accuracy in this experiment?
4. State and prove Huygens-Steiner theorem.

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2. *Стрелков С.П.* Механика. — М.: Наука, 1975. Гл. VII, §§ 52, 53, 59; гл. V, §§ 41, 42.

Lab 1.2.3

Determination of principal moments of inertia of rigid bodies by means of trifilar torsion suspension

Purpose of the lab: to determine the moments of inertia of rigid bodies and to compare the results with theoretical calculations; to verify additivity of the moments of inertia and the Huygens-Steiner theorem.

Tools and instruments: a trifilar suspension, a stopwatch, an oscillation counter, and a set of rigid bodies (a disk, a rod, a hollow cylinder etc.).

Rotational inertia is due to the moment of inertia with respect to the corresponding axis of rotation (see the introduction to this chapter). The moment of inertia with respect to an immobile axis of rotation is defined as

$$I = \int r^2\, dm. \tag{1}$$

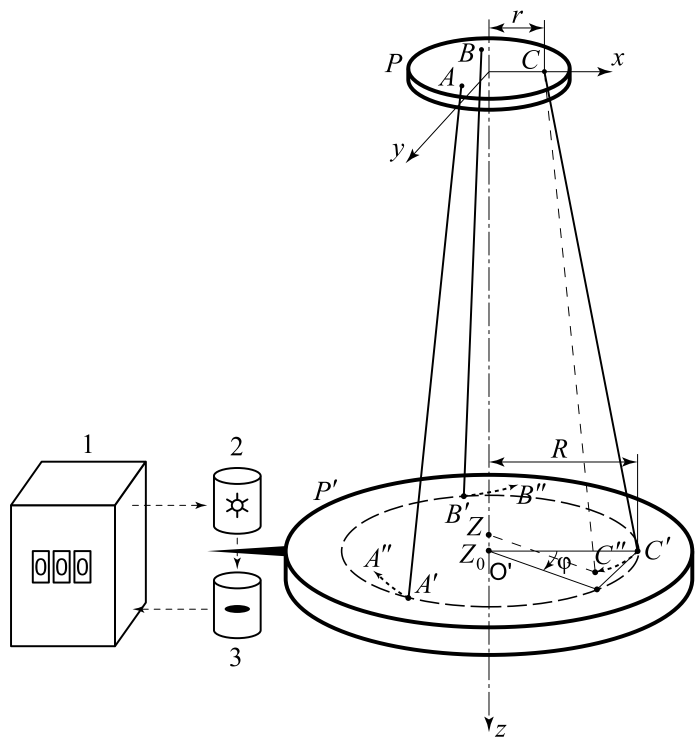

Fig. 1. Trifilar suspension

Here r is the distance of the body element dm from the axis. Integration is performed over all elements.

The moment of inertia can be calculated for uniform bodies of a simple shape. Otherwise the moment of inertia can be determined from experiment. The trifilar suspension shown in Fig. 1 is often used for this purpose. The device consists of the immobile platform P and the platform P' which is symmetrically suspended on three threads AA', BB', and CC' and can execute free oscillations.

The platform P is mounted on a bracket and is equipped with a lever (not shown) used to initiate rotational oscillations by slightly turning the upper platform. It is better to turn the upper platform which is attached to the immobile shaft since turning the lower platform would also cause pendulum-like oscillations which are difficult to account for. The upper platform remains at rest after the initial turn during the ensuing oscillations. Once the lower platform P' is turned by the angle φ with respect to the upper one, the restoring torque arises. It tends to return the lower platform to the equilibrium position that corresponds to zero rotation angle. However the platform does not remain in the equilibrium because of non-zero angular velocity (kinetic energy). This results in angular oscillations.

Neglecting the energy losses due to friction in air and at the points of suspension one can write the law of conservation of energy for the oscillations:

$$\frac{I\dot{\varphi}^2}{2} + mg(z_0 - z) = E. \tag{2}$$

Here I is the moment of inertia of the platform and the body, m is the mass of the platform and the body, φ is the platform angle of rotation (the dot stands for time derivative, so it is the angular velocity), z_0 is the vertical coordinate of the center O' of the lower platform at $\varphi = 0$, and z is the coordinate of the center that corresponds to the rotation angle φ. The first term on the left-hand side is the kinetic energy of rotation, the second term is the potential energy in the gravitational field, and E is the total energy.

It should be obvious from Eq. (2) that the restoring force is due to gravity.

Now let us choose the coordinate frame x, y, z, which is rigidly fixed to the upper platform (see Fig. 1). In this frame the coordinates of the suspension point C are $(r, 0, 0\ )$. The coordinates of the lower end C' of the corresponding thread at equilibrium are $(R, 0, z_0)$. When the platform turns by the angle φ the lower end is at the point C'' with coordinates $(R\cos\varphi, R\sin\varphi, z)$. The distance between points C и C'' is equal to the thread length L. Therefore

$$(R\cos\varphi - r)^2 + R^2\sin^2\varphi + z^2 = L^2. \tag{3}$$

Since at small angles $\cos\varphi \approx 1 - \varphi^2/2$, we obtain

$$z^2 = L^2 - R^2 - r^2 + 2Rr\cos\varphi = z_0^2 - 2Rr(1-\cos\varphi) \approx z_0^2 - Rr\varphi^2. \tag{4}$$

Taking the square root of Eq. (4) we obtain for small φ:

$$z \approx \sqrt{z_0^2 - Rr\varphi^2} \approx z_0\sqrt{1 - \frac{Rr\varphi^2}{z_0^2}} \approx z_0 - \frac{Rr\varphi^2}{2z_0}. \tag{5}$$

Substituting this value for z in Eq. (2) we get

$$\frac{1}{2}I\dot{\varphi}^2 + mg\frac{Rr}{2z_0}\varphi^2 = E. \tag{6}$$

Differentiation of the last equation with respect to time yields the equation for small angular oscillations of the platform:

$$I\ddot{\varphi} + mg\frac{Rr}{z_0}\varphi = 0. \tag{7}$$

The time derivative of E is zero since we neglected the energy losses due to friction.

One can easily check by direct substitution that the solution of this equation is

$$\varphi = \varphi_0 \sin\left(\sqrt{\frac{mgRr}{Iz_0}}t + \theta\right). \tag{8}$$

The amplitude φ_0 and the phase θ of oscillations are determined from initial conditions. The oscillation period is

$$T = 2\pi\sqrt{\frac{Iz_0}{mgRr}}. \tag{9}$$

Notice that this is the period of the simple gravity pendulum for $r = R$ and $I = mR^2$ (a thin ring).

Equation (9) gives the formula for the moment of inertia:

$$I = \frac{mgRrT^2}{4\pi^2 z_0}. \tag{10}$$

Now, the parameters R, r, and z_0 do not change during the experiment, which allows one to rewrite the last equation as:

$$I = kmT^2. \tag{11}$$

Here $k = \dfrac{gRr}{4\pi^2 z_0}$ is a constant quantity.

Thus the equations derived allow one to determine the moment of inertia of the platform with or without a body by measuring the period of angular oscillations. The moment of inertia of the body can then be calculated using additivity of moments of inertia. The additivity can be verified by performing the measurements for two bodies together and separately.

The derived equations are based on the assumption that irreversible energy losses due to friction are negligible, i.e. the oscillations decay slowly. Oscillation damping can be evaluated by comparing the time τ, which takes the oscillation magnitude to decrease by a factor of 2–3, with the oscillation period T. The irreversible energy losses are negligible providing

$$\tau \gg T. \tag{12}$$

It is recommended to determine oscillation period with a relative error of 0.5%. The number of oscillations required to measure the period is determined by this error and by accuracy of time measurement.

Oscillations are registered by a counter which consists of a light source (2), a photovoltaic cell (3), and a digital counter (1) (see Fig. 1). A leaf shutter attached to the platform crosses the beam twice a period. The signal from the cell is registered by the digital counter.

LABORATORY ASSIGNMENT

1. Before loading the lower platform check the installation, i.e. make sure that oscillations can be properly initiated and that the pendulum-like oscillations are not excited. Check operation of the oscillation counter.
2. By exciting angular oscillations check how well relation (12) is satisfied. This task does not require high accuracy of the corresponding time intervals. The measurements must be performed for the unloaded platform. Explain why.
3. Find the working range of oscillation amplitudes. In this range the oscillation period determined by 20–30 full swings is independent of the initial amplitude. This means that oscillation period remains the same when the amplitude is halved.
4. Measure parameters z_0, R, and r (see Fig. 1). Calculate the installation constant k in Eq. (11) and its error σ_k.
5. Measure the moment of inertia of the unloaded platform (hereinafter the oscillation period should be measured with a relative error less than 0,5%))

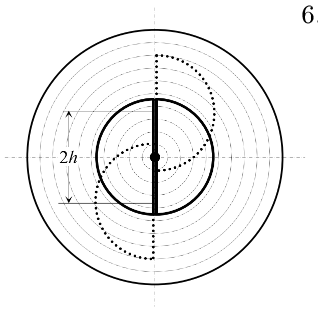

Fig. 2. Position of bodies on platform

6. Measure the moments of inertia of two bodies from the set, separately at first and then together. The bodies should be placed on the platform so that the center of mass of the system lies on the axis of rotation, i.e. no noticeable tilt of the platform is detected. For convenience a set of concentric rings is engraved on the platform. Check additivity of moments of inertia, i.e. validity of the relation $I = I_1 + I_2$, where I_1 and I_2 are the moments of inertia of the first and the second body and I is the total moment of inertia. The accuracy of this relation can be taken as the accuracy of the lab measurement. Calculate the moments of inertia I of all the bodies used and compare the results with the experimental values.
7. Place a disk which is cut in two halves on the platform. Gradually move the halves apart, so that their center of mass remains on the rotation axis (see Fig. 2), measure the moment of inertia I of the system versus the distance h between each of the halves and the rotation axis (the platform center).

Plot the dependence $I(h^2)$ and use it to determine the mass and the moment of inertia of the disk.

Questions

1. What are the assumptions used in the derivation of Eq. (10)?
2. Can the method of measuring the moments of inertia suggested in the lab be used if the axis of rotation of the platform does not pass through the center of mass?
3. Prove the Huygens-Steiner theorem.

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Lab 1.2.4

Determination of principal moments of inertia of rigid bodies by means of torsional oscillations

Purpose of the lab: to measure periods of torsional oscillations of a suspension frame with a body attached, to verify theoretical dependence between the periods of torsional oscillations with respect to different rotation axes, to determine moments of inertia with respect to different axes and to use them to determine principal moments of inertia, and to plot inertia ellipsoid.

Tools and instruments: a rigid frame suspended on a vertical wire, in which a rigid body can be fixed, a set of rigid bodies, and a stopwatch.

Rotational inertia of a rigid body is determined not only by the body mass but also by its spatial distribution. The latter is determined by the quantity called inertia tensor which can be represented by a symmetric (3×3) matrix specified by six elements. If all the matrix elements are known in some coordinate system, the moment of inertia with respect to an arbitrary axis passing through the origin can be found from Eq. (2.46). Any inertia tensor can be reduced to diagonal form like any symmetric matrix. The corresponding diagonal elements I_x, I_y, and I_z are called

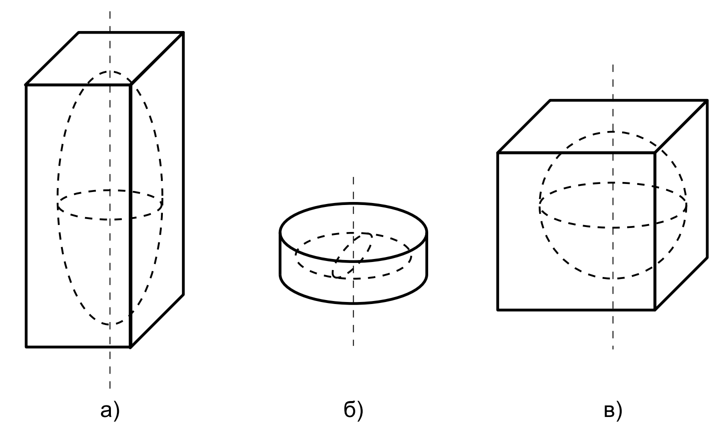

Fig. 1. Inertial ellipsoids of parallelepiped, disk, and cube

the principal moments of inertia. Inertia tensor can be visualized as an ellipsoid which in principal axes of inertia is represented by Eq. (2.51):

$$I_x x^2 + I_y y^2 + I_z z^2 = 1. \qquad (1)$$

This ellipsoid is called the inertia ellipsoid. It is rigidly fixed to the body. The coordinate axes Ox, Oy, and Oz coincide with the principal axes of inertia of the body. If the system origin O coincides with the center of mass the inertia ellipsoid is called central.

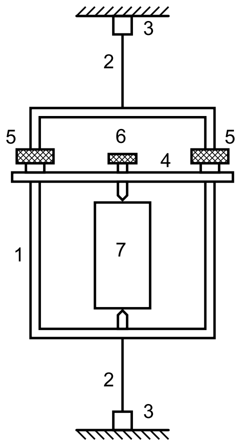

Fig. 2. Experimental setup

The inertia ellipsoid allows one to determine the moment of inertia with respect to any axis passing through the ellipsoid center. One should simply draw the radius-vector $\vec{r}$ along the rotation axis to the point of intersection with the ellipsoid surface. The length r specifies the corresponding moment of inertia according to

$$I = \frac{1}{r^2}. \qquad (2)$$

The principal axes of a body can often be determined by its symmetry. For instance, symmetry axes of cylinder and/or sphere are the principal axes of inertia because the moment of inertia with respect to any axis passing through a plane perpendicular to the symmetry axis is the same. Therefore the inertia

ellipsoid being the ellipsoid of rotation with respect to the symmetry axis has the same symmetry as the body itself.

Inertia ellipsoid turns out to be symmetric for some bodies which do not possess axial symmetry. For example consider a parallelepiped with square base or a cube. For cube the inertia ellipsoid is spherical, therefore the moment of inertia is independent of the rotation axis, just like for sphere. Figure 1 shows (not to scale) the central inertia ellipsoids for parallelepiped, disk, and cube.

Figure 2 shows the setup used to observe torsional oscillations. The frame 1 is rigidly attached to the vertical wire 2 fixed in the special clamps 3 which allow one to excite torsional oscillations around the vertical. The rigid body 7 is fixed in the frame by means of the plank 4, the nuts 5, and the screw 6. The body has special holes used to fix the body in different positions, so that the rotation axis passes through the center of mass at various angles.

Torsional oscillations of the frame and the body are described by the equation

$$(I + I_\text{p})\frac{d^2\varphi}{dt^2} = -f\varphi. \qquad (3)$$

Here I and I_p are the moments of inertia of the body and the frame, respectively, φ is the angle of rotation which depends on time t, and f is the torsion coefficient of the wire. The period of torsional oscillations is determined by the equation

$$T = 2\pi\sqrt{\frac{I + I_\text{p}}{f}}. \qquad (4)$$

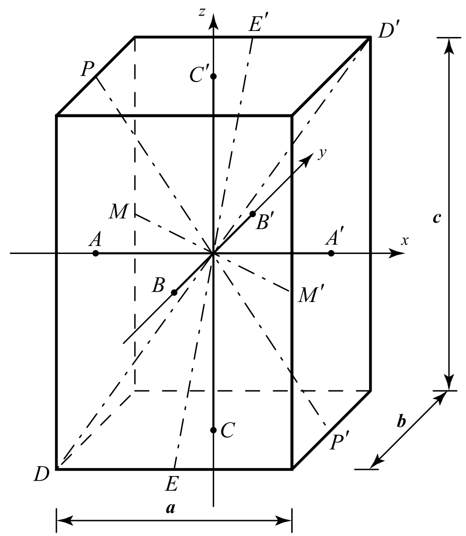

Fig. 3. Rotation axes of parallelepiped

Figure 3 shows the positions of rotation axes in parallelepiped. The principal axes are AA', BB', and CC'. The moments of inertia with respect to these axes are I_x, I_y, and I_z. The axis DD', which coincides with the main diagonal, makes the same angles with the principal axes and with the edges a, b, and c which are parallel to the axes. The cosines of the angles are a/d, b/d, and c/d, respectively, where $d = \sqrt{a^2 + b^2 + c^2}$ is the diagonal length.

The moment of inertia I_d with respect to the diagonal DD' is expressed via the principal moments of inertia as (2.53):

$$I_d = I_x \frac{a^2}{d^2} + I_y \frac{b^2}{d^2} + I_z \frac{c^2}{d^2}. \qquad (5)$$

This gives the equation:

$$(a^2 + b^2 + c^2) I_d = a^2 I_x + b^2 I_y + c^2 I_z. \qquad (6)$$

Using the relation (4) between the moment of inertia and the period of torsional oscillations one obtains the relation between the periods of oscillation:

$$(a^2 + b^2 + c^2) T_d^2 = a^2 T_x^2 + b^2 T_y^2 + c^2 T_z^2. \qquad (7)$$

Experimental verification of this relation serves to verify Eq. (5) as well. This equation also allows one to derive the relations between the moments of inertia corresponding to the axes EE', MM', and PP' and the principal moments of inertia. Using Eq. (4) one can find the corresponding oscillation periods. The reader is suggested to calculate the cosines of the angles which the above axes make with the principal axes and obtain the relations

$$(b^2 + c^2) T_E^2 = b^2 T_y^2 + c^2 T_z^2, \qquad (8)$$

$$(a^2 + c^2) T_P^2 = a^2 T_x^2 + c^2 T_z^2, \qquad (9)$$

$$(a^2 + b^2) T_M^2 = a^2 T_x^2 + b^2 T_y^2. \qquad (10)$$

These relations should be experimentally verified as well.

LABORATORY ASSIGNMENT

1. Learn how to handle the installation. Make sure that 1) the wire is tight, 2) the frame is rigidly attached to the wire, 3) the device for exciting the torsional oscillations is properly functioning, and 4) vertical vibrations are not excited together with the torsional oscillations.
2. Learn how to attach bodies to the frame. A body has special holes which must fit with the screws on the frame. To fix the body (see Fig. 2) one should do the following. Unscrew the nuts 5, pull up the plank 4 and insert the body into the frame, so that the hole on the body fits the jag on the lower side of the plank. Lower the plank and insert the screw 6 protruding from the plank by 5–7 mm into the hole on the body. Tighten the nuts 5 and then the screw 6. If the body gets loose in the frame tighten the screw 6 to fix it.

3. Before each set of measurements (the empty frame or the frame with the body) one should choose a proper amplitude of torsional oscillations (the maximum rotation angle of the frame). The amplitude is properly chosen if the oscillation period (determined by 10–15 oscillations) remains the same when the amplitude is reduced by half. One should decrease the amplitude until this condition is fulfilled.
4. Determine the oscillation periods for empty frame and for different positions of the bodies with respect to the rotation axis. A period should be measured by 10–15 oscillations, each measurement should be repeated at least 3 times.
5. Measure the parallelepiped dimensions using the caliper. Calculate the principal moments of inertia. Verify Eqs. (7) – (10) using the data obtained.
6. Draw cross-sections of inertia ellipsoid by principal planes. For this purpose take the measured oscillation periods with respect to the axis in the principal plane and for each axis calculate the quantity $1/\sqrt{T^2 - T_\mathrm{p}^2}$ which is proportional to the distance from the center of mass to the point of intersection of the ellipsoid with the axis. Here T_p is the oscillation period of the empty frame. Plot the values obtained along the directions corresponding to the axes and draw the ellipse through these points (8 points overall). The ellipse corresponds to the cross-section of the inertia ellipsoid by the principal plane (not to scale). Find the ratio of the principal moments of inertia.
7. Perform the same measurements for the cube and draw the corresponding cross-sections of the inertia ellipsoid. Verify that the central moments of inertia are equal.

Questions

1. What are the principal moments of inertia of a rigid body?
2. What does the inertia ellipsoid of a cube look like?
3. Describe the state of free (torqueless) rotation of a rigid body.

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Lab 1.2.5

Study of gyroscope precession

Purpose of the lab: to study the forced precession of gyroscope; to specify the dependence of precession velocity on the torque on the gyroscope axis; to calculate the rotational velocity of the gyroscope rotor and compare the result with that one obtained from the precession velocity.

Tools and instruments: a gyroscope in Cardan suspension, a stopwatch, a set of weights, unfastened rotor of a gyroscope, a cylinder of known mass, a torsional pendulum, a caliper, and a ruler.

The dynamical equation of a rigid body can be presented as

$$\frac{d\vec{P}}{dt} = \vec{F}, \tag{1}$$

$$\frac{d\vec{L}}{dt} = \vec{M}. \tag{2}$$

Here Eq. (1) represents dynamics of the center of inertia, and Eq. (2) is the angular momentum equation. A rigid body possesses six degrees of freedom, for this reason these two vector equations provide the complete description of its motion.

If the force $\vec{F}$ does not depend on rotational velocity and the torque $\vec{M}$ is independent of translational velocity, Eqs. (1) and (2) can be treated independently. This assumption is invalid, for example, for projectile motion in the atmosphere. But if the separation of the equations is possible, Eq. (1) describes motion of a material point and Eq. (2) regards the problem of rotation of a rigid body about a fixed point. The latter problem is considered in the lab.

The angular momentum of a rigid body written in projections on its principal axes x, y, z is

$$\vec{L} = \vec{i}\, I_x\omega_x + \vec{j}\, I_y\omega_y + \vec{k}\, I_z\omega_z, \tag{3}$$

where I_x, I_y, I_z are principal moments of inertia, ω_x, ω_y, ω_z are the components of the angular velocity vector $\vec{\omega}$. A fast-rotating body with

$$I_z\omega_z \gg I_x\omega_x, \quad I_y\omega_y,$$

is commonly referred to as gyroscope. If the gyroscope center of inertia is at rest, the gyroscope is called balanced.

According to Eq. (2), the increment of angular momentum is given by the integral

$$\Delta\vec{L} = \int \vec{M}\, dt. \tag{4}$$

If the torque is applied for a short period of time, it follows from Eq. (4) that the increment of the angular momentum $\Delta\vec{L}$ is much less than the angular momentum itself:

$$|\Delta\vec{L}| \ll |\vec{L}|.$$

This equation accounts for the remarkable dynamic stability of a fast-rotating gyroscope.

Let us figure out what forces should be applied to a gyroscope in order to change the direction of its axis. Consider a flywheel rotating about z-axis which is orthogonal to the wheel plane (Fig. 1). We assume that

$$\omega_z = \omega_0, \qquad \omega_x = 0, \qquad \omega_y = 0.$$

Now assume that the axis of rotation turns by infinitesimal angle $d\varphi$ in zx-plane in the direction of x-axis. This angular displacement represents an additional rotation of the flywheel about y-axis, such that

$$d\varphi = \Omega\, dt,$$

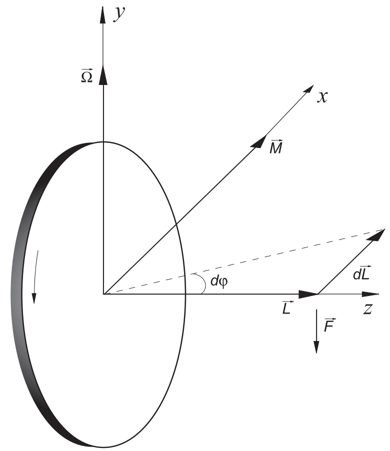

Fig. 1. Flywheel

where Ω is the angular velocity of the additional rotation. Let us assume that

$$L_\Omega \ll L_{\omega_0}. \tag{5}$$

This means that the angular momentum of the flywheel, which is equal to $I_z\omega_0$ prior to application of force, rotates in zx-plane and its magnitude remains constant. Thus

$$|d\vec{L}| = L d\varphi = L\Omega\, dt.$$

The increment of the angular momentum is directed along x-axis; for this reason one can represent vector $d\vec{L}$ as cross product of the angular velocity vector $\vec{\Omega}$ (directed along y-axis) and the vector of angular momentum of the flywheel (directed along z-axis):

$$d\vec{L} = \vec{\Omega} \times \vec{L}\, dt,$$

i.e.

$$\frac{d\vec{L}}{dt} = \vec{\Omega} \times \vec{L}.$$

Using Eq. (2) one obtains

$$\vec{M} = \vec{\Omega} \times \vec{L}. \tag{6}$$

Equation (6) is valid provided the condition (5) is fulfilled. It allows one to determine the torque $\vec{M}$ which makes the flywheel axis start rotating with velocity $\vec{\Omega}$. Thus, to turn the flywheel axis toward x-axis one needs to apply the force directed along y-axis rather than along x-axis. In this case the torque $\vec{M}$ is directed along x-axis.

The torque $\vec{M}$ on the gyroscope axis results in its slow rotation around y-axis with angular velocity Ω. This kind of motion is referred to as regular precession of gyroscope. In particular, the torque can be caused by the gravitational force if the gyroscope center of inertia does not coincide with its point of suspension. Let the gyroscope mass be m_g and its axis of rotation be deflected by angle α from the vertical. Then the velocity of precession caused by the gravitational force is

$$\Omega = \frac{M}{I_z\omega_0 \sin\alpha} = \frac{m_g g l_c \sin\alpha}{I_z\omega_0 \sin\alpha} = \frac{m_g g l_c}{I_z\omega_0}, \tag{7}$$

where l_c is the distance between the point of suspension and the center of inertia of the gyroscope, i.e. the precession velocity does not depend on the angle α.

To study the regular precession of the gyroscope one suspends additional weights on its axis. This results in displacement of the center of inertia and produces the torque of gravitational force leading to precession. The precession velocity in this case is given by the following equation:

$$\Omega = \frac{mgl}{I_z\omega_0}, \tag{8}$$

where m is the mass of the weight and l is the distance between the center of the Cardan suspension and the point of weight suspension on the gyroscope axis (see Fig. 3).

In this lab regular precession of the gyroscope is studied. The outer ring A of the suspension can freely rotate about the vertical axis aa. The inner ring B is connected to the ring A via horizontal axis bb. The gyroscope itself is mounted in the ring B, its axis cc is orthogonal to the axis bb. The center of inertia of the gyroscope coincides with the intersection point of the three axes and its spatial position is constant under arbitrary rotations of the rings. Effectively the gyroscope is suspended at the center of inertia.

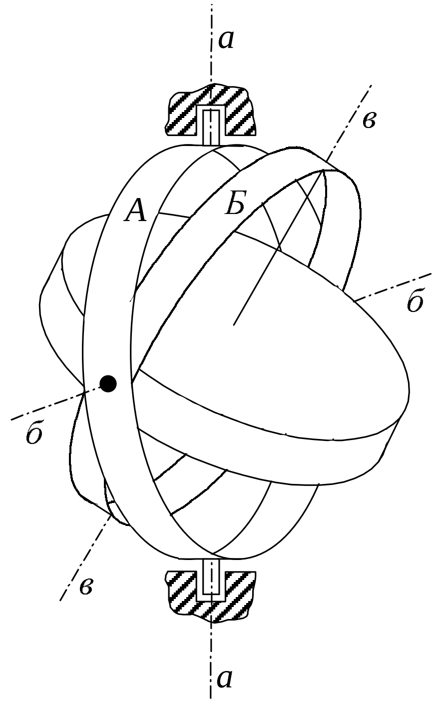

Fig. 2. Gyroscope in Cardan suspension

The experimental setup for studying the gyroscope precession is shown in Fig. 3. The gyroscope rotor is the rotor of high-speed electric motor M supplied with alternating current of the frequency of 400 Hz. The motor casing (the stator with coils supplied with 400-Hz current) is attached to the ring B (see Figs. 2 and 3). The motor and the ring B can rotate about the horizontal axis bb in the ring A which, in turn, can rotate about the vertical axis aa. The engine rotor is a massive steel cylinder with cooper veinlets like "squirrel case". The lever (marked with letter C in Fig. 3) is directed along the rotor symmetry axis, it is used for suspension of the weights W. One can alter the force F which induces precession by using different weights. The torque due to this force is determined by the distance l between the suspension point of the weights and the gyroscope center of inertia; this distance is indicated in the setup.

In the previous derivation of the equations governing gyroscope precession we assumed that the vectors of forces are coplanar to the vectors

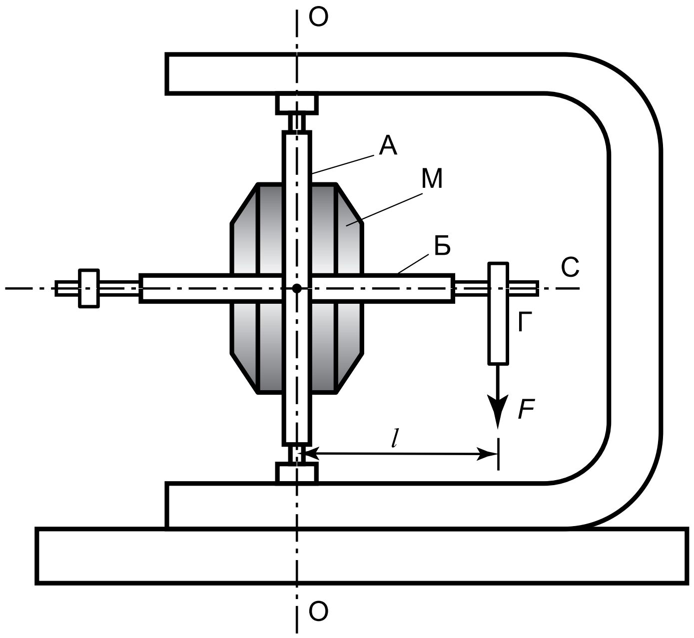

Fig. 3. Experimental setup

of self-rotation angular velocity and precession velocity (zy-plane). In this case the torque due to gravitational forces changes only the direction of the gyroscope angular momentum while the magnitude remains constant. Friction forces do not lie in the plane of axial rotation, so they can change both the magnitude and the direction of the angular momentum. The force of friction exerted on the gyroscope rotor is compensated by the motor, while the friction in the gimbal axes is not compensated. As a result the gyroscope axis will descend in the direction of gravitational force exerted on the weights. The reader is encouraged to analyze the friction forces in detail and to estimate the errors in determination of the velocity ω_0 of the gyroscope rotation around its symmetry axis due to the friction-induced lowering of the axis.

In the first part of the lab the dependence of precession velocity on the torque on the gyroscope rotation axis is studied. For this purpose one suspends the weights W on the lever C. The precession velocity is determined by measuring the number of revolutions of the lever around the vertical and the time passed. During the measurements the lever does

not only rotate but also slightly lowers, thus it should be raised by 5–6° prior to the measurements. The measurement should be stopped when the lever is lowered by the same angle.

Measurements of the gyroscope precession velocity allow one to calculate the angular velocity of its rotor. Equation (8) is used for this purpose. The moment of inertia of the rotor I_0 is measured via the torsional oscillations of the rotor replica which is suspended on a stiff wire along the rotor symmetry axis. The period of torsional oscillations T_0 depends both on the moment of inertia I_0 and the wire torsion modulus f:

$$T_0 = 2\pi\sqrt{\frac{I_0}{f}}. \tag{9}$$

To eliminate the unknown torsion modulus from Eq. (9) one measures the oscillation period of a cylinder of a given size and mass (and hence a given moment of inertia I_c). The moment of inertia of the rotor is then determined by the equation:

$$I_0 = I_c\frac{T_0^2}{T_c^2}, \tag{10}$$

where T_c is the period of torsional oscillations of the cylinder.

One can also work out the angular velocity of the rotor without the study of precession. The motor casing used in the lab has two coils which are necessary for fast spin-up of the gyroscope. In this lab the first coil is used for the spin-up while the second one can be used to measure the number of revolutions. The rotor is always slightly magnetized, for this reason its rotation leads to the induction of alternating emf in the second coil. The emf frequency equals the rotor rotation frequency; it can be measured, e.g. by observing Lissajous figures on oscilloscope screen. For this purpose one should apply the emf-signal and the sinusoidal signal from the generator to the X- and Y-inputs of the oscilloscope, respectively. If the frequencies of two signals coincide the figure on the screen is an ellipse.

LABORATORY ASSIGNMENT

1. Set the gyroscope axis horizontally by turning the lever C carefully.
2. Turn on the gyroscope power supply and wait for 4–5 minutes until the rotor motion becomes stable.
3. Make sure that the rotor rotation is fast: tapping on the lever C should not change its direction. Explain why the gyroscope axis is stable. "Play" with the gyroscope: press on the lever C with the pencil and observe the

gyroscope reaction. Determine the direction of gyroscope rotation from the observation.

4. Suspend the weight W on the lever C, which should result in gyroscope precession. Friction in the axis (in which exactly?) leads to slow lowering of the lever.
5. Lift the lever C by 5–6 degrees from the horizontal plane. Suspend the weight W and measure the precession velocity Ω with a stopwatch. Continue the measurements until the lever goes down by 5-6 degrees below the horizontal plane (the number of revolutions should be an integer). Also measure the speed of lowering. Repeat the measurement at least 5 times and average the results.
6. Repeat the experiments described in 5 for various values of the torque M (5-7 values) with respect to the gyroscope center of mass (the arm l is indicated on the setup). Plot the obtained dependence of precession velocity Ω on torque M.
7. Measure the moment of inertia of the rotor with respect to its symmetry axis I_0: suspend the rotor replica by the wire so that the symmetry axis of the replica is vertical and measure the oscillation period of the "pendulum". Replace the rotor with a cylinder of a given mass and radius and measure its oscillation period. Using Eq. (10) calculate the moment of inertia of the gyroscope rotor I_0.
8. Estimate the errors of the obtained values of I_0 and Ω.
9. Calculate the rotor rotation frequency using Eq. (8).
10. Estimate the torque due to friction using the known value of the speed of lowering.
11. Determine the rotor speed using Lissajois figures. Turn on the oscilloscope and the generator and apply the signal from the second coil of the gyroscope (from two terminals on the gyroscope base) to the oscilloscope Y-input. The signal from the generator should be applied to the X-input. The subsequent adjustment of the oscilloscope depends on its model: if "GOS-620" device is used, set the "Time/div" knob to "X-Y" mode by turning it counter-clockwise and adjust the horizontal and vertical scales using the "Volts/div" knobs. To obtain a Lissajois figure (ellipse) one should set the generator frequency equal to the rotor frequency. Make the ellipse stable by fine tuning of the generator frequency. If this is not possible turn the motor power off for a while: then the current in the first coil does not induce emf in the second one and does not interfere with the measurements. With the power off the measurements should be performed quickly due to deceleration of the rotor. Stability of the ellipse means that the generator frequency equals the rotor frequency.

12. Estimate the errors of the results and compare two values of the gyroscope angular velocity determined by different techniques.
13. Find out if Eq. (5) is applicable in the lab.

Questions

1. What is gyroscope and what are its major properties?
2. What factors does the velocity of regular precession depend on?
3. What is the dimensionality of the torsion modulus in Eq. (9)?
4. Derive Eq. (8) from Eq. (7).
5. Can you explain that a rolling coin is turning in the direction of tilt?

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Chapter III

CONTINUOUS MECHANICS

The subject of continuous mechanics is a macroscopic description of solid objects and fluids. In continuous mechanics any small volume is presumably large enough to contain a very large number of molecules. Such idealization justifies the usage of efficient mathematical methods developed for analytic functions.

Strain and stress of a deformable solid. Consider a solid object at rest which is not absolutely rigid, i.e. it can change its shape and the volume under pressure. A deformation of the solid results in internal forces which try to restore its original shape. Such a force divided by the corresponding area is called stress.

Stress is due to molecular forces, i.e. the forces between molecules. The range of molecular forces is of the order of intermolecular distance. As a macroscopic theory the continuous mechanics deals only with distances greater than distances between molecules. Therefore the «range» of intermolecular forces in continuous mechanics should be considered as negligible and so an internal force can act only through a surface.

Let some point of a solid object with coordinate x move at a distance s. If the displacement is the same for all points, this would be equivalent to a parallel transport (translation) of the object. Let us assume that the displacement of a neighboring point with coordinate $x+dx$ is different from s and it is actually $s+ds$. Strain is defined as

$$\varepsilon = \frac{ds}{dx},$$

i.e. strain is a relative displacement of two points divided by the initial distance between them. If the distance between the points increases the strain is called tensile otherwise it is called compressive.

Notice that the direction of ds is not necessarily the same as that of dx. If the strain is such that ds is perpendicular to dx it is called shear strain; the definition remains the same, $\varepsilon = ds/dx$ (see Fig. 3.1).

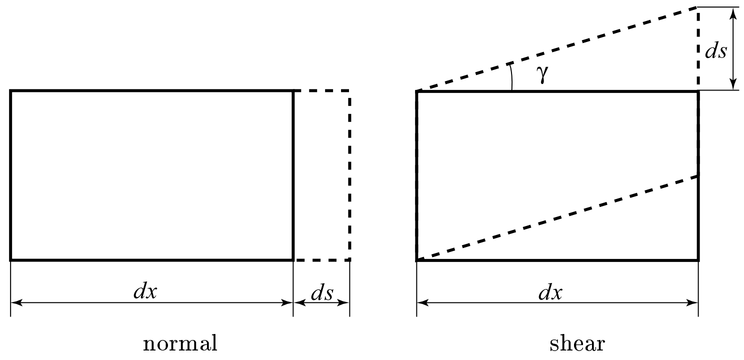

Fig. 3.1. Normal and shear strain

Actually all intermediate strain directions are possible, so in general $d\vec{s}$ и $d\vec{x}$ are vectors. The quantity ε relates the two vectors and therefore it is a second-rank tensor which can be represented as a (3×3) matrix ε_{ij}.

Consider Cartesian coordinates x, y, z and let the components of the displacement vector $\vec{s}$ be u, v, w:

$$\vec{s} = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w.$$

It could be shown that for small deformations the matrix ε_{ij} takes the form:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i}\right) \qquad i,j = 1,2,3 \quad (or\ x,y,z).$$

One can see that

$$\varepsilon_{xx} = \varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_{zz} = \varepsilon_z = \frac{\partial w}{\partial z}.$$

The forces responsible for stretch (compression) and shear distortion (see Fig. 3.2) are called tension (compression) and shear forces, respectively.

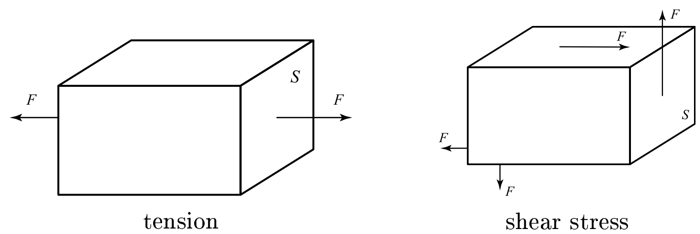

Fig. 3.2. Tension and shear stress

The corresponding stress is defined as the force divided by the area on which the force acts:

$$\sigma = \frac{F}{S}.$$

Unlike force, stress is a local quantity, i.e. it is defined at every point of an object. Stress is defined as the local force exerted on a unit area of some imaginary plane inside an object (see Fig. 3.3).

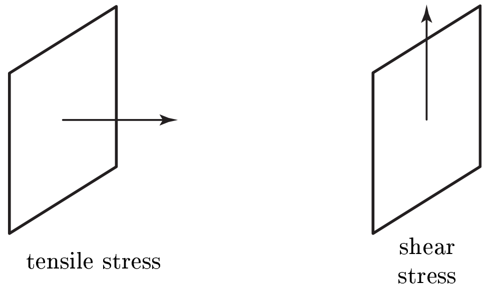

Fig. 3.3. Tensile and shear stress

In general stress depends on the plane orientation; all intermediate cases between normal tension and shear stress are possible. Therefore stress is also defined as the second-rank tensor which has nine components and relates three force components and three components of the unit vector normal to the plane the force acts upon. Figure 3.4 illustrates physical meaning of the components of stress tensor σ_{ij}.

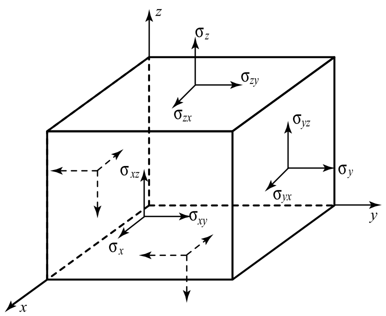

Fig. 3.4. Components of stress tensor

The figure pictures an imaginary infinitesimal parallelepiped in a solid object and the forces per unit area exerted on its faces.

Notice that an object under tension force remains at rest (see Fig. 3.2) whereas under the shear stress an object will be rotating counterclockwise. To prevent the object from rotation another pair of forces acting in the opposite direction must be applied. This can be done if the second pair of shear stress forces is exerted on the upper and lower faces of the parallelepiped in Fig. 3.2. Thus an object will remain in equilibrium providing the shear stress forces applied to the corresponding perpendicular planes are equal. The inspection of Fig. 3.4 shows that the following equations must hold:

$$\sigma_{zy} = \sigma_{yz}, \qquad \sigma_{xy} = \sigma_{yx}, \qquad \sigma_{xz} = \sigma_{zx},$$

i.e. stress tensor is symmetric. Because of this requirement only six components out of nine of any stress tensor are independent. Note that strain tensor is symmetric by definition. Overall, 12 independent variables are required to describe an equilibrium state of a deformed solid object.

Elastic modulus. Equation of state of ideal gas gives the relation between gas pressure P and its volume V at a given temperature. An equation similar to the equation of state relates the quantities σ and ε. The equation has been established empirically and it reads: for tension (compression),

$$\sigma = E\varepsilon, \tag{3.1}$$

and for shear stress,

$$\sigma = G\varepsilon = G\gamma, \tag{3.2}$$

where γ is the deformation angle (see Fig. 3.1).

The quantity E is called Young's modulus and G is called shear modulus. It is known from experiment that the moduli E and G are independent of stress in a wide range of the latter. The moduli E and G specify elastic properties of a material in the range where a linear relation between stress and strain holds.

In general, a relation between stress and strain in a crystal is determined via a forth-rank tensor which has 81 components. The tensor relates nine components of stress tensor and nine components of strain tensor, similarly to Eqs. (3.1) and (3.2). Since only six components of the stress and strain tensors are independent, there are only 36 elastic moduli. The actual number of the moduli is less due to a symmetry of the crystal and ranges from 21 to 3. Of course, this is true for single crystals. Polycrystalline bodies composed of small single crystals can be considered isotropic. This approximation is valid as long as we are interested in a large scale deformation of a crystalline solid. An isotropic body is specified by two independent elastic moduli.

Strain and stress in parallelepiped. Let a homogeneous isotropic body have a shape of a parallelepiped. Consider the forces F_x, F_y, and F_z applied to the opposite faces (see Fig. 3.5). Let the corresponding stresses be σ_x, σ_y, and σ_z and let us find the strains caused by the forces. We assume small strains, so superposition principle applies.

Let the coordinate axes be directed along the parallelepiped edges which lengths are l_x, l_y, and l_z.

If only the force F_x acts, the edge l_x is increased by $\Delta_1 l_x$:

$$\frac{\Delta_1 l_x}{l_x} = \frac{\sigma_x}{E}.$$

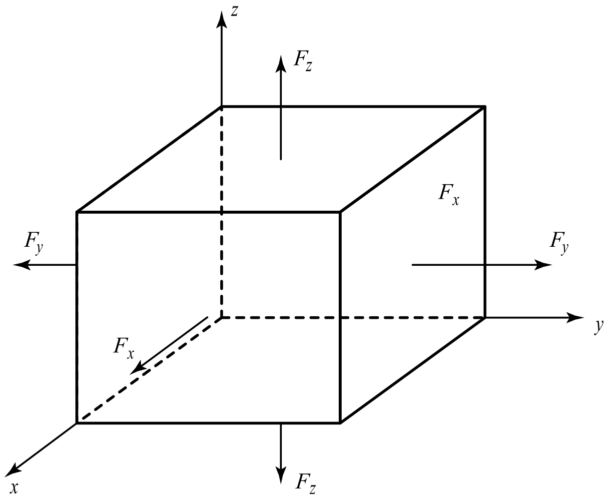

Fig. 3.5. Strains in parallelepiped

If only the force F_y acts, the dimension of the slab perpendicular to the y-axis decreases. In particular, the edge l_x would receive the decrement $\Delta_2 l_x$ which can be calculated as

$$\frac{\Delta_2 l_x}{l_x} = -\mu \frac{\sigma_y}{E},$$

where μ is called Poisson's ratio. Young's modulus E and Poisson's ratio μ specify completely elastic properties of an isotropic material. Other elastic coefficients can be expressed in terms of E and μ. The relative increment of the edge l_x due to the single force F_z would be

$$\frac{\Delta_3 l_x}{l_x} = -\mu \frac{\sigma_z}{E}.$$

If all the forces act simultaneously, the resulting increment of the edge l_x is the sum of all three increments according to the superposition principle:

$$\Delta l_x = \Delta_1 l_x + \Delta_2 l_x + \Delta_3 l_x.$$

The increments of the edges l_y and l_z can be found in a similar way. Finally:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z),$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_z + \sigma_x), \qquad (3.3)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y).$$

These equations are called generalized Hook's law.

A quasistatic stretching of the slab in the x direction does the work $A_1 = \frac{1}{2} S_x \sigma_x \Delta l_x$, where $S_x = l_y l_z$ is the area of the face orthogonal to the x-axis. The work can be written as

$$A_1 = \frac{1}{2} l_x l_y l_z \sigma_x \frac{\Delta l_x}{l_x} = \frac{1}{2} V \sigma_x \varepsilon_x,$$

where $V = l_x l_y l_z$ is the slab volume. Similarly,

$$A_2 = \frac{1}{2} V \sigma_y \varepsilon_y, \qquad A_3 = \frac{1}{2} V \sigma_z \varepsilon_z.$$

By adding all three contributions we find the density of elastic energy of the slab:

$$w_{el} = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z). \qquad (3.4)$$

Using Eq. (3.3) allows one to rewrite Eq. (3.4) as

$$w_{el} = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu(\sigma_x \sigma_y + \sigma_y \sigma_x + \sigma_z \sigma_x)\right]. \qquad (3.5)$$

Notice that an absolutely rigid slab ($E \to \infty$) does not accumulate the elastic energy ($w \to 0$) whatever forces act on it.

Strain due to uniform compression. Consider a case when all the stresses σ_x, σ_y, and σ_z are equal and negative. In this case the slab is under the uniform pressure applied to all its sides:

$$P = -\sigma_x = -\sigma_y = -\sigma_z.$$

Then it follows from Eq. (3.3) that

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -\frac{P}{E}(1 - 2\mu). \qquad (3.6)$$

Calculating the logarithmic derivative of both sides of the equation

$$V = l_x l_y l_z,$$

gives

$$\frac{\Delta V}{V} = \frac{\Delta l_x}{l_x} + \frac{\Delta l_y}{l_y} + \frac{\Delta l_y}{l_y},$$

or

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z.$$

Therefore Eq. (3.6) can be written as

$$\frac{\Delta V}{V} = -\frac{P}{K}, \tag{3.7}$$

where

$$K = \frac{E}{3(1-2\mu)}. \tag{3.8}$$

The constant K is called bulk modulus.

Then Eq. (3.5) for the elastic energy density can be rewritten as

$$w_{el} = \frac{3(1-2\mu)P^2}{2E} = \frac{P^2}{2K}.$$

Since w_{el} is positive definite, then

$$1 - 2\mu > 0,$$

or

$$\mu < \frac{1}{2}.$$

For rock Poisson's ratio μ is close to 0.25 and for metals it is 0.3.

Unilateral tension strain. Let a homogeneous rod be compressible or stretchable along its axis which is along x direction. Assume also that the transverse dimensions of the rod do not change due to the rod environment. The transversal shape of the rod is irrelevant. Then Eq. (3.3) can be used. Setting $\varepsilon_y = \varepsilon_z = 0$ gives:

$$\sigma_y - \mu(\sigma_z + \sigma_x) = 0, \qquad \sigma_z - \mu(\sigma_x + \sigma_y) = 0.$$

Then

$$\sigma_y = \sigma_z = \frac{\mu}{1-\mu}\sigma_x,$$

$$\varepsilon_x = \frac{\sigma_x}{E}\left(1 - \frac{2\mu^2}{1-\mu}\right).$$

Finally

$$\frac{\Delta l_x}{l_x} = \frac{\sigma_x}{E'}, \tag{3.9}$$

where

$$E' = E\frac{1-\mu}{(1+\mu)(1-2\mu)}. \tag{3.10}$$

The quantity E' is called P-wave modulus.

Relation between elastic moduli. As it is already mentioned a uniform isotropic elastic body is specified by two independent elastic moduli. Therefore the elastic coefficients introduced above must be related. It can be shown that

$$K = \frac{E}{3(1-2\mu)},$$

$$E' = E\frac{1-\mu}{(1+\mu)(1-2\mu)},$$

$$G = \frac{E}{2(1+\mu)},$$

$$E' = K + \frac{4}{3}G.$$

Here K is bulk modulus, E' is P-wave modulus, μ is Poisson's ratio, E is Young's modulus, and G is shear modulus. Therefore all elastic coefficients can be expressed in terms of E and G.

Pascal's law. In continuous mechanics a fluid can be defined as a medium in which a shear stress is absent in equilibrium. Therefore only the diagonal (matrix) components of the stress tensor are non-zero:

$$\sigma_{ij} = 0, \quad if \quad i \neq j; \qquad \sigma_{ii} \neq 0 \quad (i, j = 1, 2, 3).$$

Moreover all the diagonal components must be equal due to the fluid isotropy. Therefore the stress tensor of a fluid takes the form

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix},$$

where P is the pressure at a given point of the fluid.

In other words the normal stress (pressure) is independent of the orientation of a surface on which the pressure is exerted. This statement is called Pascal's law.

Pressure P in a fluid is caused by compression of the fluid. Since shear stress is absent the elastic properties of the fluid are specified by the single elastic constant called compressibility,

$$\chi = -\frac{1}{V}\frac{dV}{dP},$$

or by the inverse quantity, bulk modulus:

$$K = -V\frac{dP}{dV}.$$

It is assumed that the fluid temperature is maintained constant.

Bernoulli's equation. A fluid flow is specified if the position of any fluid parcel is known at any given time. By taking time derivative of the position it is possible to find the parcel velocity and the acceleration. Suppose that the coordinates x_0, y_0, and z_0 of a parcel at a time t_0 are given. The coordinates at a time t can be found from the following functions:

$$x = F_1(x_0, y_0, z_0, t),$$

$$y = F_2(x_0, y_0, z_0, t),$$

$$z = F_3(x_0, y_0, z_0, t).$$

This set of equations is called the Lagrange equations and the function arguments are called Lagrange variables. To specify a fluid state completely one must also know the pressure, the density, and the fluid temperature. These quantities are determined by the laws of conservation of energy and momentum and by the equation of state.

There is also another method to specify a flow that refers to what happens at any point of space at any given time. Usually three components of the velocity as functions of the coordinates and time are introduced

$$u = f_1(x, y, z, t),$$

$$v = f_2(x, y, z, t),$$

$$w = f_3(x, y, z, t).$$

This set of equations is called the Euler equations. To determine the parcel path one integrates the following set of equations:

$$dx = udt, \quad dy = vdt, \quad dz = wdt.$$

Since three constants of integration can be considered as the parcel coordinates at a given initial time the Lagrange equations are reproduced.

A pictorial representation of a fluid flow is given by the so called lines of the field flow. The tangent to a field flow line at any given point coincides with the direction of the fluid velocity. For a stationary flow, which is time independent, the field flow lines coincide with the parcel trajectories.

In a stationary flow all parcels going through the same point in space will later go along the same field flow line. A flow region swiped by the

parcel during its motion through the fluid is called material line. To derive equations which describe a flow it is convenient to consider a material line with small cross-sectional area, so that the fluid parameters can be considered constant across the line. Let ρ be the fluid density, v be the fluid velocity, and S be the cross-sectional area of the material line. Then the volumetric flow rate q, i.e. the fluid mass passing through a given cross-section per unit time, is

$$q = \rho v S. \tag{3.11}$$

Conservation of the fluid mass flowing along the material line with a varying cross-section gives:

$$\rho_1 v_1 S_1 = \rho_2 v_2 S_2. \tag{3.12}$$

As to the law of conservation of energy we take into account changes of kinetic and potential energy of a fluid caused by work of pressure forces but neglect changes of internal energy of the fluid due to compressibility, viscosity, and thermal conductivity. A fluid which viscosity and thermal conductivity can be neglected is termed perfect fluid. Consider a material line which vertical cross-section is shown in Fig. 3.6. The gravity force is directed to the figure bottom. The heights of the cross-sections 1 and 2 and the corresponding parameters of the flow are indicated. A fluid parcel traverses infinitesimal distance vdt for an infinitesimal time dt.The parcel at the cross-section S_1 moves at the cross-section S_1^1, and the parcel from S_2 moves to S_2^1. Since the displacements are small, the corresponding changes in the areas of the cross-sections are negligible. The work done by the pressure forces to displace the mass of the liquid between the cross-sections S_1 and S_2 is the sum of the positive work $p_1 S_1 v_1 dt$ and the negative work $p_2 S_2 v_2 dt$ (the displacement is opposite to the force). To calculate a change in the kinetic and potential energy notice that the energy of the liquid between the cross-sections S_1^1 and S_2 remains the same. The change is completely due to a transition of the mass between the cross-sections S_1 и S_1^1, $dm = \rho_1 S_1 v_1 dt = \rho_2 S_2 v_2 dt$, to the position between the cross-sections S_2 и S_2^1. Using the law of conservation of mass in the expression for the work due to the pressure forces and equating this work to the change in potential and kinetic energy we obtain:

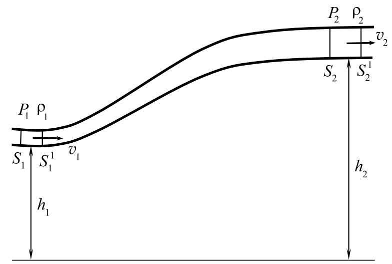

Fig. 3.6. To derivation of Bernoulli's equation

$$\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) dm = dm\left(g(h_2 - h_1) + \frac{v_2^2 - v_1^2}{2}\right). \tag{3.13}$$

This gives Bernoulli's equation:

$$\frac{v_1^2}{2} + gh_1 + \frac{p_1}{\rho_1} = \frac{v_2^2}{2} + gh_2 + \frac{p_2}{\rho_2} = \text{const.} \tag{3.14}$$

The compressibility of a liquid under standard conditions is usually small. For instance, increasing the density of water by 1% requires a pressure of 200 atm (such a pressure exists at the sea depth of 2km) and increasing by 10% requires more than 3000 atm. Therefore water is considered incompressible for small pressures. Then instead of (3.12) and (3.14) one can write

$$v_1 S_1 = v_2 S_2, \tag{3.15}$$

$$p_1 + \frac{\rho v_1^2}{2} + \rho g h_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g h_2. \tag{3.16}$$

Using Bernoulli's equation (3.16) for incompressible fluid one can derive Torricelli's equation for the velocity of a jet of liquid flowing from a vessel through an opening. The area of the opening is considered small compared to the area of the free liquid surface. Therefore the normal component of the velocity on the free surface is negligible in comparison with the jet velocity at the opening. The jet can be extended as a material line to the surface. The pressure in the jet is equal to the atmospheric pressure because the air-jet boundary is at rest, so there is no force exerted on the boundary. The pressure on the free surface is also equal to the atmospheric pressure. If the opening is below the free surface by h, Eq. (3.16) gives for the jet velocity:

$$v = \sqrt{2gh}. \tag{3.17}$$

Notice that the magnitude of the velocity is independent of its direction (the normal to the opening area). The quantity $\rho v^2/2$ is called dynamic pressure which is equal to the specific density of kinetic energy. It follows from Eq. (3.17) that the dynamic pressure equals the hydrostatic pressure $\rho g h$. The total pressure in a liquid at rest at this depth follows after adding the atmospheric pressure.

The Poiseuille equation. According to Bernoulli's equation the pressure of a stationary flow of a fluid in a horizontal tube of constant cross-section is the same along the tube. Actually the pressure decreases in the direction of the flow. To keep the flow stationary it is necessary to maintain a pressure difference at the ends of the tube that balances the forces of internal friction in the fluid.

Consider two parallel plates and a layer of liquid between them. To maintain a constant relative speed of the plates a pair of forces $\vec{F}$ and

$-\vec{F}$ must be applied to the plates. Newton found experimentally that the magnitude of the force is

$$F = \eta S \frac{v_2 - v_1}{h}, \tag{3.18}$$

where S is the plate area, h is the distance between the plates, v_1 and v_2 are the plate velocities, and η is dynamic viscosity (viscosity for short).

The force between two layers of a viscous fluid depends on the velocity gradient in the direction perpendicular to the flow (Newton's law for a viscous fluid):

$$F = S\eta \frac{dv_x}{dy}. \tag{3.19}$$

Let an incompressible fluid flow along a straight cylindrical tube of a radius R. Let abscissa be directed along the tube axis in the flow direction. Consider a cylinder of the length dx and of the radius r (see Fig. 3.7).

The lateral surface of the cylinder is subjected to the tangential force due to viscous friction, the force is directed opposite to the cylinder velocity:

$$dF = 2\pi r \eta \frac{dv}{dr} dx.$$

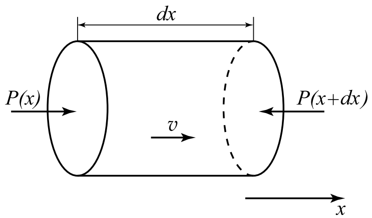

Fig. 3.7. To derivation of the Poiseuille equation

The force due to the difference in pressure acts on the cylinder bases in the direction of motion:

$$dF_1 = \pi r^2 \big(P(x) - P(x+dx)\big) = -\pi r^2 \frac{dP}{dx} dx.$$

The field flow lines are parallel, the cross-sectional area of a material line remains constant, so Eq. (3.15) shows that the acceleration of the fluid parcel under consideration is zero. Therefore the sum of the forces exerted on the parcel must vanish:

$$dF + dF_1 = 0.$$

It follows from the equation that

$$2\eta \frac{dv}{dr} = r \frac{dP}{dx}. \tag{3.20}$$

Since the velocity v as well as dv/dr are independent of x, the derivative dP/dx in Eq. (3.20) must be constant and equal to

$$\frac{P_2 - P_1}{l},$$

where P_1 and P_2 are the pressures at the tube inlet and outlet, respectively. This gives

$$\frac{dv}{dr} = -\frac{P_1 - P_2}{2\eta l} r. \tag{3.21}$$

Integration of this equation yields

$$v = -\frac{P_1 - P_2}{4\eta l} r^2 + C.$$

The constant of integration can be found by assuming that the fluid sticks to the tube walls:

$$v(R) = 0.$$

Then

$$v = \frac{P_1 - P_2}{4\eta l}(R^2 - r^2).$$

The velocity v is maximum at the tube axis and equals

$$v_0 = \frac{P_1 - P_2}{4\eta l} R^2.$$

Away from the axis the velocity decreases according to quadratic dependence.

Now let us determine the flow rate, i.e. the amount of the fluid passing through a tube cross-section per unit of time. The mass of the fluid passing through a ring-like area of internal radius r and external radius $r + dr$ equals $dQ = 2\pi r dr \cdot \rho v$. Substituting the expression for the velocity and integrating from 0 to R one finds:

$$Q = \pi\rho \frac{P_1 - P_2}{2\eta l} \int_0^R (R^2 - r^2) r \, dr,$$

or

$$Q = \pi\rho \frac{P_1 - P_2}{8\eta l} R^4. \tag{3.22}$$

Thus the flow rate is proportional to the pressure difference, to the fourth power of the tube radius, and inversely proportional to the tube length and dynamic viscosity. This law was found experimentally and derived by Poiseuille although he was not the first to discover it. Equation (3.22) is called the Hagen–Poiseuille equation.

In practice the flow rate is conveniently measured in terms of the volume of fluid flowing through cross-sectional area (volumetric flow rate). Then Eq. (3.22) becomes

$$Q_V = \frac{\pi R^4}{8\eta l}(P_1 - P_2). \tag{3.23}$$

This particular form of the Poiseuille equation is used in the lab 1.3.3.

A flow of an incompressible viscous fluid is described by the Navier-Stokes equation:

$$\frac{\partial \vec{v}}{\partial t} + v_x \frac{\partial \vec{v}}{\partial x} + v_y \frac{\partial \vec{v}}{\partial y} + v_z \frac{\partial \vec{v}}{\partial z} = -\frac{1}{\rho}\mathrm{grad}P + \frac{\eta}{\rho}\Delta \vec{v}. \tag{3.24}$$

Here

$$\mathrm{grad}P = \vec{i}\frac{\partial P}{\partial x} + \vec{j}\frac{\partial P}{\partial y} + \vec{k}\frac{\partial P}{\partial z}, \qquad \Delta \vec{v} = \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2}.$$

The equation can be reduced to a dimensionless form by introducing a typical size L and a typical velocity u of the flow. The contribution of each term is then determined by its coefficient. The contribution of the viscous term compared to the inertia terms on the left is determined by the Reynolds number:

$$\mathtt{Re} = \frac{\rho L u}{\eta}.$$

For the large Reynolds number the viscous term coefficient is small and viscosity is negligible. The Reynolds number also determines transition between the laminar and turbulent regimes of a viscous fluid flow.

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Lab 1.3.1

Determination of Young's modulus based on measurements of tensile and bending strain

Purpose of the lab: to determine experimentally the dependence between stress and strain (Hooke's law) for two simplest states of stress - normal stress and bending, and to determine Young's modulus from the results.

Tools and instruments: the first part: Lermantov' machine, a wire made of studied material, a telescope with a scale, a set of weights, a micrometer, and a ruler; the second part: a bracket for bending beams, an indicator for measuring strain, a set of beams, weights, a ruler, and a caliper.

The first part of the lab is devoted to studying normal stress described by eq. (3.1), the stress is observed in a stretched wire. Shear stress is studied in the second part, measurements are performed by bending a beam. The relation between the beam bending and the magnitude of the force applied between the points of support is expressed via Young's modulus. Therefore the modulus can be determined by measuring the bending versus the force.

I. Determination of Young's modulus by measurement of wire strain

Young's modulus is measured with the aid of Lermant's machine which design is shown in Fig. 1. The upper end of the wire Π made of material under study is attached to the bracket K, and the lower one to the cylinder at the end of the pivoted bracket III. The cylinder supports the lever r to which the mirror 3 is attached. Thus elongation of the wire can be measured by the angle of mirror rotation.

The wire strain is changed by displacing weights from the platform M to the platform O and vice versa. Under this arrangement the deformation of the bracket K remains the same and do not affect the measurement accuracy.

It should be taken into account that the wire Π is always bent if no stress is applied, which affects the results especially for moderate stress. Under small load the wire is not just stretching, it is mostly straightening up.

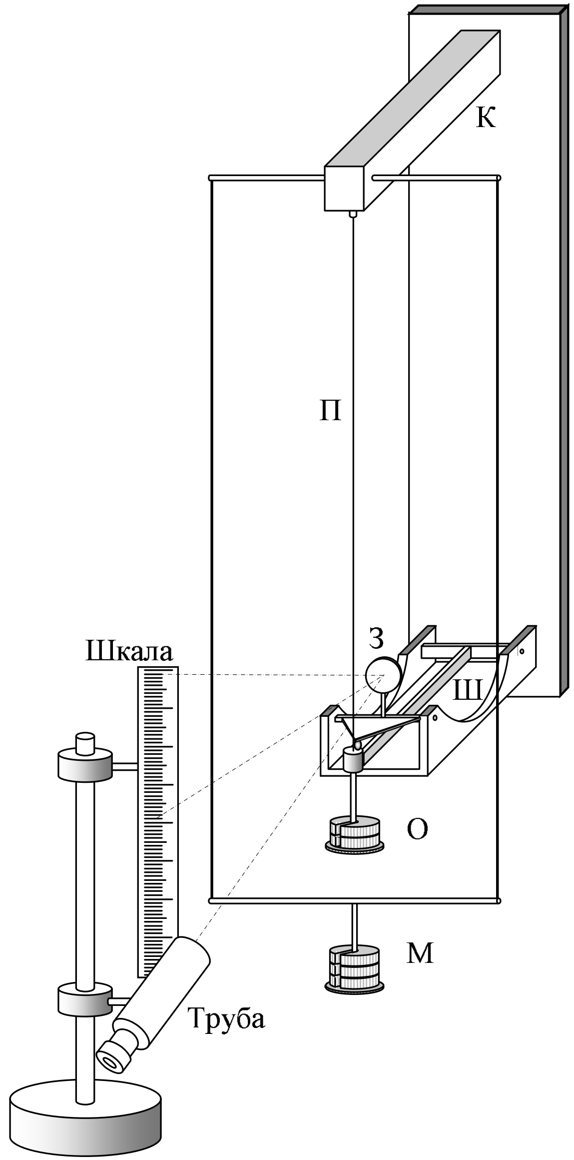

Fig. 1. Lermant's machine

LABORATORY ASSIGNMENT

1. Determine the cross-sectional area of the wire. For this purpose measure the wire diameter at least at ten different spots and in two perpendicular directions at each spot. Watch that the micrometer does not deform the wire. In the calculations that follow use the diameter averaged over all measurements.
2. Measure the wire length.
3. Train the telescope on the mirror 3. The scale reflection should be clearly visible. Derive the relation between the number n of scale graduations, the distance h between the scale and the mirror, the length of lever r and the elongation Δl of the mirror. The lever length is recorded on the machine and the distance h should be measured.
4. Make sure that wire elongation remains directly proportional to stress (elastic region) during experiment. To do so, estimate the maximum load by assuming the yield stress (at which the material begins to deform plastically) be 900 N/mm^2. The working load should not exceed 30% of the maximum. Then verify the estimate. Put a weight on the platform, remove it, and check that the wire length remains the same. Repeat the experiment with two, three, and more weights until reaching the maximum load. As soon as irreversible deformations become noticeable increasing the load must be stopped. Each time the load is changed the arising oscillations should be damped (the damper is not shown in Fig. 1).
5. Measure the dependence of wire elongation, i.e. the number n of scale graduations, on the mass m of weights by increasing and then decreasing the load. Repeat the experiment 2–3 times.
6. Using the results plot elongation Δl versus the load P. When no stretching force is applied the wire is usually bent, so for small loads its «elongation» is due to straightening rather than stretching. Therefore the elongation grows rapidly at the initial part of the curve $\Delta l(P)$ (small P) and only later the points approach a straight line (which does not pass through the origin). The line slope can be used to find elastic coefficient k of the wire and subsequently the Young's modulus. The initial part of the curve $\Delta l(P)$ should be excluded from the treatment.
7. Using the plot determine the elastic coefficient k and the Young's modulus E. Estimate the accuracy of k and E.
8. Determine the wire material by comparing the obtained value of Young's modulus with tabulated values.

II. Determination of Young's modulus by measurement of beam bending

The installation consists of a robust frame with two support prisms A and B (see Fig. 2). The beam (plank) C lies on the prism edges. The platform Π with the weights on it is suspended on the prism D at the beam center. The beam deflection is measured with the aid of the indicator I which is attached to a support separate from the frame. A complete revolution of the big indicator hand corresponds to 1 mm or one graduation of the small dial.

Young's modulus E of the beam material is related to deflection $y_{\max}$ (the displacement of the beam center) by Eq. (20) (see p. 172):

$$E = \frac{Pl^3}{4ab^3 y_{\max}}.$$

Here P is the load, l is the distance between the prisms A and B, and a and b are the width and the height of rectangular cross-section of the beam.

To exclude the error due to table deflection which changes under the load, the weights should be placed on the plank above the lower shelf of the support frame before the experiment.

Equation (20) is derived under the following conditions: firstly, the edges of the support prisms A and B are at the same height and, secondly, the force P is applied precisely at the beam center. The reader is recommended to verify how significantly this equation changes if the above conditions are not satisfied within the accuracy of experiment.

LABORATORY ASSIGNMENT

1. Measure the distance between the prisms A and B.
2. Determine width and thickness of the beam. To do so, measure these parameters at least at ten different spots. The averaged values should be used in calculations.
3. Put the beam on the frame. Set the indicator at the beam center and measure the deflection $y_{\max}$ versus the load P. Perform the measurements by increasing and then decreasing the load. Check that the beam restores its initial shape when the load is removed.
4. Study how the result depends on the position of the point where the force P is applied. Displace the prism D by 2–3 mm from the beam center and measure the deflection again. Compare the value obtained with the previous result.

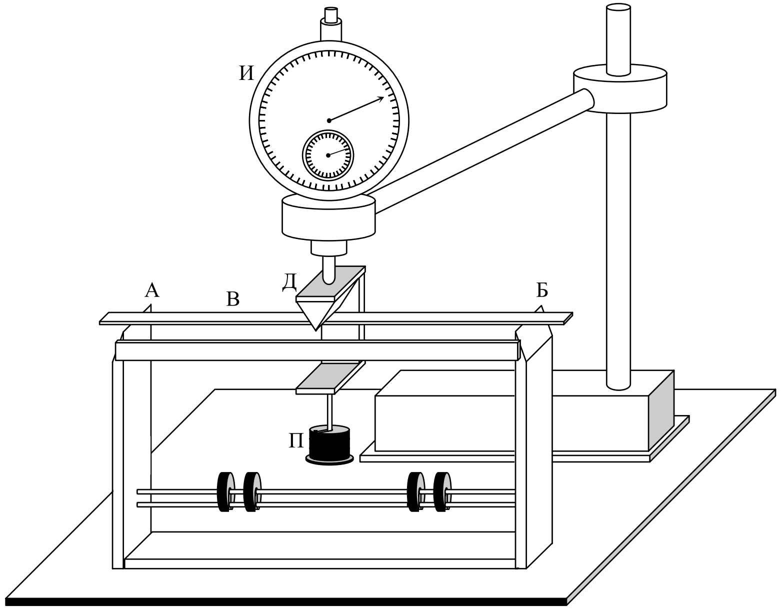

Fig. 2. Installation for measurement of Young's modulus

5. Overturn the beam upside down and repeat the measurements. Compare the results with the previous ones.
6. Perform the measurements for two or three wooden beams and for one made of metal.
7. For each beam plot the dependence of «load» versus «deflection» both for increasing and decreasing loads. Determine the average Young's moduli from the slopes of the curves.
8. Estimate the measurement errors and compare the Young's moduli obtained with the corresponding tabulated values.

Questions

1. What are the main sources of measurement errors? How can the errors be diminished?
2. Estimate the maximum accuracy of measurement of wire elongation and beam deflection which is reasonable in this experiment.
3. What is the difference between the state of normal stress and the state of normal deformation?
4. For which stress and strain does Hooke's law hold?
5. Which deviations from Hooke's law are possible in deformation of solids?

6. What is Poisson's ratio?
7. Which assumptions are made to obtain the relation between the maximum beam deflection and Young's modulus?
8. What function $y(x)$ describes the shape of the middle line of beam under perfect bending?
9. What is the use of platform M in Lermant's machine?

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Appendix

Figure 3a shows the beam under the load P applied in the middle between supports A and B. Each support exerts the force $P/2$ at points A and B. The beam is bent so that upper layers become compressed and lower ones stretched. It is reasonable to assume that the magnitude of stress in a layer is proportional to the distance between the layer and the middle line of the beam, as it is shown by the arrows in Fig. 3b for some beam element. Since the middle line of the beam is not stressed, the length dl_0 of the element middle line does not change under deformation (which is also true for the middle line of the beam). This stressed state of beam is called pure bending. We assume that stresses in layers are related to their deformations by Hooke's law:

$$\sigma = E\frac{dl - dl_0}{dl_0}. \tag{1}$$

The slope of middle line of the beam element (see Fig. 3c) changes from α to $\alpha - d\alpha$ along the distance dl_0. The corresponding arc length can be expressed via curvature radius R:

$$dl_0 = -R d\alpha. \tag{2}$$

Here the minus sign is taken because R is considered positive and the slope of middle line in the coordinates of Fig. 3a decreases along the beam (as it is shown in Fig. 3c). Let $y(x)$ be the equation of the middle line in the coordinates x, y (notice that the ordinate points downward), then the slope of the middle line is determined by the expression:

$$\frac{dy(x)}{dx} = \tan\alpha. \tag{3}$$

The length of the element middle line can be written as (see Fig. 3d):

$$dl_0 = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

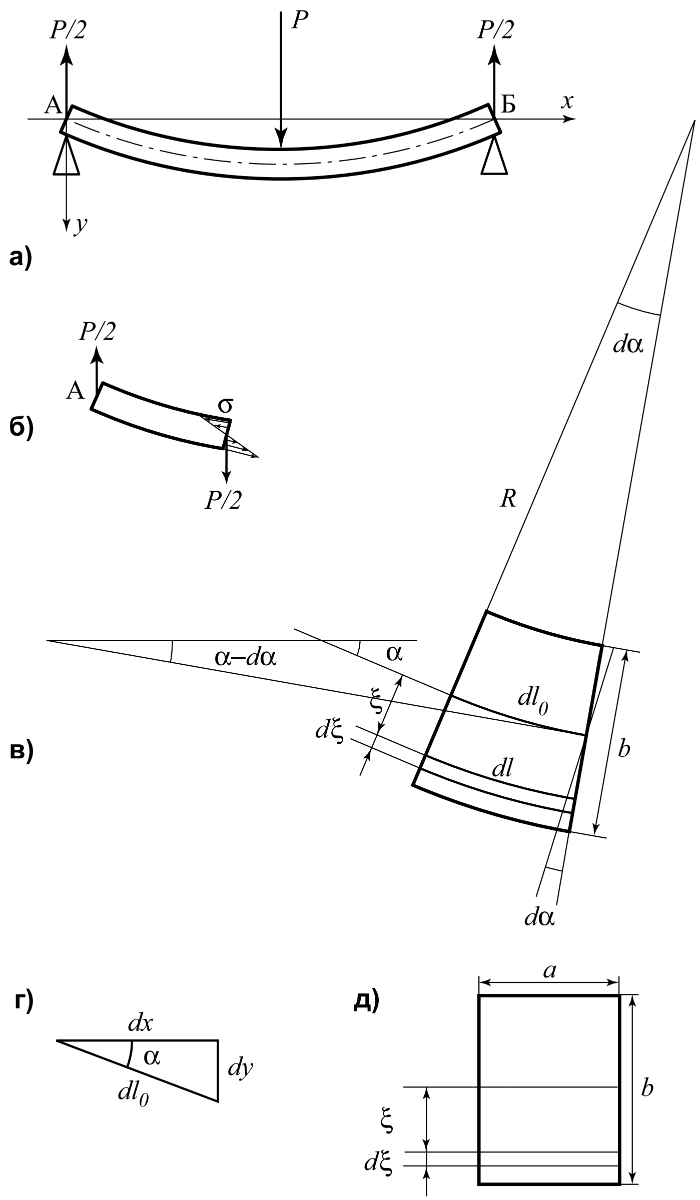

Fig. 3. Beam bending

From the same triangle it follows that

$$\frac{dx}{dl_0} = \cos\alpha. \tag{5}$$

Differentiating Eq. (3) with respect to x and using Eq. (2) one obtains:

$$\frac{d^2y}{dx^2} = \frac{1}{\cos^2\alpha}\frac{d\alpha}{dx} = \left(\frac{dl_0}{dx}\right)^2 \frac{d\alpha}{dl_0}\frac{dl_0}{dx} = -\left(\frac{dl_0}{dx}\right)^3 \frac{1}{R}. \tag{6}$$

Together with Eq. (4) this gives:

$$\frac{1}{R} = -\frac{y''}{(1+y'^2)^{3/2}}. \tag{7}$$

The stress in the layer located at the distance ξ from the middle line of the beam (see Fig. 3c) is given by Eq. (1) which can be rewritten as

$$\sigma = E\frac{dl - dl_0}{dl_0} = \frac{E}{R}\xi. \tag{8}$$

This formula makes use of the relation following from similarity of the triangles in Fig. 3c:

$$\frac{dl - dl_0}{\xi} = \frac{dl_0}{R}. \tag{9}$$

The net elastic force acting in a beam cross-section is zero, so the net torque due to the forces is independent of the point used to calculate the torque. Let us choose the point at the beam middle line. This gives:

$$M = \int\limits_{-b/2}^{b/2} \xi\sigma\, dS = \frac{E}{R}\int\limits_{-b/2}^{b/2} \xi^2\, dS = \frac{E}{R}I, \tag{10}$$

where $dS = a\,d\xi$, a is the width, and b is the height of the beam cross-section (see Fig. 3). I is called moment of inertia of the beam cross-section with respect to the axis passing through the beam middle line. It follows from Fig. 3b that the beam section from $x = 0$ to x is in equilibrium provided the forces applied at the point of support and at the cross-section are equal as well as the corresponding torques and the torque determined by Eq. (10). Torque equality gives:

$$\frac{EI}{R} = \frac{xP}{2}. \tag{11}$$

Now using Eq. (7) one can write the equation for the beam middle line:

$$y'' = -(1+y'^2)^{3/2}\frac{P}{2EI}x. \tag{12}$$

For small deflection

$$y'^2 \ll 1. \tag{13}$$

In this case it follows from Eq. (12) that

$$y'' = -\frac{P}{2EI}x. \tag{14}$$

Integrating this equation one gets:

$$y' = -\frac{P}{4EI}x^2 + C. \tag{15}$$

Here C is the constant determined by the condition that the beam is symmetrically bent, $y' = 0$ at $x = l/2$. Then Eq. (15) gives

$$y' = -\frac{P}{4EI}\left(x^2 - \frac{l^2}{4}\right). \tag{16}$$

Integrating one more time and taking into account that $y = 0$ at $x = 0$ one obtains the equation for the beam middle line:

$$y = \frac{Px}{48EI}(3l^2 - 4x^2). \tag{17}$$

The maximum deflection of the beam is determined by the value of y at $x = l/2$:

$$y_{\max} = \frac{Pl^3}{48EI}. \tag{18}$$

For beam of rectangular cross-section

$$I = \int\limits_{-b/2}^{b/2} \xi^2\, dS = a\int\limits_{-b/2}^{b/2} \xi^2\, d\xi = \frac{ab^3}{12}. \tag{19}$$

The value of Young's modulus follows from Eqs. (18) and (19):

$$E = \frac{Pl^3}{4ab^3 y_{\max}}. \tag{20}$$

Lab 1.3.2

Determination of torsional rigidity

Purpose of the lab: to measure the dependence of twist angle of an elastic rod on torque applied, to measure torsion and shear moduli of a rod using static method, and to measure the same moduli using torsional oscillations.

Tools and instruments: part 1: a rod, an eyeglass with a scale, a tape measure, a micrometer, and a set of weights; part 2: a wire made of the studied material, weights, a stopwatch, a micrometer, a tape measure, and a ruler.

The distribution of deformations and stresses in a twisted cylindrical rod of circular cross section is uniform along the rod only far from the points of force application. In these regions of uniform deformation one can consider every cross section as absolutely rigid, i.e. rod particles are not displaced from the radial lines on which they are located prior to the deformation; all radial lines in a given cross-section are thus turned by the same angle. This stressed state of the material is referred to as pure torsion. In what follows it will be shown that the tangential stresses in the cross section are directly proportional to the distance to the rotation axis.

Consider a part of length l of a twisted cylinder shown in Fig. 1a. A straight line drawn parallel to the axis of an unstrained cylinder becomes a helix after a twisting torque is applied. Cross sections separated by the distance l are rotated by the angle φ.

To derive equations describing torsion it is convenient to consider a part of cylinder: a ring of arbitrary radius r, infinitesimal thickness dr, and infinitesimal height dl, as shown in Fig. 1b. The top of the ring under torsion is rotated by the angle $d\varphi$ relative to the bottom while the generatrix of the ring cylindrical surface dl (an infinitesimal part of the helix mentioned above) is tilted by the angle $a\alpha$ from the vertical.

For small torsion angles α one can write down the relation

$$\alpha dl = r d\varphi. \tag{1}$$

One can readily see that α grows with the distance to the cylinder axis. An infinitesimal part of the deformed ring is shown in Fig. 1c. The tangential stress τ is directly proportional to the twist angle α, the proportionality constant is shear modulus G (see Eq. (3.2)):

$$\tau = G\alpha. \tag{2}$$

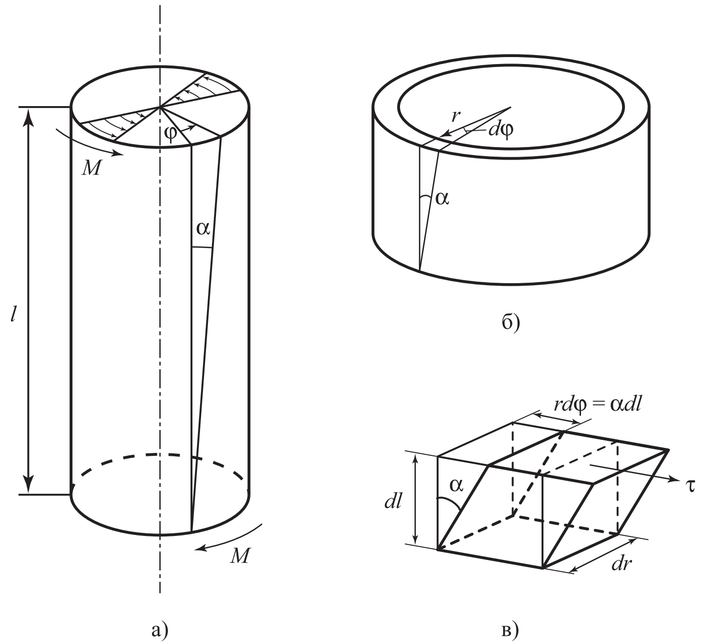

Fig. 1. Twisted cylinder

The tangential stress τ is directly proportional to α, hence it increases proportionally to the distance to the axis of the cylinder, as it was mentioned above. Using Eq. (1) one obtains

$$\tau = Gr\frac{d\varphi}{dl}. \tag{3}$$

These tangential stresses provide the torque about the cylinder axis:

$$dM = 2\pi r dr \cdot \tau \cdot r. \tag{4}$$

The total torque on the whole cross section can be obtained by integrating Eq. (4) over r from zero to the cylinder radius R:

$$M = 2\pi G\frac{d\varphi}{dl}\int_0^R r^3\,dr = \pi G\frac{d\varphi}{dl}\frac{R^4}{2}. \tag{5}$$

This torque is constant over the cylinder length. Torques acting on the face planes of any given part of cylinder are balanced, thus there is no rotation.

Then using Eq. (5) one readily obtains a linear relation between the relative twist φ of two cross sections and the distance l between them. Therefore we obtain the relation between the applied torque M, the relative twist angle φ of the cross sections, and the distance l between them:

$$M = \frac{\pi R^4 G}{2l}\varphi = f\varphi. \tag{6}$$

Here the torsion modulus f is introduced which is related to shear modulus G by the following equation:

$$f = \frac{\pi R^4 G}{2l}. \tag{7}$$

It is worth empathizing that Eq. (6) is valid only for the stresses much less than the shear modulus, i.e. at small angles α.

I. Static method of determination of torsion modulus of a rod

The experimental setup for the study of static torsion is shown in Fig. 2. The top end of the vertical rod R is rigidly attached to the bar while the bottom end is jointed to the disc D. The twisting moment is provided by two wires wound around the disc and passed over the blocks B; the wires are loaded by identical weights W. The mirror M mounted on the disc is used to measure the twist angle. To determine the angle one should adjust the eyeglass to observe a sharp reflection of the scale in the mirror M. The scale and the eyeglass are mounted on single support. Measurement of displacement of the scale image allows one to determine the twist angle of the rod.

LABORATORY ASSIGNMENT

1. By adjusting the eyeglass observe a clear image of the scale reflected by the mirror M. Measure the distance between the mirror and the scale and the diameters of rod R and disc D.
2. Gradually increase the load on the wires and obtain the dependence $\varphi = \varphi(M)$. Carry out the measurements by decreasing the torque. Repeat the measurements at least three times.
3. Plot the results in the $(\varphi,\ M)$ – coordinates. Using the plot obtained determine the torsion modulus f and estimate the error.
4. Using Eq. (7) calculate the shear modulus G and compare its value with the tabulated one.

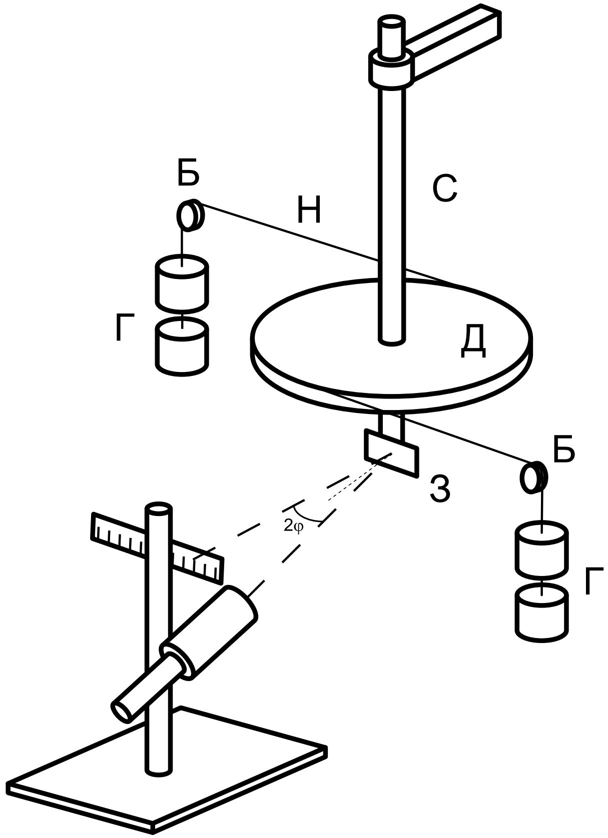

Fig. 2. Experimental setup

II. Dynamic measurement of the shear modulus (using torsional oscillations)

The experimental setup used in this part of the lab is shown in Fig. 3. The setup includes the vertical wire and the horizontal metal rod R attached to its lower end. Two identical movable weights W are symmetrically attached to the rod. The upper end of the wire is securely clamped by a collet; a special mechanism allows conjoint rotation of the wire end and the collet about the vertical axis, thus it is possible to excite torsional oscillations of the system. Rotation of the rod R and the weights W is due

to the elastic torque of the wire. The rotation is described by Eq. (2.35):

$$I\frac{d^2\varphi}{dt^2} = -M. \tag{8}$$

Here I is the moment of inertia of the rod and weights about the rotation axis, φ is the rotation angle measured from the equilibrium, and M is the torque which at small angles φ is well described by Eq. (6). Introducing the notation

$$\omega^2 = \frac{f}{I}, \tag{9}$$

one obtains from Eqs. (6) and (8):

$$\frac{d^2\varphi}{dt^2} + \omega^2\varphi = 0. \tag{10}$$

This is the equation of harmonic oscillations (4.4). Its solution is

$$\varphi = \varphi_0 \sin(\omega t + \theta), \tag{11}$$

where amplitude φ_0 and phase θ are determined by the initial conditions. The oscillation period T equals

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{f}}. \tag{12}$$

Equation (10) together with Eqs. (11) and (12) describe free oscillations. In order to apply them to a real process one should ascertain that the damping of oscillations is negligible. If the amplitude of oscillations decreases less than by half after 10 full swings one can use the equations for free oscillations. Also one should make sure that the oscillation period does not depend on the initial amplitude, otherwise the amplitude should be decreased until this dependence vanishes.

LABORATORY ASSIGNMENT

1. Estimate experimentally the working range of amplitudes in which the results derived for free oscillations are valid. For this purpose fix the weights on the rod symmetrically and excite torsional oscillations. Measure the time of several full swings (at least ten) and calculate the period T_1. Halve the initial amplitude and determine the corresponding period T_2. If $T_1 = T_2$ one can work with any amplitude not exceeding the first one. Otherwise decrease the initial amplitude and repeat the measurements until the equality is obtained.

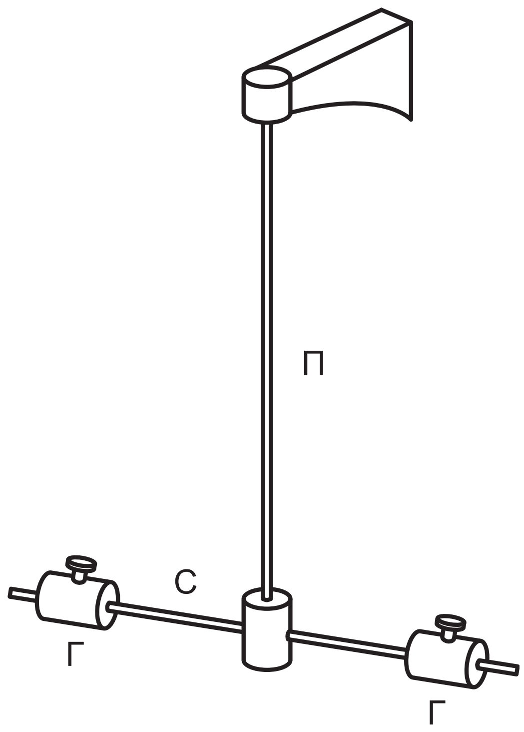

Fig. 3. Experimental setup

2. Make sure that after 10 full swings the amplitude is decreased less than by half.
3. Fix the weights on the rod at equal distances l from the rotation axis (wire) to the centers of inertia of the weights and measure the oscillation period T. Repeat the measurement for 4–6 different values of l. The torsion modulus can be obtained from the experimental data plotted in coordinates (l^2, T^2).
4. Measure the wire length and diameter. Using the obtained torsion modulus f calculate the shear modulus G (see Eq. (7)), estimate the error, and compare the result with the tabulated value.

Questions

1. How does friction in the axes of blocks B affect the results of static measurements? How can one minimize this influence?
2. How does the oscillation period change when damping is increased?
3. Which method of measurement of shear modulus is preferable in practice: the static or dynamic one?

4. How can one estimate the error of shear modulus from the plot in (l^2, T^2)-coordinates?

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Lab 1.3.3

Determination of air viscosity by measuring a rate of gas flow in thin pipes

Purpose of the lab: determine a domain of stationary flow, regimes of laminar and turbulent flows, air viscosity, and the Reynolds number.

Tools and instruments: metal pipes mounted on a horizontal support, gas flow meter, micrometer-type manometer, U-shaped glass pipe, stopwatch.

Consider a flow of viscous fluid in a circular pipe. At small velocities of the flow its motion is laminar (streamline), velocities of flow parcels are parallel to the pipe axis and their magnitude is a function of radius. Increasing of the velocity makes the flow turbulent, so layers of different velocities mix. In turbulent regime the velocity at any point of the fluid chaotically changes its magnitude and direction while the average velocity remains constant.

Particular regime of the fluid flow through a pipe is determined by a specific value of the dimensionless Reynolds number:

$$\mathtt{Re} = \frac{v r \rho}{\eta}, \tag{1}$$

where v is the flow velocity, r is the pipe radius, ρ is the fluid density, and η is its viscosity. In circular pipes with smooth walls transition from laminar to turbulent regime occurs at $\mathtt{Re} \approx 1000$.

In the laminar regime the volume of gas V flowing through a pipe of length l during a time period t is given by Poiseuille equation (3.23):

$$Q_V = \frac{\pi r^4}{8 l \eta}(P_1 - P_2). \tag{2}$$

In this equation $P_1 - P_2$ is the pressure difference between cross sections 1 and 2 of the pipe and l is the distance between the cross sections. The quantity Q is referred to as the volumetric flow rate. Equation (2) allows one to determine the gas viscosity once the flow rate is known.

Let us specify the conditions for Eq. (2) to be valid. First, the inequality $\mathtt{Re} < 1000$ should be satisfied. Second, the specific volume (or density) of the gas should be almost constant throughout the pipe (the specific volume is assumed to be constant in (2)). For a liquid flow this assumption is usually well satisfied; for a gas flow the pressure difference between the pipe ends must be small compared to the pressure itself. In the experimental setup the gas pressure equals the atmospheric pressure (10^3 cm of water) while the pressure difference does not exceed 10 cm of water, i.e. it is less than 1% of the atmospheric pressure. Third, Eq. (2) is valid for the pipe regions in which the radial distribution of gas velocities does not change along the pipe.

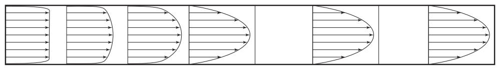

Fig. 1. Formation of gas flow in a circular pipe

When gas flows into a pipe from a bulk reservoir the velocities of gas layers are constant throughout the pipe cross section (Fig. 1). The velocity distribution pattern gradually changes along the pipe as the wall friction drags the adjacent layers. The parabolic velocity distribution typical for a laminar flow is formed at a certain distance a from the pipe entry point. This distance depends on the pipe radius r and the Reynolds number and can be estimated as

$$a \approx 0{,}2r \cdot \mathtt{Re}. \tag{3}$$

The pressure gradient in the flow formation domain is greater than that in the laminar flow domain. This fact allows one to distinguish these domains experimentally.

Laboratory setup. The measurements are performed by means of the experimental setup shown in Fig. 2. Pressurized air (an extra pressure exceeds the atmospheric one by 5-7 cm of water) flows through the gas meter GM into the reservoir A to which two metal pipes are soldered. The approximate dimensions of the pipes are given in the figure; the exact dimensions are marked on the setup. Both pipes are supplied with end caps blocking the air flow. During the measurements the end cap is removed only from the working pipe while the other pipe should be tightly sealed.

Previous to the gas meter a U-shaped pipe half-filled with water is set up. It is used for two purposes: first, it measures the pressure of the incoming gas; second, it preserves the gas meter from a possible breakdown. The gas meter operates normally providing the input pressure does not exceed 600 mm of water. The height of the U-shaped pipe is about 600

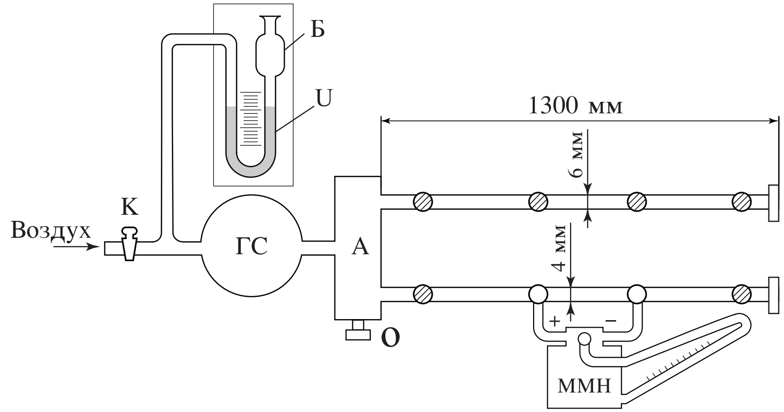

Fig. 2. Setup for measurement of air viscosity

mm, thus if the input pressure exceeds 600 mm the water spills out from the pipe into the tank T thereby attracting the experimenter's attention. Such situation can occur if gas is supplied to the system while the pipe ends are sealed.

There are several millimeter-wide openings in the pipe walls for measuring a pressure difference. To measure the difference, manometer inlets are connected to two adjacent openings while the other ones are sealed. Air supply is adjusted by the valve V.

In the lab *micrometer-type manometer*MTM (Fig. 3) is used; it allows one to measure the pressure difference up to 200 mm of water. To increase the manometer sensitivity its pipe is slanted. The marks 0.2, 0.3, 0.4, 0.6, and 0.8 on the stanchion 4 are the coefficients which must be multiplied by the manometer readings to obtain the pressure in millimeters of water (at a given slope). The working liquid is ethanol. The manometer zero is adjusted by shifting ethanol level in the vessel 1 using the instrumentality of cylinder 6. A driving depth of the cylinder is controlled by the screw 7.

The manometer is supplied with two inclinometers 9 placed on the plate 3 orthogonal to each other. Level adjustment is performed by two legs 10. The three-way cock 8 is mounted on the gauge top; it has two operating positions: «0» and «+» (see Fig. 3). Position «0» is used for adjusting the zero level of the meniscus. Position «+» is used for the pressure measurements. The rod 5 is used to switch between the positions (Fig. 3), this does not change a level of the working liquid in the reservoir.

The gas flow meter (shown in Fig. 4) is used for measuring small amounts of gas. Its casing is a cylinder with a mechanical counter and

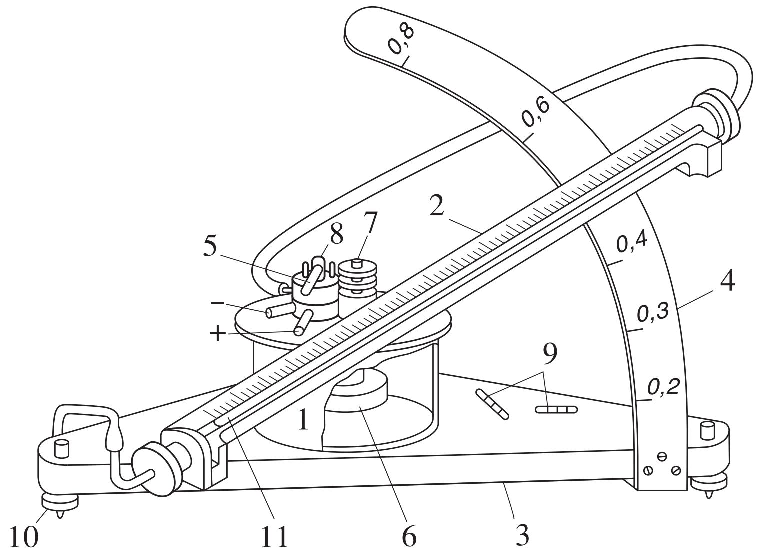

Fig. 3. Micrometer-type manometer MTM

a dial on its front face. One revolution of the pointer corresponds to 5 liters of gas passed through the meter.

The gas flow meter is filled with water up to a level determined by the gauge 1. The gas inlet and outlet pipes 2 and 3 are located on the rear and top sides of the meter, respectively. The U-shaped manometer is connected to the pipe sockets 4, the socket 5 is used for the thermometer. The valve 6 is used as a drain. The meter has an inclinometer and retractable legs for level adjustment.

The operating principle of the gas flow meter is illustrated in Fig. 5. Several light cups are attached to the shaft on the cylinder axis line (for simplicity only two cups are shown). Incoming air from the pipe 2 fills a cup located above the pipe. The air-filled cup rises to the surface while the next cup takes its place and so on. Shaft rotation is transmitted to the counter.

LABORATORY ASSIGNMENT

1. Check the setup and make necessary level adjustments, check water level in the gas flow meter and adjust the zero of the manometer meniscus. Choose one of the pipes for the complete set of measurements (the pipe of $d = 4$ mm is preferable).

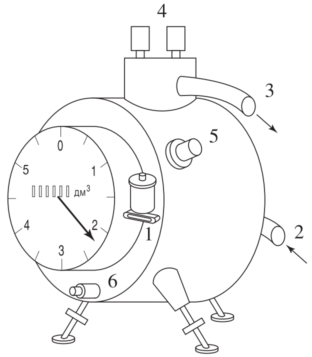

Fig. 4. Schematic view of the gas flow meter

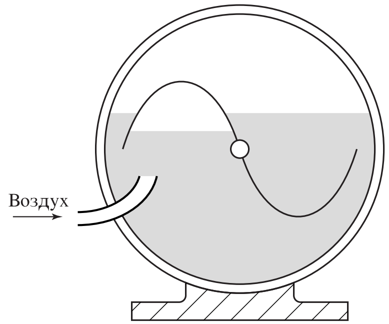

Fig. 5. Interior of the gas flow meter

2. Using Eq. (3) estimate the length of the region of flow formation. Take $\mathtt{Re} = 1000$.
3. Connect the manometer inlets to a pair of adjacent openings in the selected pipe (in the region of the formed flow). Uncap the pipe outlet; all the other outlets should be sealed.
4. Gradually open the valve V (Fig. 2) feeding the setup with air. Carefully track the manometer readings since at a high pressure difference the ethanol can spill out from the manometer through the pipe 11.

   This undesirable situation often occurs when working with thin pipes. In this case ethanol not only floods an elastic pipe which connects the manometer pipe 11 with the three-way cock, but it can also leak into the pipe connected with (–). Drops of liquid in the pipe result in incorrect measurements of $\Delta P = P_1 - P_2$. For this reason before the measurements (or if the ethanol has flooded the pipes) one should ascertain that there are no drops of liquid in the connecting pipes. The drops can be detected by observing sudden leaps of manometer readings when slowly moving the connecting pipes. If this is the case the pipes should be removed and dried out.
5. Determine the air viscosity. For this purpose measure the dependence of the pressure difference ΔP on the air flow rate $Q = \Delta V/\Delta t$. The gas volume ΔV is measured with the gas flow meter and Δt - with the stopwatch. Set the slope coefficient on the manometer stanchion equal to 0.2. Start the measurements from small pressure differences (2–3 mm of water) and gradually increase the gas flow rate Q.

   Within the range from 0 to 100 of the manometer dial (Fig. 3) one should perform not less than 5 measurements to survey the laminar regime.

Subsequent measurements could be sparse but they should cover a wider pressure range to examine the turbulence regime. Using the data obtained plot the dependence $\Delta P = f(Q)$ which should be linear in the laminar regime (see Eq. 2). The dependence becomes non-linear for a turbulent flow since the pressure difference grows faster than the gas flow rate.

6. Calculate the slope of the curve $\Delta P = f(Q)$ in the linear domain and determine the air viscosity η. Estimate the error of the slope and find the error of the obtained value of the viscosity.
7. Calculate the Reynolds number Re corresponding to transition between laminar and turbulent regimes.
8. Measure the pressure distribution along the pipe in the laminar regime. Connect the manometer to all pipe openings one by one (including the opening «0», see Fig. 2). Plot the pressure vs. the distance from the pipe inlet ($P = f(l)$). Using the plot estimate the length of the flow formation region. Compare the result with Eq. (3).
9. Measure the dependence $Q = f(P)$ for all pipes in the formed flow region (at the end of a pipe) in the laminar regime (Re < 500). Using the data calculate the following quantity:

$$\frac{8l\eta Q}{\pi(P_1 - P_2)} = r^n.$$

Plot the obtained function on a log-log graph, i.e. plot the values of $\ln\big(8l\eta Q/\pi(P_1 - P_2)\big)$ on the Y-axis and $\ln r$ on the X-axis. Obviously the curve slope equals n and for the Poiseuille equation $n = 4$. Verify it. Estimate the error of the result.

Questions

1. Write the equation which describes the radial distribution of laminar flow velocity in a circular pipe. What is the ratio of the average and maximum velocities?
2. How is the Reynolds number defined? How can it be determined experimentally?
3. Describe the method of graphical treatment of the experimental data (see 8) that allows one to distinguish the regions of formed and non-formed flow clearly.

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Lab 1.3.4

Study of stationary flow of liquid through pipe

Purpose of the lab: to measure liquid flow velocity using Venturi and Pitot methods and to compare the results with those obtained by direct measurement of volumetric flow rate.

Tools and instruments: a setup that includes venturi and pitot tubes and a stopwatch.

A flow of liquid through a pipe of constant cross-section is studied in the lab.

The main purpose of experimental study of fluid flow through a pipe is the measurement of flow velocity and volumetric (mass) flow rate. Accurate measurement of the flow rate is important in practical applications: operation of oil and gas pipelines, plumbing, and central heating.

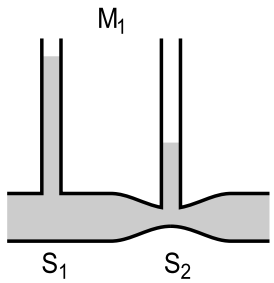

Fig. 1. Venturi tube

A lot of different methods have been developed to measure fluid flow rate and flow velocity. The most simple and accurate ones rely on measurement of the pressure difference due to detector positioning (toward or along the flow in the pitot tube) or due to an obstacle impeding the flow (the narrowing of Venturi tube or a washer).

A venturi tube (see Fig. 1) is a horizontal tube which cross-sectional area gradually changes along the tube. Wide (S_1) and narrow (S_2) sections are connected to water manometer M_1. The pressure in a section is determined by the height of the corresponding water column.

Since water is incompressible ($v_1S_1 = v_2S_2$) and the tube is horizontal ($z_1 = z_2$) Bernoulli' equation (3.14) allows one to express the flow velocity in section S_1 in terms of the pressure in sections S_1 and S_2:

$$v_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left[(S_1/S_2)^2 - 1\right]}}. \qquad (1)$$

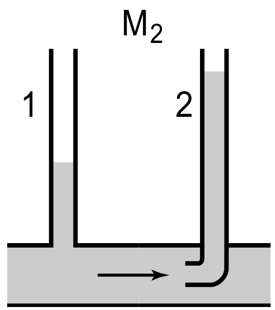

Fig. 2. Pitot tube

A pitot tube is shown in Fig. 2. Tube T is connected to two tubes of water manometer M_2. Tube 1 is connected to the surface of the tube T while the tip of the tube 2 is bent toward the flow. Obviously the liquid is at rest, $v_2 = 0$, at the opening of tube 2.

Let the pressures measured by means of the tubes 1 and 2 be p_1 and p_2, respectively. Bernoulli' equation (3.14) gives $p_1 + \rho v_1^2/2 = p_2$, so

$$v_1 = \sqrt{2(p_2 - p_1)/\rho}. \tag{2}$$

Equation (2) relates flow velocity to the difference in liquid heights in the tubes 1 and 2.

The pitot tube allows one to measure the local flow velocity at the tube location. Using the venturi tube one can determine only the velocity averaged over tube cross-section. Therefore the venturi tube is predominantly used for flow rate measurements. The pitot tube is used to measure flow velocity; more often it is an open flow rather than a flow in pipe. The pitot tube is used for velocities ranging from those of viscous boundary layers to supersonic velocities.

Operation of pipelines requires constant monitoring of volumetric or mass fluid flow rate. The measurements are complicated by viscosity which results in the fluid «sticking» to pipe wall, so fluid velocity next to the wall vanishes. Therefore the velocity always increases along the pipe radius from the wall to the center. For stationary flow and a low Reynolds number one can apply the Poiseulle equation, so it would suffice to measure the flow velocity at any point, e.g. at the pipe axis. Otherwise an accurate measurement of the flow requires integrating the flow velocity over a pipe cross-section, so the velocity must be measured at several points. In the monograph «Hydrodynamics» by T. Ye. Faber it is recommended to use 20 pitot tubes located at different distances from the pipe axis in two perpendicular directions.

One of physical methods of measurement of fluid flow rate is realized in an ultrasonic flow meter. The method is based on the observation that speed of sound propagating in a fluid is constant with respect to the fluid, so the speed of sound is greater in the direction of fluid flow and it is less if the sound propagates against the flow. An ultrasound emitter and receiver are mounted on the opposite walls of the pipe although not facing each other, so the sound propagates at some angle with respect to the flow. Therefore the speed of sound in the direction of the fluid flow exceeds that in the fluid at rest and vice versa. A difference between the speeds allows one to determine the flow velocity even if the speed of sound itself is not known. Operation of ultrasonic flow meter is not affected by fluid viscosity, which is an advantage. However the meter measures some average velocity on the path of the sound, so for precise measurements the device has to

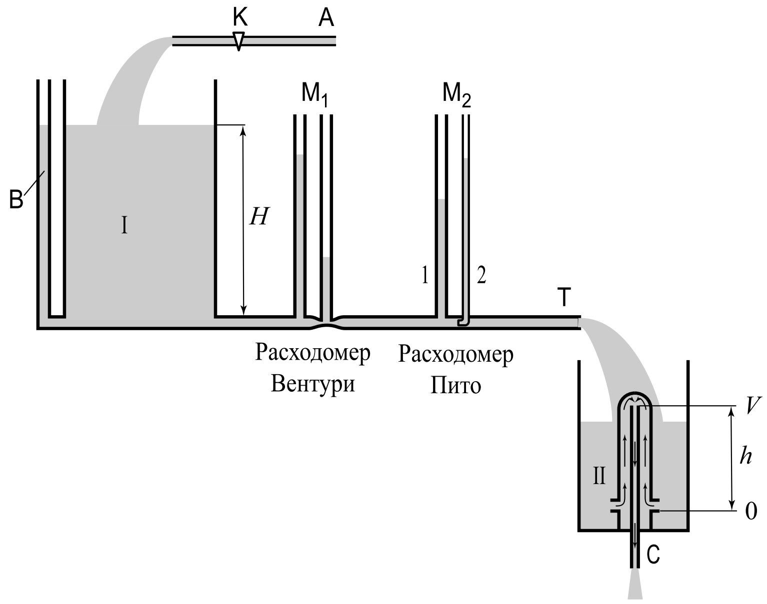

Fig. 3. Experimental installation for studying stationary flow of liquid through pipe

be calibrated. The calibration depends on Reynolds number because it determines a velocity profile of the flow.

There is also a turbine flow meter in which flow rate is directly proportional to the number of revolutions of a turbine. However the meter readings depend on fluid viscosity.

Laboratory setup. An experimental installation for studying liquid flow is shown in Fig. 3. Water enters tube T from cylindric vessel I equipped with glass tube B serving as a water meter. The vessel is filled with tap water via tube A, the influx is controlled by tap K. Water flowing out of tube T fills receiver vessel II which has siphon C mounted on the bottom.

The siphon preserves the receiver from overflowing by emptying it as soon as water level reaches the height h. Tube T is equipped by venturi and pitot meters.

The flow rate averaged over the tube cross-section can be determined by measuring the time required to fill the receiver II which volume is known. On the other hand the rate can be found using the readings of the manometers with the aid of Eqs. (1) and (2). Comparison of the rates found by different methods allows one to check whether Bernouilli' equation can be applied and to assess a role of viscosity that changes velocity profile across the flow. It is convenient to compare the flow rates by plotting the rate of filling the receiver on abscissa and the venturi and pitot rates on ordinate. For ideal liquid the plot would be a straight line at 45° to the abscissa.

Viscosity can be estimated by observing the water levels in the reservoir and two manometer tubes. For ideal liquid the levels would be the same. Due to viscosity the levels decrease along the flow.

Up to this point we assumed that the liquid is ideal, so there is no friction due to viscosity in tube T and no associated losses. The following experiment allows one to estimate viscosity quantitatively. Fill the reservoir I to some level z_1, measure the flow velocity in tube T using receiver II (since water is essentially incompressible it enters and leaves the tube at the same speed). Using Torricelli's law evaluate the height z_2 that results in the same speed for an ideal liquid. The difference $z_1 - z_2$ is a measure of internal losses due to viscosity. Moreover it is safe to assume that the losses occur mostly in tube T since velocity of water in reservoir I is much less.

Viscosity changes the readings of the venturi manometer by a quantity Δh which can be estimated as the product of the difference $z_1 - z_2$ and the ratio of the distance Δl between the manometer entries to the tube length L. If

$$\Delta h \gg (z_1 - z_2)\frac{\Delta l}{L},$$

water can be considered as ideal liquid at the scale of Δl. If Δh is comparable to $(z_1 - z_2)\frac{\Delta l}{L}$ one should subtract the quantity $\Delta z \frac{\Delta l}{L}\rho g$ from $p_1 - p_2$. The same applies to the pitot tube. In addition, one should estimate the correction to manometer readings due to a finite size of the bent section of tube 2 inserted in the flow.

It is important to ensure that the flow remains stationary during the experiment. This is achieved by maintaining water level in the reservoir I at the same height H by adjusting tap K. The glass tube used as a meter has millimeter graduations for convenience. Before the experiment one should make sure that the manometer tubes are not clogged.

LABORATORY ASSIGNMENT

1. Pour some water in reservoir 1. Plug tube T and make sure that water levels in manometer tubes and in the reservoir are the same. Make necessary adjustments if this not so.
2. Measure the flow rate for several water levels H in reservoir I starting from ~1 cm. A flow must be stationary, so a water level should be maintained constant during the measurement. The rate is determined by the time t required to fill reservoir II. Estimate the error of t. For every H record the readings of the venturi and pitot manometers.
3. Calculate average flow velocity $v_\text{p} = V_0/(tS_1)$, where V_0 is the volume of reservoir II, t is the time required to fill the reservoir, and S_1 is the cross-sectional area of tube T. Estimate the error of v_p.
4. Measure the length L of tube T and Δl of the venturi and pitot manometers.
5. Plot the quantity v_p^2 versus water level H. Plot the errors as cross-bars. Plot also the height calculated according to Torricelli' equation, $z_2 = = v_p^2/(2g)$, on the same graph. Do the points coincide? What is the reason of the discrepancy?
6. Using Eqs. (1) and (2) and the readings of venturi and pitot manometers calculate velocities v_V and v_P (taking the losses into account and without them). Estimate the errors of the velocities. Compare the velocities with v_p and plot them versus v_p. How do the errors of S_1 and S_2 in Eq. (1) and the narrowing of the tube cross-section where the pitot tube 2 is inserted affect the dependence obtained?
7. Plot v_p versus H. Determine graphically the regions of laminar and turbulent flow. Determine the Reynolds number at the point of transition from the laminar to turbulent regime:

$$\mathtt{Re} = \frac{v_\text{p} r \rho}{\eta},$$

where ρ is the water density, r is the radius of tube T,$\eta = 1{\cdot}10^{-3}\ kg/m{\cdot}s$ -is the water viscosity.

Questions

1. Specify the assumptions used to derive Bernouilli's equation.
2. How does viscosity affect the readings of venturi and pitot flow meters?
3. Which water levels H in reservoir 1 correspond to laminar or turbulent flow in tube T?
4. Suppose there is a laminar fluid flow through a tube and the viscosity decreases gradually while other flow parameters remain constant. How does the flow change?
5. Which flow regime, laminar or turbulent, provides a better agreement between the values of flow velocity determined by venturi and pitot tubes and that one obtained by using reservoir II?
6. Derive Torricelli's equation and use it to estimate the velocity of liquid flowing out a very short pipe for different levels H. Why are the experimental values of the velocities of water flowing out a long pipe sufficiently less?

7. Estimate the difference of water levels Δh in the left tubes of the manometers (see Fig. 3) attached to tube T where the cross-sectional areas are the same. How can the pressure difference be explained? Can the pressure difference between the inlet and outlet of tube T be found by linearly extrapolating the pressure difference between the tubes?

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Chapter IV

MECHANICAL OSCILLATIONS AND WAVES

Free harmonic oscillations. Mechanical motion and the processes which can be regarded as periodical are usually called oscillations. Such processes can be related to different phenomena of nature, economics, or society. Oscillation takes place provided there is a process that returns perturbed system to equilibrium (restoring force). This feature makes it possible to give universal mathematical description of oscillations. Some examples of restoring force in mechanics include elastic force of spring, gravity force, elastic force of twisted rod or wire, etc.

A simple example of oscillation is the motion of a weight suspended on elastic spring. But we start with even a simpler system. Let us put a weight and a spring on a horizontal smooth (frictionless) surface. One end of the spring is fixed while the weight of mass m is attached to the other end. Let the length of the undeformed spring be l_0. The weight starts moving along the spring axis (let it be x-axis) if it is displaced from the equilibrium or it receives some initial velocity along the axis. Now let us assume that the reaction force F of the spring is proportional to its elongation $l-l_0$ which is equal to the displacement $x = l-l_0$ of the weight from the point of equilibrium:

$$F = -kx. \tag{4.1}$$

The minus sign indicates that force is opposite to displacement. The constant k is the so called spring elastic constant. It should be noted that for large deformations spring rigidity depends on the deformation magnitude. This results in non-linearity discussed later in this chapter.

Equation of motion of mass m follows from Newton's second law of motion:

$$m\ddot{x} = -kx. \tag{4.2}$$

Hereinafter the dots over variables stand for time derivative.

Let us introduce the notation

$$\omega_0^2 = \frac{k}{m}. \tag{4.3}$$

Then Eq. (4.2) becomes

$$\ddot{x} + \omega_0^2 x = 0. \tag{4.4}$$

This is an ordinary differential equation of the second order. The general solution of Eq. (4.4) depends on two constants determined by two conditions. In particular one can impose initial (i.e. at $t = 0$) conditions. For instance, at $t = 0$: $x = x_0$ and $\dot{x} = 0$ or $x = 0$ and $\dot{x} = v_0$.

To integrate Eq. (4.2) let us multiply it by $\dot{x}$. Since $\ddot{x} = d\dot{x}/dt$ and $\dot{x} = dx/dt$, this gives

$$m\dot{x}\frac{d\dot{x}}{dt} + kx\frac{dx}{dt} = \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \frac{kx^2}{2}\right) = 0. \tag{4.5}$$

Then

$$\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = E. \tag{4.6}$$

Here the first term is kinetic energy of mass m and the second term is elastic energy of the deformed spring. Constant of integration E is the total mechanical energy of the weight and the spring. Equation (4.6) shows that E is a positive quantity which can be found from initial conditions. If the initial velocity vanishes,

$$E = \frac{kx_0^2}{2}. \tag{4.7}$$

If the initial displacement vanishes,

$$E = \frac{mv_0^2}{2}. \tag{4.8}$$

Thus the first integral (4.6) of Eq. (4.2) is the law of conservation of mechanical energy. For further integration let us write Eq. (4.6) as

$$\dot{x} = \pm\sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2}. \tag{4.9}$$

Let us introduce the notation

$$x\sqrt{\frac{k}{2E}} = \sin y. \tag{4.10}$$

Using Eqs. (4.9), (4.10), and (4.3) one obtains

$$\dot{y} = \pm\sqrt{\frac{k}{m}} = \pm\omega_0.$$

Integration of this equation gives

$$x_1 = \sqrt{\frac{2E}{k}}\sin(\omega_0 t + \alpha),$$

$$x_2 = -\sqrt{\frac{2E}{k}}\sin(\omega_0 t + \beta) = \sqrt{\frac{2E}{k}}\sin(\omega_0 t + \pi + \beta).$$

Both solutions can be written in the same form:

$$x = \sqrt{\frac{2E}{k}}\sin(\omega_0 t + \varphi_0), \tag{4.11}$$

where φ_0 is the constant determined from initial conditions. It is often convenient to write Eq. (4.11) as

$$x = \sqrt{\frac{2E}{k}}\cos(\omega_0 t + \varphi_0). \tag{4.12}$$

The argument of the sine, $\omega_0 t + \varphi_0$, is called oscillation phase and the constant φ_0 is called initial phase of oscillations. The value of sine is the same for two phases which differ by a multiple of 2π, so Eq. (4.12) describes a periodic process. The period T is determined by the relation

$$2\pi = \omega_0(t + T) + \varphi_0 - (\omega_0 t + \varphi_0) = \omega_0 T.$$

The quantity ω_0 introduced in Eq. (4.3) is called cyclic frequency of oscillations. It is related to the number of oscillations per second (temporal frequency or frequency for short) and to period T as

$$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}. \tag{4.13}$$

Equation (4.6) shows that velocity $\dot{x}$ decreases when displacement x grows. A halt ($\dot{x} = 0$) occurs at the maximum displacement $x = a$ which is called amplitude of oscillations:

$$\frac{ka^2}{2} = E. \tag{4.14}$$

The amplitude a is positive by definition. Substitution of Eq. (4.14) to (4.12) gives

$$x = a\sin(\omega_0 t + \varphi_0). \tag{4.15}$$

Therefore the velocity is

$$\dot{x} = a\omega_0 \cos(\omega_0 t + \varphi_0). \tag{4.16}$$

Obviously the maximum displacement in the positive direction of x lags behind the maximum velocity in the same direction by a phase of $\pi/2$ (or $90°$).

In general, when both x_0 and v_0 are non-zero at $t = 0$ we have

$$a = \sqrt{x_0^2 + v_0^2/\omega_0^2}, \qquad \varphi_0 = \arctan\left(\frac{\omega_0 x_0}{v_0}\right). \tag{4.17}$$

Oscillations described by Eq. (4.15) are called harmonic (or sinusoidal), since sine and cosine are harmonic functions. Harmonic oscillations are isochronous, i.e. their period does not depend on amplitude. A system which executes harmonic oscillations described by Eq. (4.4) is called harmonic oscillator. Notice that circular motion at constant speed can be considered as the sum of two harmonic perpendicular oscillations which have the same amplitude and the phases differing by $\pi/2$. The cyclic frequency in this case coincides with angular velocity of the circular motion, therefore the name. In general, addition of two perpendicular oscillations with different amplitudes and phases results in a complicated trajectory called Lissajous curve.

Equation (4.15) can be written as

$$x = A\sin\omega_0 t + B\cos\omega_0 t. \tag{4.18}$$

This relation depends on two constants of integration determined from initial conditions as in Eq. (4.15).

Using Eqs. (4.15) and (4.16) one can obtain the following expressions for kinetic and potential (elastic) energy of the oscillator:

$$K = \frac{m\dot{x}^2}{2} = \frac{m\omega_0^2 a^2}{2}\cos^2(\omega_0 t + \varphi_0) = \frac{m\omega_0^2 a^2}{4}[1 + \cos(2\omega_0 t + 2\varphi_0)],$$

$$U = \frac{kx^2}{2} = \frac{m\omega_0^2 a^2}{2}\sin^2(\omega_0 t + \varphi_0) = \frac{m\omega_0^2 a^2}{4}[1 - \cos(2\omega_0 t + 2\varphi_0)].$$

Notice that

$$K + U = E = \frac{m\omega_0^2 a^2}{2}.$$

The values of K and U averaged over the period are

$$\bar{K} = \frac{1}{T}\int_0^T K(t)\,dt, \qquad \bar{U} = \frac{1}{T}\int_0^T U(t)\,dt.$$

Integration gives

$$\bar{K} = \bar{U} = \frac{m\omega_0^2 a^2}{4} = \frac{E}{2}. \tag{4.19}$$

Therefore the average kinetic and potential energies of the oscillator are equal.

Now consider oscillations of the weight of mass m suspended on a spring with elastic coefficient k in gravitational field with free-fall acceleration g. In this case instead of Eq. (4.2) one gets

$$m\ddot{x} = -kx + mg. \tag{4.20}$$

Here the x-axis is directed downwards along the gravity force.

Let x_0 be the spring elongation in equilibrium, then

$$mg = kx_0. \tag{4.21}$$

Using Eqs. (4.20) and (4.21) one obtains for deviation $\xi = x - x_0$ from the equilibrium:

$$m\ddot{\xi} = -k\xi. \tag{4.22}$$

This is the equation of harmonic oscillator (4.4).

Phase portrait of harmonic oscillator. There is a remarkable representation of harmonic oscillations in the so-called phase plane. Coordinate axes on the plane are coordinate x and a quantity proportional to its time derivative, e.g. momentum $m\dot{x}$. A point on the phase plane specifies the state of a mechanical system with one degree of freedom at a given time. Now consider the phase plane of a harmonic oscillator which executes the motion

$$x = a\cos(\omega_0 t + \varphi).$$

Let the abscissa represent the coordinate x and the ordinate represent the quantity $y = \dot{x}/\omega_0$. This choice is convenient since both x and y have the same dimension. Obviously

$$y = -a\sin(\omega_0 t + \varphi).$$

One can see that

$$x^2 + y^2 = a^2. \tag{4.23}$$

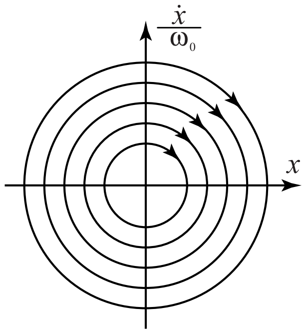

Fig. 4.1. Phase portrait of harmonic oscillator

This is the equation of the circle of radius a. A point (x, y) on the plane represents the state of oscillator at a given time. Let us refer to this point as representing point. There is one-to-one correspondence between the motion of oscillator and the motion of representing point along the phase trajectory which is circular in our case. Oscillations of different amplitudes are represented by a family of circles centered at the origin. Figure 4.1 shows the portrait of harmonic oscillator.

Oscillations of the same amplitude but of different initial phases are represented by the same circle, however simultaneous positions of the representing points on the circle are different. The phase difference is equal to the angle between radius vectors of the points. It is easy to verify that representing points run clockwise. A full revolution is completed for oscillation period $T = 2\pi/\omega_0$.

Free motion of damped harmonic oscillator. Consider oscillations which in addition to restoring force are also subjected to a force impeding the motion, i.e. the force directed opposite to velocity. Such a force arises when the oscillation proceeds in a medium that resists motion. At a small velocity the force is directly proportional to it:

$$F_c = -b\dot{x}. \tag{4.24}$$

In this case instead of (4.2) one obtains:

$$m\ddot{x} = -kx - b\dot{x}. \tag{4.25}$$

Let us introduce the notation

$$\frac{b}{m} = 2\beta. \tag{4.26}$$

Then Eq. (4.4) can be rewritten as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{4.27}$$

Let us check that the solution of this equation has the form

$$x = a_0 e^{-\beta t}\sin(\omega t + \varphi_0). \tag{4.28}$$

Indeed, substitution of this ansatz to Eq. (4.27) shows that the equation holds provided

$$\omega^2 = \omega_0^2 - \beta^2. \tag{4.29}$$

Therefore increase in the viscous damping coefficient decreases the oscillation frequency, so the period which is inversely proportional to the frequency grows. Strictly speaking, this motion is not periodic. Nevertheless the period of damped oscillations can be defined as the time interval between two consecutive passages in the same direction through the equilibrium:

$$T = \frac{2\pi}{\omega}.$$

For small damping ($\beta \ll \omega_0$) it is reasonable to assume that the maximum deviation occurs whenever the sine in Eq. (4.28) equals unity:

$$a = a_0 e^{-\beta t}. \tag{4.30}$$

The ratio of two consecutive maxima of deviation in the same direction is called decrement:

$$D = \frac{a_i}{a_{i+1}} = e^{\beta T}. \tag{4.31}$$

The natural logarithm of this ratio δ is called damping ratio:

$$\delta = \beta T. \tag{4.32}$$

For some systems oscillation amplitude increases and the ratio is negative, then it is called increment. For small positive δ the amplitude decreases slowly and damping is small. It follows from Eq. (4.29) that for $\beta \ll \omega_0$ the oscillation frequency is close to ω_0.

Let us determine the rate of energy dissipation of the oscillator for small damping. It follows from Eq. (4.19) that the energy depends on the amplitude as

$$E = \frac{1}{2}m\omega_0^2 a^2. \tag{4.33}$$

Substituting Eq. (4.30) in Eq. (4.33), taking logarithm, and differentiating one obtains the relative change in the energy averaged over the period:

$$\frac{dE}{E} = -2\beta dt. \tag{4.34}$$

Therefore the energy decrement ΔE during the period T is:

$$\frac{\Delta E}{E} = 2\beta T = 2\delta. \tag{4.35}$$

The important parameter of damped oscillations is Q-factor which is defined as the ratio of oscillation energy to its losses per period multiplied by 2π. Several useful expressions of Q-factor in terms of oscillation parameters at small damping are given below:

$$Q = 2\pi\frac{E}{\Delta E} = \frac{\pi}{\delta} = \frac{\pi}{\beta T} = \frac{\omega_0}{2\beta} = \frac{m\omega_0}{b} = \frac{\sqrt{km}}{b} = \frac{k}{b\omega_0} = \pi n. \qquad (4.36)$$

Here n is the number of oscillation cycles executed before the amplitude decreases by a factor of e ($e = 2.71828\ldots$).

Phase portrait of damped oscillations is a spiral approaching the origin as it revolves around. The motion becomes aperiodic for strong damping, $\beta \geqslant \omega_0$. When $\beta = \omega_0$ the damping is called critical.

Compound pendulum. Any rigid body that executes oscillations around a pivot or a rotation axis due to restoring force is called compound pendulum. Consider, for example, a case when the restoring force is due to gravity. The center of mass of the pendulum is below the pivot on the same vertical. During oscillations the line connecting the pivot and the center of mass deflects from the vertical. Let the instantaneous value of the deflection angle be φ. Then according to Eq. (2.35) the equation of motion for this angle is

$$I\ddot{\varphi} = -mga\sin\varphi. \qquad (4.37)$$

Here I is the moment of inertia around the pivot (rotation axis), a is the distance from the rotation axis to the center of mass.

If the deflection angle remains small, so that $\sin\varphi \approx \varphi$, the equation of harmonic oscillator follows which gives the period of compound pendulum as

$$T = 2\pi\sqrt{\frac{I}{mga}}. \qquad (4.38)$$

If the size of the body suspended on a thread or a weightless rod of length l is much less than the length, the body is called point particle and the pendulum is called simple gravity pendulum. In this case $I = ml^2$ and $a = l$ and the expression for the period of simple gravity pendulum is reproduced:

$$T = 2\pi\sqrt{\frac{l}{g}}. \qquad (4.39)$$

If the period of simple gravity pendulum coincides with the period of compound pendulum, l is called equivalent length l_{eq}:

$$l_{eq} = \frac{I}{ma}. \qquad (4.40)$$

Center of oscillation of a compound pendulum (Fig. 4.2) is the point O' located at the distance l_{eq} from the pivot O on the vertical passing through the pivot and the center of mass. If the pendulum mass is concentrated at the center of oscillation the compound pendulum becomes simple gravity pendulum with the same period. Let the moment of inertia of compound pendulum around the center of mass be I_0. Then according to Huygens-Steiner theorem (2.31) the moment of inertia around the pivot is

$$I = I_0 + ma^2. \qquad (4.41)$$

Substitution of Eq. (4.41) to (4.40) gives

$$l_{eq} = a + \frac{I_0}{ma}. \qquad (4.42)$$

Obviously the center of oscillation is farther away from the pivot than the center of mass. It also follows from the above equations that the equivalent length l'_{eq} of the pendulum suspended at the center of oscillation coincides with l_{eq}. To prove this statement notice that the distance from the center of oscillation, which is now the pivot, to the center of mass is

$$a' = l_{eq} - a = \frac{I_0}{ma}. \qquad (4.43)$$

Then

$$l'_{eq} = \frac{I_0}{ma'} + a' = a + l_{eq} - a = l_{eq}. \qquad (4.44)$$

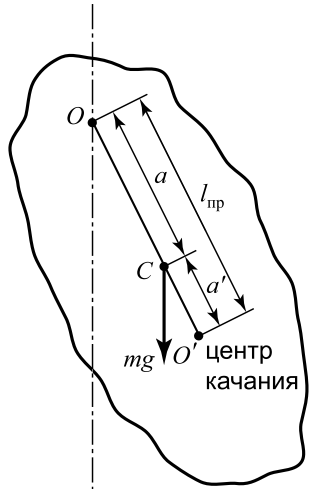

Fig. 4.2. Compound pendulum

Since the equivalent lengths are the same, the period of compound pendulum does not change if the pendulum is suspended at the center of oscillations.

When deflection angle is large oscillations of simple gravity pendulum become non-linear, i.e. the oscillation period exhibits dependence on amplitude (the maximum deflection angle). Equation (4.37) is integrated in the introduction to the lab 1.4.3. For small amplitudes it reads:

$$T \approx T_0\left(1 + \frac{\varphi_m^2}{16}\right). \qquad (4.45)$$

Here T_0 is the period at zero amplitude given by Eq. (4.38) and φ_m is the maximum deflection angle.

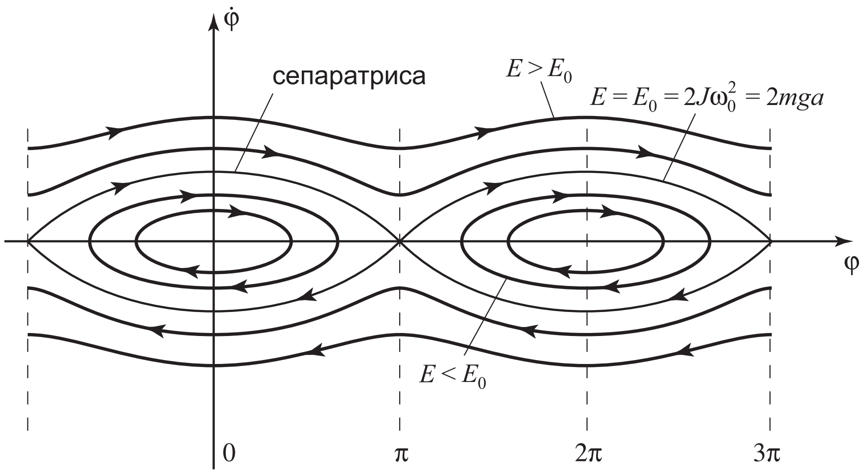

Fig. 4.3. Phase portrait of pendulum

Equation (4.37) represents the law of conservation of mechanical energy (the first integral of motion) for non-linear oscillations:

$$\frac{\dot{\varphi}^2}{2} - \omega_0^2 \cos\varphi = \frac{E_0}{I} - \omega_0^2.$$

Here $\omega_0^2 = mga/I$ is the oscillation frequency for small amplitudes when nonlinearity can be neglected and E_0 is total energy (the potential energy is zero at the equilibrium). The phase portrait of the pendulum is shown in Fig. 4.3. The elliptic trajectories at small angles become the circles of Fig. 4.1. The trajectories cease to be ellipses when the energy (or amplitude) gets large because oscillation becomes rotation. The trajectory that separates finite (bounded) motion of pendulum from rotation is called separatrix. A trajectory corresponding to infinite motion is called a runaway trajectory.

Driven oscillator with viscous damping. Stationary oscillations of a system subjected to external periodic force are called driven. We consider the most important case of a force which time dependence is described by harmonic function, $F = F_0 \sin\omega_0 t$. Any force can be represented as a linear superposition of harmonic forces using Fourier series. Since the equation of harmonic oscillations is linear we can use the principle of superposition.

An external force initiates oscillations of different frequencies. During the transition process only those oscillations survive which frequency coincides with the frequency of the driving force. The rest of the oscillations decay during the transition.

When the driving force depends on time harmonically, the equation of motion reads:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin\omega t. \qquad (4.46)$$

To solve Eq. (4.46) for stationary oscillations let us substitute the oscillation which has the same frequency as the driving force:

$$x = x_0 \sin(\omega t + \varphi). \qquad (4.47)$$

Here φ is the phase shift between the displacement x and the force F. The phase shift is to be found from Eq. (4.46). Notice that the phase shift in Eq. (4.15) is determined by the initial conditions which are not essential for the stationary driven oscillations.

Differentiation of Eq. (4.47) and substitution to (4.46), gives

$$\left\{\left[(\omega_0^2-\omega^2)\cos\varphi-2\beta\omega\sin\varphi\right]x_0-\frac{F_0}{m}\right\}\sin\omega t+\left[(\omega_0^2-\omega^2)\sin\varphi+2\beta\omega\cos\varphi\right]x_0\cos\omega t = \qquad (4.48)$$

Since functions $\sin\omega t$ and $\cos\omega t$ are linearly independent,

$$\left[(\omega_0^2 - \omega^2)\cos\varphi - 2\beta\omega\sin\varphi\right]x_0 = \frac{F_0}{m},$$
$$\left[(\omega_0^2 - \omega^2)\sin\varphi + 2\beta\omega\cos\varphi\right]x_0 = 0. \qquad (4.49)$$

The second equation of (4.49) can be rewritten as

$$\tan\varphi = -\frac{2\beta\omega}{\omega_0^2 - \omega^2}. \qquad (4.50)$$

Using the trigonometric formulae

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha}, \qquad \sin^2\alpha = \frac{1}{1+\cot^2\alpha},$$

one can derive from Eq. (4.50) that

$$\cos\varphi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \quad \sin\varphi = -\frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.$$

Substituting these expressions to the first of Eqs. (4.49) one can find the amplitude x_0 of the stationary oscillations:

$$x_0 = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}. \qquad (4.51)$$

Equations (4.50), (4.51), and (4.47) give the desired solution for driven oscillations.

Figures 4.4 and 4.5 show the amplitude and phase shift of driven oscillations versus the frequency of external force.

When the frequency of driving force tends to zero the amplitude tends to the constant

$$\frac{F_0}{m\omega_0^2} = \frac{F_0}{k}. \tag{4.52}$$

Thus for slow motion, i.e. at small frequency (or large period), the displacement is determined by the spring constant.

At high frequency

$$x_0 \to \frac{F_0}{m\omega^2}, \tag{4.53}$$

i.e. the amplitude falls when the frequency grows. The larger the oscillator mass, the greater the rate of the fall.

Calculating the extremum of Eq. (4.51) one can find the maximum amplitude of the oscillations and the corresponding frequency of the driving force:

$$\omega_{\max} = \sqrt{\omega_0^2 - 2\beta^2}, \qquad x_{0\max} = \frac{F_0/m}{2\beta\sqrt{\omega_0^2 - \beta^2}}. \tag{4.54}$$

For small damping

$$\omega_{\max} \approx \omega_0, \qquad x_{0\max} \approx \frac{F_0}{2\beta\omega_0 m}. \tag{4.55}$$

The less the damping, the greater the amplitude. Amplitude enhancement of driven oscillations at frequencies close to the eigenfrequency is called resonance. As it follows from Eqs. (4.55), (4.52), and (4.36) the ratio of the amplitude at the resonance to the amplitude at small frequencies is equal to Q-factor.

The Q-factor specifies the function (4.51) close to the resonance frequency and also the width of resonance peak. For small difference $\omega_0 - \omega$ and using Eq. (4.36) one can obtain from Eq. (4.51) :

$$x_0(\omega) = \frac{F_0}{2m\beta\omega_0\sqrt{1 + \left(\dfrac{2\omega_0\Delta\omega}{2\beta\omega_0}\right)^2}} = \frac{x_{0\max}}{\sqrt{1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2}}. \tag{4.56}$$

The function

$$\frac{1}{\sqrt{1 + \dfrac{(\omega_0 - \omega)^2}{\beta^2}}}$$

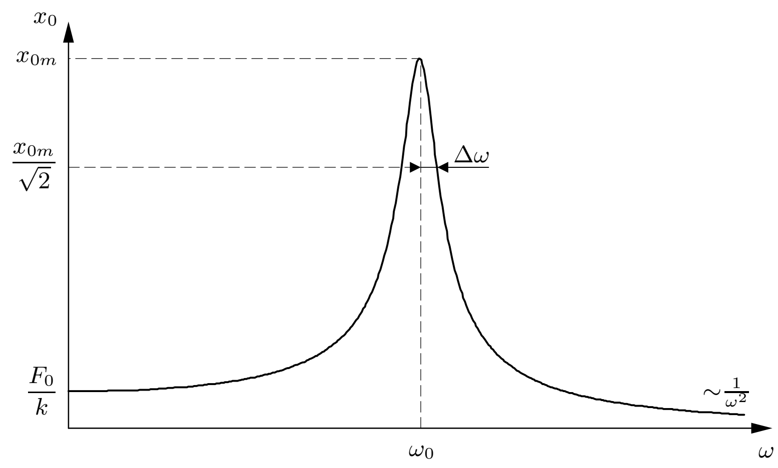

Fig. 4.4. Amplitude-frequency response ($Q = 10$)

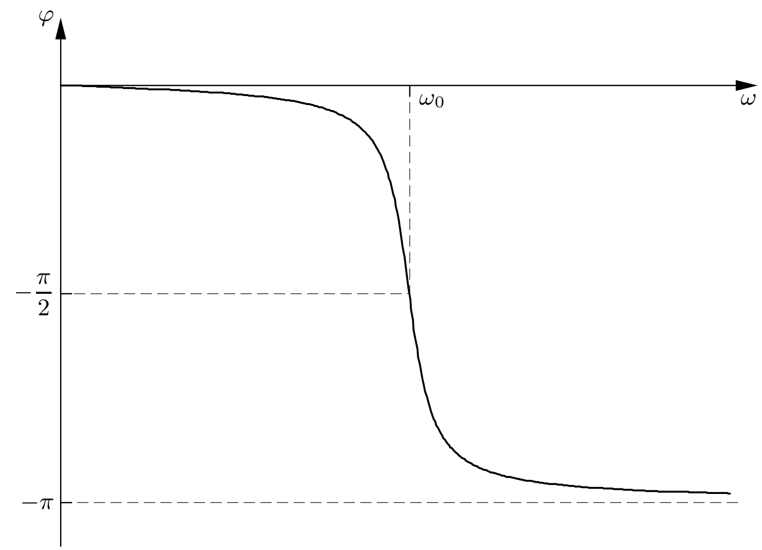

Fig. 4.5. Phase-frequency response ($Q = 10$)

is called Lorentz function. It is often used to analyze spectral lines.

Equation (4.56) gives the width of the peak at $x_0 = x_{0\max}/\sqrt{2}$ as

$$2\Delta\omega = \frac{\omega_0}{Q}. \qquad (4.57)$$

Equation (4.50) shows that the phase shift between displacement and driving force tends to zero for vanishing force frequency. The phases are the same. At resonance the displacement lags behind the driving force by $\pi/2$, but the phase of velocity and the phase of force coincide. It should be clear that maximum amplitude is attained when the maximum force is collinear with the maximum velocity. At high frequency of the force the displacement lags behind by π (they are in antiphase).

Resonance dependences of velocity amplitude v_0 and acceleration a_0 can be figured out similarly. Since $v_0 = x_0\omega$ and $a_0 = x_0\omega^2$, then $v_0 = 0$ and $a_0 = 0$ at $\omega = 0$. The maximum velocity amplitude is attained at $\omega = \omega_0$ and the maximum acceleration amplitude at $\omega_0^2/\sqrt{\omega_0^2 - 2\beta^2}$. When the frequency of the driving force grows the velocity amplitude decreases while the acceleration amplitude tends to F_0/m.

Energy of oscillator driven by external force remains constant. At the same time the oscillator consumes energy from external source. The energy is converted to work against friction and dissipates into heat. The rate of energy consumption per unit time is

$$I(\omega) = \overline{F \cdot \dot{x}} = F_0\omega x_0\overline{\cos(\omega t + \varphi)\sin\omega t} = -\frac{1}{2}F_0\omega x_0 \sin\varphi. \qquad (4.58)$$

Suppose that the system is close to resonance, i.e. $|\omega - \omega_0| = |\Delta\omega| \ll \omega_0$. Then

$$\frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \approx \frac{Q}{\omega_0^2\sqrt{1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2}},$$

which gives

$$x_0 = \frac{F_0 Q}{m\omega_0^2\sqrt{1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2}},$$

$$\sin\varphi = -\frac{1}{\sqrt{1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2}}.$$

Substitution of these expressions in (4.58) yields

$$I(\Delta\omega) = \frac{F_0^2 Q}{2m\omega_0\left[1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2\right]} \qquad (4.59)$$

or

$$I(\Delta\omega) = \frac{I(0)}{1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2},$$

where

$$I(0) = \frac{F_0^2 Q}{2m\omega_0}.$$

Equation (4.59) shows that the energy consumption versus the frequency of external force is also of resonant nature. Let us determine the width of the curve. At 1/2 we have

$$\frac{I(0)}{2} = \frac{I(0)}{1 + Q^2\left(\dfrac{2\Delta\omega}{\omega_0}\right)^2}.$$

Therefore

$$\frac{\Delta\omega}{\omega_0} = \pm\frac{1}{2Q},$$

i.e. the width of the resonant curve is

$$2|\Delta\omega| = \frac{\omega_0}{Q}.$$

Thus both the maximum of energy consumption and the width of the curve is determined by Q-factor.

Free oscillations of coupled pendulums. Up to this point we discussed only the systems with one degree of freedom. Now consider the simplest system with two degrees of freedom, namely, two identical pendulums connected by a spring which execute oscillations in the same plane (see Fig. 4.6). A pendulum consists of a massless rod with a small massive bob at the end.

The notations are shown in the figure. If the deflection angles from the vertical are small ($\sin\varphi \approx \varphi$, $\cos\varphi \approx 1 - \varphi^2/2$), the torque on the first pendulum due to the spring is

$$M_{21} = ka^2(\varphi_2 - \varphi_1).$$

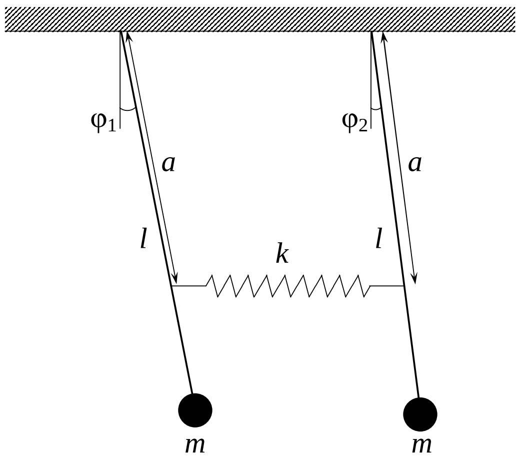

Fig. 4.6. Coupled pendulums

The torque on the second pendulum has the same magnitude and the opposite sign:

$$M_{12} = -ka^2(\varphi_2 - \varphi_1).$$

The pendulums are coupled via these torques.

Equations of motion of the pendulums are

$$ml^2 \frac{d^2\varphi_1}{dt^2} = -mgl\varphi_1 + ka^2(\varphi_2 - \varphi_1), \tag{4.60}$$

$$ml^2 \frac{d^2\varphi_2}{dt^2} = -mgl\varphi_2 - ka^2(\varphi_2 - \varphi_1). \tag{4.61}$$

Adding the equations one obtains:

$$ml^2 \frac{d^2}{dt^2}(\varphi_1 + \varphi_2) = -mgl(\varphi_1 + \varphi_2). \tag{4.62}$$

Subtracting Eq. (4.61) from (4.60) gives

$$ml^2 \frac{d^2}{dt^2}(\varphi_1 - \varphi_2) = -(mgl + 2ka^2)(\varphi_1 - \varphi_2). \tag{4.63}$$

Notice that addition and subtraction of Eqs. (4.60) and (4.61) allows one to decouple them. Solutions of Eqs. (4.62) and (4.63) are

$$\varphi_1 + \varphi_2 = A\cos(\omega^+ t + \alpha), \tag{4.64}$$

$$\varphi_1 - \varphi_2 = B\cos(\omega^- t + \beta), \tag{4.65}$$

$$\omega^+ = \sqrt{\frac{g}{l}}, \qquad \omega^- = \sqrt{\frac{g}{l} + \frac{2ka^2}{ml^2}},$$

where A, B, α, and β are some constants. Adding and subtracting Eqs. (4.64) and (4.65) one obtains

$$\varphi_1 = \frac{1}{2}A\cos(\omega^+ t + \alpha) + \frac{1}{2}B\cos(\omega^- t + \beta), \tag{4.66}$$

$$\varphi_2 = \frac{1}{2}A\cos(\omega^+ t + \alpha) - \frac{1}{2}B\cos(\omega^- t + \beta). \tag{4.67}$$

Therefore the angular velocities are

$$\dot\varphi_1 = -\frac{1}{2}\omega^+ A\sin(\omega^+ t + \alpha) - \frac{1}{2}\omega^- B\sin(\omega^- t + \beta), \tag{4.68}$$

$$\dot\varphi_2 = -\frac{1}{2}\omega^+ A\sin(\omega^+ t + \alpha) + \frac{1}{2}\omega^- B\sin(\omega^- t + \beta). \tag{4.69}$$

Let us analyze the obtained solutions. Suppose the pendulums have the same initial (at $t = 0$) deflections and zero velocities:

$$\varphi_1(0) = \varphi_2(0) = \varphi_0, \qquad \dot\varphi_1(0) = \dot\varphi_2(0) = 0.$$

Then from Eqs. (4.66) – (4.69) one gets

$$\sin\alpha = 0, \qquad A = 2\varphi_0, \qquad B = 0,$$

i.e.

$$\varphi_1 = \varphi_0\cos\omega^+ t, \qquad \varphi_2 = \varphi_0\cos\omega^+ t. \tag{4.70}$$

Therefore the pendulums oscillate with the same amplitude and phase (in-phase oscillations).

If at $t = 0$

$$\varphi_1(0) = -\varphi_2(0) = \varphi_0, \qquad \dot\varphi_1(0) = \dot\varphi_2(0) = 0,$$

then it follows from Eqs. (4.66) – (4.69) that

$$\sin\beta = 0, \qquad A = 0, \qquad B = 2\varphi_0,$$

i.e.

$$\varphi_1 = \varphi_0\cos\omega^- t, \qquad \varphi_2 = -\varphi_0\cos\omega^- t = \varphi_0\cos(\omega^- t + \pi). \tag{4.71}$$

The relations show that the pendulums oscillate with the same amplitude but their phases differ by π (antiphase oscillations). Two types of motion described by Eqs. (4.70) and (4.71) are called normal modes of coupled oscillators. Normal mode of oscillation is a collective motion in which the amplitude of oscillation of each degree of freedom remains constant. The concept of normal mode is very important for modern physics.

Now consider the case when only one pendulum is initially deflected, i.e.

$$\varphi_1(0) = \varphi_0, \qquad \varphi_2(0) = 0, \qquad \dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0.$$

It can be shown that in this case

$$\varphi_1 = \frac{\varphi_0}{2}(\cos\omega^+ t + \cos\omega^- t), \tag{4.72}$$

$$\varphi_2 = \frac{\varphi_0}{2}(\cos\omega^+ t - \cos\omega^- t). \tag{4.73}$$

Using trigonometric formulae

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2},$$

one can write Eqs. (4.72) and (4.73) as

$$\varphi_1 = \varphi_0 \cos\frac{\omega^+ - \omega^-}{2}t \cdot \cos\frac{\omega^+ + \omega^-}{2}t, \tag{4.74}$$

$$\varphi_2 = \varphi_0 \sin\frac{\omega^- - \omega^+}{2}t \cdot \sin\frac{\omega^+ + \omega^-}{2}t. \tag{4.75}$$

Let us analyze Eqs. (4.74) and (4.75). Notice that the oscillation frequency of the even mode (labeled by «+»), $\omega^+ = \sqrt{g/l}$, equals ω_0 where ω_0 is the eigenfrequency of a solitary pendulum (the so-called partial frequency). On the other hand, the frequency of the odd mode (labeled by «−») is

$$\omega^- = \omega_0\sqrt{1+2\varepsilon},$$

where the parameter $\varepsilon = ka^2/mgl$ specifies pendulum coupling. For small coupling, $\varepsilon \ll 1$,

$$\omega^- \approx \omega_0(1+\varepsilon),$$

i.e.

$$\omega^- - \omega^+ \approx \omega_0\varepsilon, \qquad \omega^- + \omega^+ \approx 2\omega_0.$$

In this approximation Eqs. (4.74) and (4.75) become

$$\varphi_1 = \varphi_0 \cos\frac{\omega_0\varepsilon}{2}t\,\cos\omega_0 t, \tag{4.76}$$

$$\varphi_2 = \varphi_0 \sin\frac{\omega_0\varepsilon}{2}t\,\sin\omega_0 t = \varphi_0 \sin\frac{\omega_0\varepsilon}{2}t\,\cos\left(\omega_0 t - \frac{\pi}{2}\right). \tag{4.77}$$

Thus we deal with harmonic oscillations of frequency ω_0 which amplitude varies periodically with time at a much less frequency $\omega_0\varepsilon/2$. This is the so-called amplitude modulated oscillation or beat. The phase shift is $\pi/2$. The modulated amplitude of oscillations of the first pendulum is

$$A_1(t) = \varphi_0 \cos\frac{\omega_0\varepsilon}{2}t. \tag{4.78}$$

Similarly the oscillation amplitude of the second pendulum is

$$A_2(t) = \varphi_0 \sin\frac{\omega_0\varepsilon}{2}t = \varphi_0\cos(\frac{\omega_0\varepsilon}{2}t - \frac{\pi}{2}).$$

Initially, at $t = 0$:

$$A_1 = \varphi_0, \qquad A_2 = 0.$$

At $t = \frac{\pi}{\omega_0\varepsilon}$:

$$A_1 = 0, \qquad A_2 = \varphi_0.$$

At $t = 2\frac{\pi}{\omega_0\varepsilon}$:

$$A_1 = -\varphi_0, \qquad A_2 = 0.$$

Notice that the amplitude of harmonic oscillation is positive by definition. The negative sign here means that the phase shift changes by π. At $t = 3\frac{\pi}{\omega_0\varepsilon}$:

$$A_1 = 0, \qquad A_2 = -\varphi_0.$$

At $t = 4\frac{\pi}{\omega_0\varepsilon}$

$$A_1 = \varphi_0, \qquad A_2 = 0.$$

Thus pendulums exchange energy of oscillations. At $t = 0$ the energy is accumulated in the first pendulum. Then the energy is gradually transfered via the spring to the second pendulum until it accumulates all the energy. The time τ of the transfer can be estimated as

$$\frac{\omega_0\varepsilon}{2}\tau = \pi,$$

i.e.

$$\tau = \frac{2\pi}{\omega_0\varepsilon}. \tag{4.79}$$

The frequency of the energy exchange between oscillators is

$$\frac{2\pi}{\tau} = \omega_0\varepsilon = \omega^- - \omega^+.$$

Notice that oscillations in a system consisting of a large number of coupled oscillators can be regarded as propagation of waves of a certain kind.

Plane wave. In physics any time variation and spatial alternation of maxima and minima of any quantity, e.g. matter density, pressure, temperature, electric field, etc., is called a wave. Such alternation is essentially an oscillation process in a system with infinite number of degrees of freedom. However, propagation of a short time perturbation, a «pulse», is often called a wave as well. The simplest mathematical model of a wave process is a plane wave.

Suppose that some scalar quantity s depends on time t and position x (but it is independent of y and z) as

$$s = f(x - ut), \tag{4.80}$$

where f is an arbitrary function and $u = \text{const}$. Consider a snapshot of the wave process at $t = 0$. In this case

$$s(0, x) = f(x). \tag{4.81}$$

Then consider a snapshot of the same wave at $t = t_1$. It is described by the equation

$$s(t_1, x) = f(x - ut_1). \tag{4.82}$$

Comparing Eqs. (4.81) and (4.82) one can see that two snapshots differ by the displacement ut_1 in the positive direction of x. Therefore the wave propagates to the right at the speed u while *retaining its shape*. A wave process described by the function (4.80) is called plane wave. The wave specified by

$$s = f(x + ut),$$

propagates in the opposite direction.

Plane sinusoidal wave. A case of sinusoidal function f is of special interest. Consider

$$s = A\cos(\omega t - kx) = A\cos\big[k(x - ut)\big], \tag{4.83}$$

where $u = \omega/k$ is the velocity of wave propagation. At any point x the value of s executes simple harmonic motion with the amplitude A and the circular frequency ω. Both quantities are the same for all x. The oscillation period is $T = 2\pi/\omega$ and the phase is kx.

A snapshot of (4.83) is a spatial sinusoid. For instance, at $t = 0$

$$s = A\cos kx.$$

The minimum distance λ, so that

$$s(x + \lambda) = s(x)$$

for any x, is called wavelength. The quantity k is called wave vector or spatial frequency. Obviously

$$\lambda = \frac{2\pi}{k}.$$

Standing wave. Let a scalar quantity s depend on position coordinates x, y, and z and time t as

$$s = F(x, y, z)\cos(\omega t + \varphi),$$

where $F(x, y, z)$ is an arbitrary function and ω and φ are constants. According to the equation s executes simple harmonic motion of the same frequency and phase at any point in space. But the oscillation amplitude varies. Such a process is called standing wave.

Let us show that superposition of two plane waves of the same amplitude, wavelength, and phase and propagating in opposite directions is a sinusoidal standing wave.

Indeed let

$$s_1 = A\cos(\omega t - kx + \alpha_1), \qquad s_2 = A\cos(\omega t + kx + \alpha_2).$$

Their sum

$$s = s_1 + s_2$$

in accordance with the trigonometric formula

$$\cos x + \cos y = 2\cos\frac{x + y}{2}\cos\frac{x - y}{2}$$

can be written as

$$s = 2A\cos\left(kx - \frac{\alpha_1 - \alpha_2}{2}\right)\cos\left(\omega t + \frac{\alpha_1 + \alpha_2}{2}\right). \tag{4.84}$$

This equation describes a sinusoidal standing wave.

Now consider

$$s_1 = A_1 \cos(\omega t - kx + \alpha_1), \qquad s_2 = A_2 \cos(\omega t + kx + \alpha_2).$$

It can be shown that in this case

$$s = 2A_2 \cos\left(kx - \frac{\alpha_1 - \alpha_2}{2}\right) \cos\left(\omega t + \frac{\alpha_1 + \alpha_2}{2}\right) + a\cos(\omega t - kx + \alpha_1).$$

Here $a = A_1 - A_2$. The quantity a/A_2 is called coefficient of running.

Wave equation. Consider a function which describes plane wave:

$$f(x,t) = f(x - ut). \tag{4.85}$$

Differentiating it with respect to time t one gets:

$$\frac{\partial f}{\partial t} = f'(x - ut)\cdot(-u), \qquad \frac{\partial^2 f}{\partial t^2} = f''(x - ut)\cdot u^2. \tag{4.86}$$

Here the prime stands for derivative with respect to $x - ut$. Now let us differentiate the function (4.85) twice with respect to x:

$$\frac{\partial f}{\partial x} = f'(x - ut), \qquad \frac{\partial^2 f}{\partial x^2} = f''(x - ut). \tag{4.87}$$

Comparing Eqs. (4.86) and (4.87) one can see that the function (4.85) satisfies the following equation

$$\frac{\partial^2 f}{\partial t^2} = u^2 \frac{\partial^2 f}{\partial x^2}. \tag{4.88}$$

Equation (4.88) is the partial differential equation termed wave equation which plays an important role in physical applications. It can be proven that the general solution of the equation is

$$f(x,t) = f_1(x - ut) + f_2(x + ut),$$

where f_1 and f_2 are arbitrary functions determined by initial or boundary conditions.

Longitudinal waves in elastic body. Consider dynamics of longitudinal waves in elastic rod. Let x-axis be directed along the rod. Assume that the rod elements which lie in a plane perpendicular to x at $t = 0$ also remain in a plane perpendicular to x at any $t \neq 0$. A cross-section with coordinate

x at $t = 0$ has a different coordinate x' at $t = t'$. In the following the quantity (positive or negative)

$$s = x' - x$$

is called the displacement of x. Now consider the cross-section between the planes x and $x + \Delta x$. In the non-deformed rod the cross-section thickness is Δx. A deformation displaces the planes which coordinates become x' and $x' + \Delta x'$, respectively.

Let

$$x' = x + s(x),$$

$$x' + \Delta x' = x + \Delta x + s(x + \Delta x),$$

where $s(x)$ is the displacement of the plane x and $s(x + \Delta x)$ is the displacement of the plane $x + \Delta x$. Then the thickness of the rod section equals

$$(x' + \Delta x') - x' = \Delta x'.$$

The increment of the section thickness is

$$\Delta x' - \Delta x = s(x + \Delta x) - s(x).$$

The average longitudinal strain of the rod section between x and $x + \Delta x$ is

$$\frac{s(x + \Delta x) - s(x)}{\Delta x}.$$

The longitudinal strain ε at a given plane is defined as the limit

$$\varepsilon = \lim_{\Delta x \to 0} \frac{s(x + \Delta x) - s(x)}{\Delta x} = \frac{\partial s}{\partial x}. \tag{4.89}$$

According to Hooke's law

$$\sigma = E\varepsilon, \tag{4.90}$$

where σ is the stress and E is the bulk modulus. Now let us apply Newton's law of motion to the rod section between the planes x and $x + \Delta x$. The section mass is $\rho S \Delta x$ where ρ and S are the density and the cross-sectional area in the absence of deformation. Let s be the displacement of the center of mass of the section. Then

$$\rho S \Delta x \frac{\partial^2 s}{\partial t^2} = S\sigma(x + \Delta x) - S\sigma(x).$$

The left-hand side is the mass multiplied by acceleration while the right-hand side equals the net force exerted on the section. Let us divide the equation by $S\Delta x$:

$$\rho\frac{\partial^2 s}{\partial t^2}=\frac{\sigma(x+\Delta x)-\sigma(x)}{\Delta x}.$$

Taking the limit $\Delta x\to 0$ one obtains the equation

$$\rho\frac{\partial^2 s}{\partial t^2}=\frac{\partial\sigma}{\partial x}. \tag{4.91}$$

Substitution of Eq. (4.90) to (4.91) gives

$$\rho\frac{\partial^2 s}{\partial t^2}=E\frac{\partial\varepsilon}{\partial x}.$$

According to Eq. (4.89)

$$\frac{\partial\varepsilon}{\partial x}=\frac{\partial^2 s}{\partial x^2},$$

i.e.

$$\frac{\partial^2 s}{\partial t^2}=\frac{E}{\rho}\frac{\partial^2 s}{\partial x^2}. \tag{4.92}$$

This is wave equation. Therefore a deformation propagates along the rod either as a plane wave $s=f(x\mp ut)$ or a superposition of such waves. The speed of wave propagation (speed of sound) is

$$u=\sqrt{\frac{E}{\rho}}.$$

For steel $u=5200$ m/s, for copper $u=3700$ m/s, for aluminum $u=$ $=5100$ m/s, and for rubber $u=46$ m/s.

Notice that the wave equation is derived under assumption that the wavelength is large compared to the rod cross-section. The opposite limit corresponds to unbounded elastic medium. It can be shown that the speed of longitudinal elastic wave in that case is

$$u_1=\sqrt{\frac{E'}{\rho}}=\sqrt{\frac{E(1-\mu)}{\rho(1+\mu)(1-2\mu)}},$$

where μ is the Poisson ratio.

Energy density. Consider a small section of the rod which volume in the non-deformed state is $S\Delta x$, so its mass is $\rho S\Delta x$. Kinetic energy of the section moving in x-direction is

$$\frac{1}{2}\rho S\Delta x\left(\frac{\partial s}{\partial t}\right)^2,$$

where $\partial s/\partial t$ is the instantaneous velocity of the section. Then the kinetic energy per unit volume is

$$w_{\text{к}}=\frac{1}{2}\rho v^2.$$

This quantity is called kinetic energy density.

It can be shown that the section has also the potential energy which density equals (consult the derivation of Eq. (3.4)):

$$w_{\text{п}}=\frac{1}{2}E\varepsilon^2.$$

The total energy density is

$$w=w_{\text{к}}+w_{\text{п}}=\frac{1}{2}(\rho v^2+E\varepsilon^2).$$

The total mechanical energy of the rod section bounded by the planes $x=x_1$ and $x=x_2$ is:

$$W=\int\limits_{x_1}^{x_2}wS\,dx=\frac{S}{2}\int\limits_{x_1}^{x_2}(\rho v^2+E\varepsilon^2)\,dx.$$

An energy change equals the work done by the forces exerted by the adjacent sections. Let indices 1 and 2 refer to quantities related to the sections $x=x_1$ and $x=x_2$, respectively. The force acting on the left is $F_1=-S\sigma_1$ (the sign is negative since for $\sigma_1>0$ the force F_1 is directed to the left). The force acting on the right is $F_2=S\sigma_2$ (if $\sigma_2>0$ the force is directed to the right). The work done by the forces F_1 and F_2 during the time dt equals F_1v_1dt and F_2v_2dt, respectively. Therefore the net work is

$$(F_1v_1+F_2v_2)dt=-(\sigma_1v_1-\sigma_2v_2)Sdt.$$

According to the law of conservation of mechanical energy this work is equal to energy increment dW, therefore

$$\frac{dW}{dt}=Q_1-Q_2,$$

where

$$Q_1=-S\sigma_1v_1,\qquad Q_2=-S\sigma_2v_2.$$

It should be clear that the quantity $Q=-S\sigma v$ specifies energy flow through a given cross-section. The corresponding unit of measurement is $[Q]=1$ erg/s or 1 $J/s=1$ W.

Energy flow density is defined as

$$q = -\sigma v = Pv,$$

where $-\sigma = P$ is the pressure in a given cross-section. The corresponding unit of measurement is $[q] = 1\ erg/(cm^2 \cdot s)$ or $1\ W/m^2$. Let us calculate the density of energy flow of the plane sinusoidal wave described by the equation

$$s = A\cos(\omega t - kx).$$

Obviously

$$\sigma = E\varepsilon = E\frac{\partial s}{\partial x} = EkA\sin(\omega t - kx),$$

$$v = \frac{\partial s}{\partial t} = -A\omega\sin(\omega t - kx).$$

Therefore

$$q = -\sigma v = Ek\omega A^2\sin^2(\omega t - kx) = \frac{1}{2}Ek\omega A^2\big(1 - \cos(2\omega t - 2kx)\big).$$

One can see that the energy flow attains its maximum twice per period and its frequency is 2ω at any point of the rod. The value of q averaged over the period is

$$\bar{q} = \frac{1}{T}\int_0^T q(t)\,dt = \frac{1}{2}Ek\omega A^2.$$

In acoustics the value $\bar{q}$ is called sound volume. Usually the volume is measured in decibels (dB) according to

$$D = 10\lg\left(\bar{q}, \frac{\mu W}{cm^2}\right) + 100\ (dB).$$

For example, if $\bar{q} = 10^{-10}\ \mu W/\mathrm{cm}^2$ then $D = 0$ (the initial value). For $\bar{q} = 10^{-6}\ \mathrm{W/cm}^2$, $D = 100$ dB. The threshold of pain, i.e. the value of $\bar{q}$ at which sound becomes painful for a listener is

$$\bar{q} = 10^{-4}\ \frac{\text{Вт}}{\text{см}^2} = 10^2\ \frac{\mu W}{cm^2}.$$

This corresponds to $D = 120$ dB.

Now consider a standing wave

$$s = A\sin kx\cos(\omega t + \varphi),$$

so that

$$\sigma = EkA\cos kx\cos(\omega t + \varphi),$$

$$v = -A\omega\sin kx\sin(\omega t + \varphi),$$

$$q = \frac{1}{4}Ek\omega A^2\sin 2kx\sin(2\omega t + 2\varphi).$$

One can see that the density of energy flow through the cross-sections with coordinates

$$x_1 = 0,\quad x_2 = \frac{\lambda}{4} = \frac{\pi}{2k},\quad x_3 = 2\frac{\lambda}{4} = 2\frac{\pi}{2k},\quad x_4 = 3\frac{\lambda}{4} = 3\frac{\pi}{2k},\quad \ldots$$

is always zero. Therefore any section of the rod of the length $\lambda/4$ enclosed between a stress nod and a velocity nod next to it does not exchange energy with the neighbors. Its energy is constant.

Transversal waves on string. In acoustics a uniform elastic thread tightened by an external force is called a string. It can be a stretched wire, cable, or a violin string.

Consider a string which equilibrium position coincides with abscissa. Assume that the string elements move only in the plane $(x,\ y)$. Let $s(x,t)$ be the displacement of the element which position in equilibrium is x. Now let us write Newton's law of motion for the element enclosed in the interval $x, x+\Delta x$. The element mass is $\rho S\Delta x$ where ρ is specific mass of the string material and S is cross-sectional area. The product of the element mass by its acceleration $\partial^2 s/\partial t^2$ is equal to the y-component of the net force applied to the ends of the element:

$$\rho S\Delta x\frac{\partial^2 s}{\partial t^2} = -S\sigma(x)\sin\alpha(x) + S\sigma(x+\Delta x)\sin\alpha(x+\Delta x). \qquad (4.93)$$

Here $\sigma(x)$ is tension at x and $\alpha(x)$ is the angle between the tangent to the string at x and the abscissa. Obviously,

$$\tan\alpha = \frac{\partial s}{\partial x}.$$

Now suppose that displacement $s(x,t)$ is small, so it is safe to assume that: 1) the string tension $\sigma(x)$ is approximately equal to the tension σ in equilibrium, 2) $\sin\alpha$ approximately equals $\tan\alpha$.

Then Eq. (4.93) is simplified and becomes:

$$\rho\Delta x\frac{\partial^2 s}{\partial t^2} = \sigma\left[\left(\frac{\partial s}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial s}{\partial x}\right)_x\right]. \qquad (4.94)$$

Dividing Eq. (4.94) by Δx and taking the limit $\Delta x \to 0$ one obtains the wave equation:

$$\frac{\partial^2 s}{\partial t^2} = \frac{\sigma}{\rho}\frac{\partial^2 s}{\partial x^2}. \tag{4.95}$$

According to the equation a transversal wave propagating on string retains its shape, the wave speed is

$$c_s = \sqrt{\frac{\sigma}{\rho}} = \sqrt{\frac{F}{\rho S}},$$

where F is string tension and ρS is the mass per unit length.

String eigenmodes. Under certain conditions string vibration becomes standing transversal wave which is described by the equation

$$s = A \sin kx \cos(\omega t + \varphi), \tag{4.96}$$

where $k = \omega/u$. Let us separate a string segment by fixing the string at the points $x = 0$ and $x = n(\lambda/2) = n\pi/k$. Since the points are at rest (these are the nodes of s), their fixing does not change the vibration pattern. Therefore a string of length l with its ends fixed can execute sinusoidal standing vibrations with nodes at the ends. The string length is then a multiple integer of half-wavelengths:

$$l = n\frac{\lambda}{2} = n\frac{\pi u}{\omega}, \quad n = 1, 2, \ldots$$

The frequency of n-th eigenmode can be easily found:

$$\omega_n = \frac{n\pi}{l}\sqrt{\frac{F}{\rho S}}, \qquad \nu_n = \frac{n}{2l}\sqrt{\frac{F}{\rho S}} \quad n = 1, 2, \ldots \tag{4.97}$$

If the frequency of external transversal sinusoidal force coincides with the frequency of an eigenmode, resonance occurs. The resulting wave is the standing wave corresponding to the vibrational eigenmode.

Passage of longitudinal wave through boundary between two media. Let the plane $x = 0$ be the boundary between two different elastic media. The quantities referred to the media on the left and on the right with respect to the boundary will be labeled with indices 1 and 2, respectively. Suppose an elastic wave is coming from the left:

$$s_1 = A_1 \cos(\omega t - k_1 x). \tag{4.98}$$

Here s_1 is a displacement in the x-direction. What happens on the boundary?

To answer this question one should invoke physical properties of the boundary. Firstly, continuity requires the displacement on the both sides of the boundary ($x = 0$) to be the same:

$$s_1(0, t) = s_2(0, t), \tag{4.99}$$

Secondly, according to third Newton's law the stress on the both sides must be equal as well:

$$\sigma_1(0, t) = \sigma_2(0, t). \tag{4.100}$$

Now suppose that the wave penetrates from the first medium to the second,

$$s_2 = A_2 \cos(\omega t - k_2 x), \tag{4.101}$$

but this process does not affect the first medium, so that Eq. (4.98) holds. Substitution of Eqs. (4.98) and (4.101) to (4.99) and (4.100) yields

$$A_1 = A_2, \qquad A_1 = \gamma A_2,$$

where

$$\gamma = \frac{E_2 k_2}{E_1 k_1} = \frac{E_2 c_{l1}}{E_1 c_{l2}} = \frac{\sqrt{E_2 \rho_2}}{\sqrt{E_1 \rho_1}}.$$

Here c_l is the speed of longitudinal wave. Notice that the quantity $\sqrt{E\rho} = \rho c_l$ is often called acoustic impedance. However the above equations are incompatible unless there is no boundary,

$$\gamma = 1.$$

Equations (4.99) and (4.100) can be simultaneously satisfied by taking into account the experimental observation that there is also a reflected wave in the first medium,

$$A_1' \cos(\omega t + k_1 x),$$

so that

$$s_1 = A_1 \cos(\omega t - k_1 x) + A_1' \cos(\omega t + k_1 x). \tag{4.102}$$

Substituting Eqs. (4.101) and (4.102) to (4.99) and (4.100) one obtains:

$$\left.\begin{array}{l} A_1 + A_1' = A_2, \\ A_1 - A_1' = \gamma A_2 \end{array}\right\}. \tag{4.103}$$

Equations (4.103) can always be solved for A_1' and A_2. For a given amplitude A_1 of the incident wave Eqs. (4.103) determine the amplitudes of the reflected and refracted waves:

$$A_1' = \frac{1-\gamma}{1+\gamma} A_1, \qquad A_2 = \frac{2}{1+\gamma} A_1. \tag{4.104}$$

Notice that

$$\frac{k_2}{k_1} = \frac{\lambda_1}{\lambda_2} = \frac{c_{l1}}{c_{l2}}.$$

Wavelengths are different in both media. The wavelength is greater in the medium in which the speed of sound is greater. Let us introduce the notations:

$$R = \frac{\overline{q_1'}}{\overline{q_1}}, \qquad T = \frac{\overline{q_2}}{\overline{q_1}}.$$

The quantities R and T are called reflection and transmission coefficient, respectively. It is not difficult to show that

$$R = \left(\frac{1-\gamma}{1+\gamma}\right)^2, \qquad T = \frac{4\gamma}{(1+\gamma)^2}. \tag{4.105}$$

As expected,

$$R + T = 1.$$

This relation follows from the law of conservation of mechanical energy:

$$\overline{q_1'} + \overline{q_2} = \overline{q_1}.$$

For $\gamma = 0$ and $\gamma = \infty$ we have $R = 1$, $T = 0$: the energy is reflected back to the first medium. Notice that Eqs. (4.105) are invariant under replacement of γ with $1/\gamma$. Therefore the introduced reflection and transmission coefficients are the same regardless of the direction of propagation of the incident wave.

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Lab 1.4.1

Compound pendulum

Purpose of the lab: to study the dependence of oscillation period of compound pendulum on its moment of inertia.

Tools and instruments: a compound pendulum (uniform steel rod), a knife edge, a simple gravity pendulum, an oscillation counter, a ruler, and a stopwatch.

A compound pendulum is a rigid body which can freely swing about a stationary horizontal axis in the gravitational field. The motion of pendulum is described by the following equation:

$$I\frac{d^2\varphi}{dt^2} = M, \tag{1}$$

where I is the moment of inertia of the pendulum, φ is the deviation angle measured from the equilibrium position, t is time, and M is the torque acting on the pendulum.

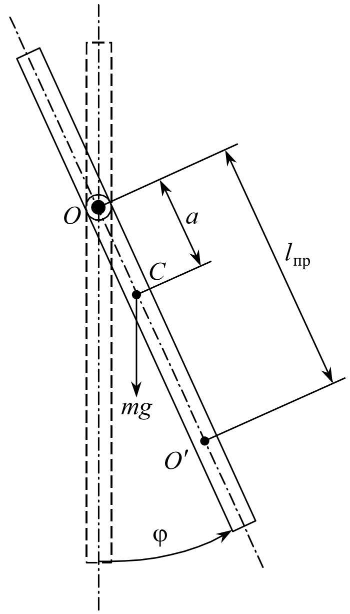

Fig. 1. Compound pendulum

A uniform steel rod of length l is used as a compound pendulum in this lab (see Fig. 1). A knife edge is fixed on the rod, so its arris is the pivot axis. The knife edge can be shifted along the rod thereby altering the distance $OC \equiv a$ between the pivot of the pendulum and its center of gravity. Using the Huygens-Steiner theorem (2.31) one can find the moment of inertia of the pendulum:

$$I = \frac{ml^2}{12} + ma^2,$$

where m is its mass. The torque on the pendulum is due to the gravitational force:

$$M = -mga\sin\varphi.$$

If the deviation angle φ is small one can set $\sin\varphi \approx \varphi$ and hence obtain

$$M \approx -mga\varphi.$$

The pendulum can exhibit hundreds of oscillations without notable damping provided the experimental setup is in good order. In this case friction can be neglected. Substituting the expressions for I and M into Eq. (1) one obtains

$$\ddot{\varphi} + \omega^2 \varphi = 0, \tag{2}$$

where

$$\omega^2 = \frac{ga}{a^2 + \dfrac{l^2}{12}}. \tag{3}$$

The solution is given by Eq. (4.15):

$$\varphi(t) = A \sin(\omega t + \alpha).$$

The amplitude A and the initial phase α depend on the way the oscillations started, i.e. they are determined by initial conditions; the frequency ω according to Eq. (3) depends only on the free fall acceleration g and the pendulum parameters l and a.

The oscillation period equals

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{a^2 + \dfrac{l^2}{12}}{ag}}. \tag{4}$$

We can see that the period of small oscillations of a compound pendulum depends neither on the phase nor on the amplitude. This statement manifests the isochronism of oscillations, it is valid for processes described by Eq. (2). In fact, this description of the pendulum motion is approximate since the equality $\sin\varphi \approx \varphi$ used in the derivation of Eq. (2) is approximate as well.

The oscillation period of a simple gravity pendulum is given by (4.39):

$$T' = 2\pi \sqrt{\frac{l'}{g}},$$

where l' is the pendulum length. For this reason the quantity

$$l_{eq} = a + \frac{l^2}{12a} \tag{5}$$

is referred to as the equivalent length. The point O' separated by the distance l_{eq} from O is called the center of oscillations. The pivot point and the oscillation center are reversible, i.e. the periods of oscillations about O' and O are the same.

An experimental verification of the above statement is a good way of testing the theory. Another way is to test validity of Eq. (4). The latter contains the quantity a which changes when the edge is moved along the rod. In this lab a lead ball suspended on two diverging wires (as shown in Fig. 2) is used as a simple gravity pendulum. The wires are wound on a horizontal axis and their length can be varied.

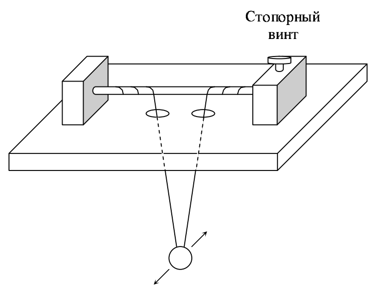

Fig. 2. Simple gravity pendulum

LABORATORY ASSIGNMENT

1. Set the working range of the amplitudes so that the oscillation period T is approximately amplitude-independent. For this purpose deflect the pendulum from its equilibrium position by the angle φ_1 ($\sim 10°$) and measure the time of 100 full swings. The number of oscillations is counted by an electronic or mechanical counter and the time is measured with a stopwatch. To decrease the error of time measurements start and stop the stopwatch at the moment of pendulum crossing the point of equilibrium. Using the data obtained calculate the oscillation period T_1.

   Repeat the experiment for the initial deflection angle of 1.5–2 times less than that in the first experiment. If the periods are equal within the experimental error the working range of the amplitudes lies within $(0,\varphi_1)$. If the periods differ one should repeat the experiment for smaller angles.

   Identify the source of the largest error of the measurement of the period and try to reduce it.
2. Shift the knife edge along the rod and study the dependence of the oscillation period T on the distance a between the pivot point and the center of mass. Plot the values T^2 vs a^2 and obtain the values $g/4\pi^2$ and $l^2/12$ by performing a linear fit (use Eq. (4)). Compare the obtained value of g with the tabulated one and verify the value of l by direct measurement.
3. Find the appropriate length of the simple gravity pendulum for a particular position of the knife edge so that the periods of both pendulums coincided within the error. Measure the length of the simple gravity pendulum and compare it with the equivalent length calculated from Eq. (5).
4. Verify experimentally reversibility of the pivot point and the oscillation center. What pivot position ensures the most accurate verification?

Questions

1. What are the simplifications used in deriving Eq. (4)?

2. What distance between the pivot point and the center of mass corresponds to the minimum period of oscillations?
3. Describe the behavior of the compound pendulum which pivot point and the center of mass coincide.
4. Why is the simple gravity pendulum suspended on two wires?
5. Formulate and prove the Huygens-Steiner theorem.

Literature

1. *Сивухин Д.В.* Общий курс физики. Т. I. — М.: Наука, 1996. §§ 30, 33, 35, 36, 40, 41.
2. *Стрелков С.П.* Механика. — М.: Наука, 1975. §§ 52, 59, 124.

Lab 1.4.2

Measurement of gravitational acceleration by means of Kater's pendulum

Purpose of the lab: to determine the local acceleration of gravity using Kater's pendulum

Tools and instruments: Kater's pendulum, an oscillation counter, a stopwatch, a caliper with 1 m scale.

Free fall is a motion near Earth's surface such that forces resisting the motion can be neglected. The gravitational acceleration near Earth's surface which is usually called g is then determined by gravity force F exerted on a body of mass m,

$$\vec{g} = \frac{\vec{F}}{m}. \tag{1}$$

A reference frame related to the Earth is not inertial. In such a frame there are also centrifugal force and Coriolis force in addition to gravity force. The Coriolis force is always perpendicular to the velocity of the body, so the force changes only the velocity direction while the magnitude remains intact. Usually the gravitational acceleration is identified with the acceleration component which is tangential to the body trajectory, so the Coriolis force does not contribute. Obviously the normal force exerted on a body that rests upon Earths' surface equals the sum of gravity and centrifugal forces (the body weight).

A gravity pull exerted on a body by the Earth is equal to the product of the body mass m by gravitational acceleration $\vec{g}_0$:

$$\vec{F}_0 = m\vec{g}_0. \tag{2}$$

Gravitational acceleration is determined by distribution of mass inside the Earth. If the Earth were a solid sphere of constant density, the acceleration inside the sphere would be directly proportional to the distance towards the Earth center and the acceleration outside would fall according to the inverse-square law. Actually the Earth mass density is not uniform and grows with depth. Because of that the gravitational acceleration slightly increases up to the depth of 2800 km (which corresponds to the distance towards the center of 3600 km) and then falls linearly with the distance to the center. Above the surface and close to it the gravitational acceleration is well approximated by that of the uniform sphere. The acceleration decreases by 10% at the height of 300 km which corresponds approximately to a satellite orbit. Observation of satellite motion allows one to determine the distribution of mass inside the Earth, which is used, e.g. for search of ore bodies.

The net gravity force also includes gravitational attraction to the Moon and the Sun. Although their contribution to the net force is small these forces are responsible for global effects such as tides.

Earth rotation around its axis resulted in the Earth deformation because of centrifugal force. The distance from the Earth center to a pole is approximately 21 km less than the distance to equator which is equal to 6 378 140 m. As it was already mentioned the centrifugal force is combined with the gravity force for a body residing on the Earth surface. It is called the net gravitational acceleration g and its values are given in the tables of local acceleration of gravity. On a pole $g = 983.2155$ cm/s^2 and it decreases towards the equator where $g = 978.0300$ cm/s^2. Therefore a pendulum clock on the equator lags behind the one on a pole by 3.8 min. The direction of the gravitational acceleration is always perpendicular to the surface of a body of water and does not deviate significantly from the direction to the Earth center.

The mass distribution inside the Earth is not spherically symmetric, which also results in local variations of g. Extensive and precise measurements of g on the Earth surface showed that gravitational acceleration depends on time as well. Periodic variations related to the Moon and Sun tides are approximately $2.49 \cdot 10^{-4}$ cm/s^2 and $9.6 \cdot 10^{-5}$ cm/s^2, respectively. There are also periodic variations of the same order due to geological processes inside the Earth (the so-called secular variations).

Measurements of g on the Earth surface are recorded on the gravimetric maps to be used in searching for ore bodies and studying internal composition of the Earth.

The first measurements of g with an accuracy of up to 10^{-3} cm/s^2 (milligal) were performed at the beginning of the 20-th century by means of Kater's pendulums. Such an accuracy requires the accuracy of pendulum periods of 10^{-6} s and the accuracy of equivalent length of $1\mu m$. Modern methods of measurement of g are divided into dynamic and static. The dynamic methods include the measurements with the aid of pendulums, in particular, Kater's pendulums. However these measurements can be made precise only in laboratory conditions and take a lot of time. This is also true for string gravimeters in which the frequency of string oscillations is determined by its tension due to a suspended weight.

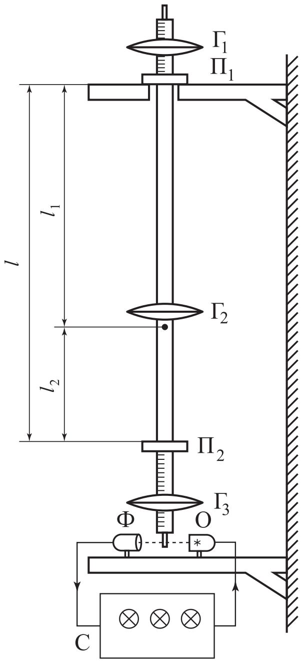

Fig. 1. Kater's pendulum

Recently the accuracy of measurement of length and time intervals has been significantly improved, so it becomes possible to measure free fall acceleration directly. For example, using a laser interferometer and an atomic clock to measure the path and time interval covered by a body equipped with a corner reflector which falls in an evacuated tube allows one to reach the accuracy of $3 \cdot 10^{-6}$ cm/s^2. The dynamic methods are used to measure the absolute value of free fall acceleration. Static methods allow one to measure a relative difference in the gravitational acceleration with an accuracy of up to $1.5 \cdot 10^{-5}$ cm/s^2. The static methods employ measurements of spring deformations or torsional deformations of horizontal strings due to suspended weights. To reduce temperature effects the springs and strings are made of quartz. The static method is difficult to use for precise measurements of the absolute value of gravitational acceleration because a dependence of the load on deformation deviates from Hooke's law. The relative variations of g measured by a static method are then compared to the reference points in which the absolute values are obtained by dynamical methods. This is how gravimetric maps are produced.

The equivalent length of a compound pendulum is determined by Eq. (4.38):

$$T = 2\pi\sqrt{\frac{I}{mga}}. \tag{3}$$

Here I is the moment of inertia of the pendulum about the pivot, m is the pendulum mass, and a is the distance from the pivot to the center of mass.

The pendulum mass and oscillation period can be measured with a high accuracy while the moment of inertia cannot. Usage of Kater's pendulum allows one to exclude the moment of inertia from the equation for g.

The method of Kater's pendulum is based on the observation that the period of a compound pendulum remains the same when the pivot is placed in the center of oscillation, i.e. the point separated from the pivot at the distance equal to the equivalent length and located on the same vertical with the pivot and the center of mass.

The pendulum used in the lab (see Fig. 1) consists of a steel plate (or a rod) to which two identical prisms Π_1 and Π_2 are attached. The oscillation period of the pendulum can be varied by means of movable weights Γ_1, Γ_2, and Γ_3.

Suppose one has attached the weights so that the periods T_1 and T_2 of pendulum oscillations on the prisms Π_1 and Π_2 are the same, i.e.

$$T_1 = T_2 = T = 2\pi\sqrt{\frac{I_1}{mgl_1}} = 2\pi\sqrt{\frac{I_2}{mgl_2}}, \tag{4}$$

where l_1 and l_2 are the distances from the center of mass to prisms Π_1 and Π_2.

This condition is met providing the equivalent lengths, I_1/ml_1 and I_2/ml_2, are the same. According to Huygens-Steiner theorem

$$I_1 = I_0 + ml_1^2, \qquad I_2 = I_0 + ml_2^2, \tag{5}$$

where I_0 is the moment of inertia of pendulum about the axis through the center of mass and parallel to the pivot. Excluding I_0 and m from Eqs. (4) and (5) one obtains the equation for g:

$$g = \frac{4\pi^2}{T^2}(l_1 + l_2) = 4\pi^2\frac{L}{T^2}. \tag{6}$$

Here $L = l_1 + l_2$ -is the distance between prisms Π_1 and Π_2 which can be measured with an accuracy of 0.1 mm with the aid of a large caliper. Summation of the lengths l_1 and l_2 is less accurate since the corresponding error is several millimeters.

Notice that Eq. (6) follows from Eqs. (4) and (5) providing

$$l_1 \neq l_2, \tag{7}$$

since Eqs. (4) and (5) become identities for $l_1 = l_2$.

Equation (6) is derived under assumption that $T_1 = T_2$. Actually it is not possible to equate the periods precisely. In general

$$T_1 = 2\pi\sqrt{\frac{I_0 + ml_1^2}{mgl_1}}, \qquad T_2 = 2\pi\sqrt{\frac{I_0 + ml_2^2}{mgl_2}}.$$

Then

$$T_1^2 g l_1 - T_2^2 g l_2 = 4\pi^2(l_1^2 - l_2^2),$$

and

$$g = 4\pi^2 \frac{l_1^2 - l_2^2}{l_1 T_1^2 - l_2 T_2^2} = 4\pi^2 \frac{L}{T_0^2}, \tag{8}$$

where

$$T_0^2 = \frac{l_1 T_1^2 - l_2 T_2^2}{l_1 - l_2} = T_2^2 + \frac{l_1}{l_1 - l_2}(T_1 + T_2)(T_1 - T_2). \tag{9}$$

The error of g can be found from Eq. (8):

$$\frac{\sigma_g}{g} = \sqrt{\left(\frac{\sigma_L}{L}\right)^2 + 4\left(\frac{\sigma_{T_0}}{T_0}\right)^2}. \tag{10}$$

To evaluate the error σ_{T_0} let us examine how the period of oscillation depends on the distance l between the center of mass and the pivot. To do so we express moment of inertia I via I_0 using Eq. (5):

$$T = 2\pi\sqrt{\frac{I_0 + ml^2}{mgl}}. \tag{11}$$

This function is shown in Fig. 2. When $l \to 0$ the period goes to infinity as $l^{-1/2}$. When $l \to \infty$ the period goes to infinity as $l^{1/2}$. The minimum of the period is at $l_{\min} = \sqrt{I_0/m}$. Every value of T for $T > T_{\min}$ is repeated twice for two different values of l, one of them is greater than $l_{\min}$ and the other is less. These values were used in Eqs. (4) – (6). The plot shows that the values of the quantities l_1 and l_2 diverge when T grows.

Let us determine how the error of T_0 depends on the difference $l_1 - l_2$. To this end let us find how σ_{T_0} depends on the error of T_1. Differentiating the first equation of (9) at constant T_2 we obtain:

$$2T_0(dT_0)_{T_2} = \frac{l_1}{l_1 - l_2} 2T_1 dT_1, \qquad (dT_0)_{T_2} = \frac{l_1}{l_1 - l_2} \cdot \frac{T_1}{T_0}\, dT_1.$$

Similarly we obtain at constant T_1:

$$(dT_0)_{T_1} = -\frac{l_2}{l_1 - l_2} \cdot \frac{T_2}{T_0}\, dT_2.$$

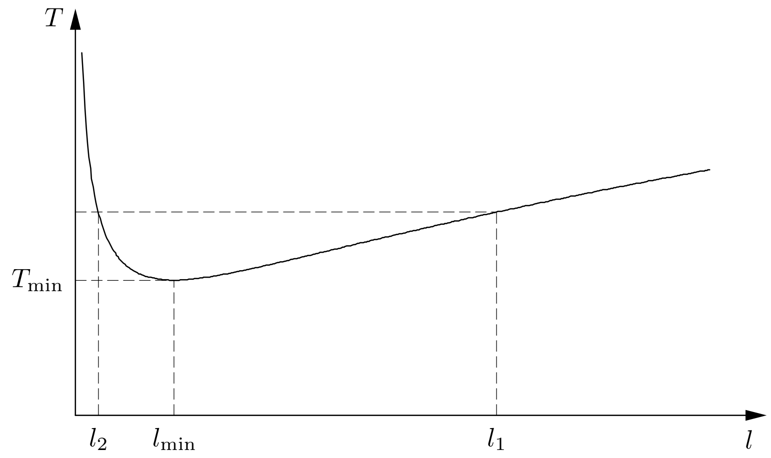

Fig. 2. Oscillation period versus distance between center of mass and pivot

Now consider the case when l_1 and l_2 are close. The denominator is small and the error of T_0 grows sharply. Therefore the period of oscillations must be chosen so that l_1 and l_2 are significantly different. If they differ by a factor of 1.5 the error of T_0 exceeds the error of T_1 by less than an order of magnitude.

Let us derive the equation for dT_0. Consider the second equality in Eq. (9). Notice that $T_1 \approx T_2$, so the difference $T_1 - T_2$ is small. Therefore the second term in the equation can be regarded as a minor correction as long as $l_1 - l_2$ is not large.

Therefore the errors of l_1 and l_2, if taken into account, will be multiplied by a small difference $T_1 - T_2$ and can be neglected in calculation of σ_{T_0}. This is true even for the errors of several millimeters typical for this lab. Now, since the errors of T_1 and T_2 are independent and approximately equal the general formula (1.33) gives finally:

$$\sigma_{T_0} \approx \frac{\sqrt{l_1^2 + l_2^2}}{l_1 - l_2}\sigma_T, \tag{12}$$

where σ_T is the error of the period.

One can see that the error does not significantly depend on the accuracy of the equality $T_1 = T_2$. Therefore, as soon as the equality holds within several percent, a further improvement is not necessary.

Finally notice that the ratio l_1/l_2 should not be too large. Indeed l_1 is always less than the distance L between the prisms. The quantity l_2

becomes small for large l_1/l_2 and the period of oscillations grows sharply (recall that I is always greater than I_0). This increases the duration of experiment and an uncertainty due to friction which is not taken into account in derivation of Eq. (3).

Let us quantify this statement. The contribution due to friction can be determined as the ratio of the work done by friction forces to the energy of oscillation. The work of friction depends on l_2 only slightly because the work is the product of the torque due to friction (which is almost independent of l_2) and deflection angle which is completely independent of l_2. The energy of oscillation equals the potential energy of the pendulum, i.e.

$$W_{osc} = mgl_2(1 - \cos\varphi),$$

where φ is the deflection angle of the pendulum. So, the less l_2, the less W_{osc}.

Thus we conclude that the ratio of l_1 to l_2 should be neither too small nor too large. A preferred value lies in the range:

$$1{,}5 < \frac{l_1}{l_2} < 3. \tag{13}$$

Laboratory setup. The design of Kater's pendulum is shown in Fig. 1. The distance L between the prisms Π_1 and Π_2 is fixed. Distances l_1 and l_2 can be varied by moving weights Γ_1, Γ_2 and Γ_3.

The number of oscillations is measured by a counter which consists of a spotlight, a photocell, and a digital counter. A light rod attached to the pendulum end crosses the beam of light twice a period. Pulses generated by the photocell are registered by the digital counter. If n_1 and n_2 are the initial and final readings of the counter during time t, the number of periods is, obviously, equal to $N = (n_2 - n_1)/2$ and the oscillation period is $T = t/N$. Time t is measured by the stopwatch mounted on the counter. To measure l_1 and l_2 one should remove the pendulum from its support and place it on the special horizontal bar which has a sharp edge. Then one should find the position of the center of mass by balancing the pendulum on the bar. The distances from the bar to the prisms are l_1 and l_2. If they differ significantly (see Eq. (13)) and the periods T_1 and T_2 are close, the accuracy of measurement of l_1 and l_2 need not be high according to Eq. (9).

LABORATORY ASSIGNMENT

1. Study Kater's pendulum design.

2. Find the working range of oscillation amplitudes in which oscillation period can be considered as independent of the amplitude. To do so put the pendulum on a prism, deflect it from the vertical by an angle φ_1 ($\sim 10°$), and measure the time of 100 full swings. Find the period T_1. Repeat the experiment by decreasing the initial deflection by a factor of 1.5–2 and find the period T_1'. If the periods coincide within the measurement accuracy, any initial amplitude φ which does not exceed φ_1 can be chosen for further measurements. If it turns out that $T_1 \neq T_1'$, take the second value of the initial amplitude as φ_1 and repeat the experiment. It is not recommended to take the initial amplitude greater than 10° since the prism can possibly slide on the support.
3. Figure out how the oscillation periods T_1 and T_2 (the pivot point on the prism Π_1 and Π_2, respectively) depend on the position of weights Γ_1, Γ_2 and Γ_3. It would suffice to measure the time of 10–15 full swings. It is necessary to determine

   a) which of the weights has the greatest effect on T_1 and T_2, and which one has the least;

   b) which of the weights has the greatest effect on the difference $|T_1 - T_2|$. Does a weight displacement changes the periods T_1 and T_2 in the same direction? Do the experiments for all the weights.
4. By moving the weight which has the greatest effect on the difference $|T_1 - T_2|$ (usually it is Γ_2) make the periods roughly coincide. Determine T_1 and T_2 by 10–15 full swings. Remove the pendulum from the support, locate its center of mass, and measure the distances l_1 and l_2. As it was already mentioned, they should differ by a factor of no less than 1.5 and no more than 3.
5. By moving the weight which has the least effect on the periods, make T_1 and T_2 coincide within one percent accuracy. Check whether the values l_1 and l_2 satisfy inequalities (13). The final measurement should be performed using 200–300 full swings. By the way make sure that friction has no significant effect on the oscillations, i.e. the amplitude of oscillations decreases no more than by a factor of 2–3 during the 200–300 full swings.
6. Using Eqs. (8) and (9) calculate the gravitational acceleration. Evaluate the error and compare the result with the tabulated value.

Questions

1. How do temperature variations, friction, and the amplitude of oscillations affect the accuracy of the experiment?
2. What distance from the pivot to the center of mass corresponds to the minimum oscillation period?
3. Show that the center of mass lies between the pivot and the center of oscillations.

4. Prove Huygens-Steiner theorem.
5. Show that if the pivot is placed in the center of oscillations the period of oscillations remains the same.

Literature

1. *Сивухин Д.В.* Общий курс физики. Т. I. — М.: Наука, 1996. §§ 35, 36, 41, 66.
2. *Стрелков С.П.* Механика. — М.: Наука, 1975. §§ 50, 124.

Lab 1.4.3

Study of non-linear oscillations of a long-period pendulum

Purpose of the lab: determination of the dependence of oscillation frequency on amplitude

Tools and instruments: long-period pendulum, stopwatch

Equation of the pendulum motion. A pendulum used in the lab consists of two identical weights fixed on a rigid rod; the rod can rotate about a horizontal axis which is slightly off the center of mass of the system. An arrangement of the rod and the weights is shown in Fig. 1, the names of the variables used are indicated in the same figure.

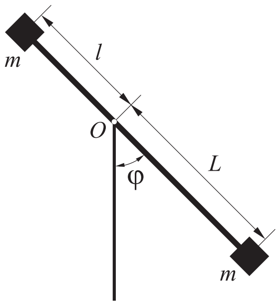

Fig. 1. Long-period pendulum

The pendulum oscillations are due to the torque of the gravitational force. The motion of the pendulum is specified by the dependence of the deviation angle φ (measured from the equilibrium position) on time. The torque M_g due to the gravitational force, which tends to return the system to the equilibrium, can be written as

$$M_g = -\left(m + \rho\frac{L+l}{2}\right) g(L-l)\sin\varphi.$$

In what follows we shall assume that the drag force, which is responsible for the oscillation damping, is directly proportional to the velocity. This fact is consistent with the experimental data providing the velocity is not large. In the case considered the drag force depends on the air viscosity and on the friction in the bearings of the pendulum axis which is too small to be taken into account. Thus the torque M_d of the drag force equals

$$M_d = -b(L^2 + l^2)\dot\varphi,$$

where b is a constant.

The pendulum moment of inertia I about the rotational axis is equal to the sum of the moments of inertia of its constituents (the weights and the rod) about the same axis:

$$I = m(L^2 + l^2) + \rho\frac{L^3 + l^3}{3},$$

where m is the mass of each of the weights (considered as point masses) and ρ is the linear density of the rod.

Consequently, the equation of the rotational motion of the pendulum,

$$I\ddot\varphi = M_g + M_d,$$

becomes

$$I\ddot\varphi = -\left(m + \rho\frac{L+l}{2}\right) g(L-l)\sin\varphi - b(L^2 + l^2)\dot\varphi,$$

or

$$\ddot\varphi + 2\beta\dot\varphi + \omega_0^2\sin\varphi = 0, \qquad (1)$$

where

$$2\beta = b\frac{L^2 + l^2}{I} \quad \text{и} \quad \omega_0^2 = \left(m + \rho\frac{L+l}{2}\right)\frac{g(L-l)}{I}.$$

Neglecting the rod mass compared to the masses of the weights one can write down

$$2\beta = \frac{b}{m}, \qquad \omega_0^2 = g\frac{L-l}{L^2+l^2}.$$

Small-amplitude oscillations. For small deviation angles $\sin\varphi \approx \varphi$; when plugged into Eq. (4.27) it gives the equation of small-amplitude damped oscillations

$$\ddot\varphi + 2\beta\dot\varphi + \omega_0^2\varphi = 0. \qquad (2)$$

The solution of the equation is given by (4.28)

$$\varphi = ae^{-\beta t}\cos(\omega t + \alpha), \qquad (3)$$

where (see eq. (4.29))

$$\omega^2 = \omega_0^2 - \beta^2. \qquad (4)$$

The constants a and α are determined by initial conditions.

From Eq. (4) one can see that damping decreases the oscillation frequency and thus increases the period. To estimate the magnitude of the effect (assuming a small damping: $\beta^2 \ll \omega^2$) we rewrite Eq. (4) as $\Delta\omega^2 = -\beta^2$ and obtain

$$\frac{\Delta T}{T} = -\frac{\Delta\omega}{\omega} = -\frac{2\omega\Delta\omega}{2\omega^2} = -\frac{\Delta\omega^2}{2\omega^2} = \frac{\beta^2}{2\omega^2} = \frac{\beta^2 T^2}{8\pi^2}.$$

Using Eqs. (4.31) and (4.32) one finally obtains

$$\frac{\Delta T}{T} = \frac{\delta^2}{8\pi^2}, \quad \textit{where} \quad \delta = \beta T = \ln\frac{a_i}{a_{i+1}}. \tag{5}$$

Equation (5) allows one to estimate the influence of the damping on the oscillation period as the amplitudes a_i can be easily measured. We assume that the correction (5) due to damping is small compared to the correction due to the non-linearity of the oscillations. However, this assumption should be experimentally verified.

Non-linear oscillations. An equation of large-amplitude undamped oscillations can be obtained by setting $\beta = 0$ in (1)

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0. \tag{6}$$

This equation is non-linear[1]. For small deviation angles $\sin\varphi \approx \varphi$ eq. (6) is linear and coincides with the equation of the harmonic oscillator (4.4).

The dependence of the period of non-linear oscillations on the amplitude can be obtained by integrating the relation

$$dt = \frac{d\varphi}{\dot{\varphi}} \tag{7}$$

from $t = 0$ to, e.g. $t = T/4$. To find the angular velocity $\dot{\varphi}$ and the deviation angle φ one should multiply Eq. (6) by $\dot{\varphi}$

$$\dot{\varphi}\ddot{\varphi} + \dot{\varphi}\omega_0^2 \sin\varphi = 0$$

and integrate once:[2]:

$$\frac{\dot{\varphi}^2}{2} + \omega_0^2(\cos\varphi_m - \cos\varphi) = 0, \tag{8}$$

[1] We remind that linear equations are those in which all terms are the first powers of functions and their derivatives. In eq. (6) the non-linearity is due to the sine function. In other cases there could be polynomial or more complicated functions

[2] One can also obtain (8) from the energy conservation law

where φ_m is the maximum deviation angle. From here it follows that

$$\dot{\varphi}^2 = 2\omega_0^2(\cos\varphi - \cos\varphi_m) = 4\omega_0^2\left(\sin^2\frac{\varphi_m}{2} - \sin^2\frac{\varphi}{2}\right),$$

$$\dot{\varphi} = 2\omega_0 \sin\frac{\varphi_m}{2}\sqrt{1 - \frac{\sin^2\frac{\varphi}{2}}{\sin^2\frac{\varphi_m}{2}}}. \tag{9}$$

Using Eqs. (9) and (7) one obtains

$$T = 4\int_0^{T/4} dt = 4\int_0^{\varphi_m}\frac{d\varphi}{\dot{\varphi}} = \frac{4}{2\omega_0\sin\frac{\varphi_m}{2}}\int_0^{\varphi_m}\frac{d\varphi}{\sqrt{1 - \frac{\sin^2\frac{\varphi}{2}}{\sin^2\frac{\varphi_m}{2}}}}. \tag{10}$$

Introducing a new variable θ

$$\sin^2\theta = \frac{\sin^2\frac{\varphi}{2}}{\sin^2\frac{\varphi_m}{2}} \tag{11}$$

we can rewrite an expression for the oscillation period T as

$$T = T_0 \cdot \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - \sin^2\frac{\varphi_m}{2}\sin^2\theta}}. \tag{12}$$

Here $T_0 = 2\pi/\omega_0$ is the period of small-amplitude (linear) oscillations. The integral (12) is not expressed via primitive functions but it can be worked out by Taylor expanding of the integrand. This gives the following dependence of the oscillation period on the amplitude

$$\frac{T}{T_0} = 1 + \frac{1}{4}\sin^2\frac{\varphi_m}{2} + \frac{9}{64}\sin^4\frac{\varphi_m}{2} + \ldots \tag{13}$$

For relatively small angles one obtains:

$$T \approx T_0\left(1 + \frac{\varphi_m^2}{16}\right). \tag{14}$$

In Fig. 2 the rigorous solution (13) (solid line) and the approximate one (14) (dashed) are depicted. At 90°-amplitudes the discrepancy between the solutions is about 2% while a non-linear contribution to the period is about 15–20% and can be measured with a simple stopwatch if the oscillation period is about 10 seconds.

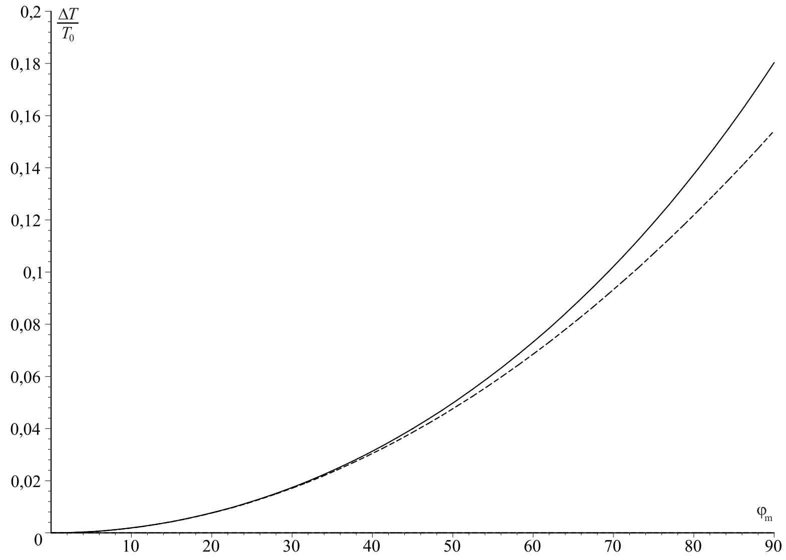

Fig. 2. Dependence of oscillation period on amplitude

Both non-linearity and damping affect the pendulum oscillation period (6). We have considered the contribution of each of the factors independently assuming that the other one is negligible. In fact these factors act simultaneously and the oscillation period is a complicated function of the damping decrement and the amplitude. But if the correction to the period is small, one can use the Taylor expansion of the function of two variables:

$$f(x,y) \approx f(0,0) + \frac{\partial f(0,0)}{\partial x}x + \frac{\partial f(0,0)}{\partial y}y,$$

which gives for eqs. (5) and (14):

$$T(\delta^2, \varphi_m^2) \approx T_0\left(1 + \frac{\delta^2}{8\pi^2} + \frac{\varphi_m^2}{16}\right). \tag{15}$$

One can see that in the first order the damping and the non-linearity contributions are independent.

LABORATORY ASSIGNMENT

1. Adjust the position of the weights on the rod so that the pendulum oscillation period is 5-10 seconds.

2. Release the pendulum without pushing from the initial 80–90° and deviation angle and start the stopwatch simultaneously.
3. Each time the angle reaches its maximum value tabulate the number of the periods n passed from the start of motion, the maximum deviation angle φ_n, and the stopwatch readings.
4. Repeat the experiment several times for various oscillation periods.
5. Estimate the effect of damping on the pendulum oscillations. For this purpose plot the values $\ln\varphi_n$ vs n, calculate the slope of the line and extract the value of the logarithmic decrement δ (see Eq. (5)). Using (5) estimate the contribution of the damping to the oscillation period and ascertain that it is small compared to the effect observed (or compared to the expected value calculated from Eq. (14)). Otherwise one should introduce the correction for the damping using Eq. (15) and use the value $T_0 - \frac{\delta^2}{8\pi^2}$ instead of the small-oscillation period T_0.
6. Plot the dependence of the oscillation period T vs. the maximum deviation angle squared φ^2 (measured in radians). Compare your result with the theoretical prediction (14).

Questions

1. How does the pendulum oscillation period depend on damping?
2. Discuss the design of a moderate size pendulum which has a large oscillation period. Could a conventional pendulum be used in the lab instead?
3. Discuss the dependence of the pendulum oscillation period on the amplitude.

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Lab 1.4.4

Study of oscillations of coupled pendulums

Purpose of the lab: to study an oscillator with two degrees of freedom

Tools and instruments: a setup of two identical bifilar gravity pendulums suspended on a tight horizontal string, a stopwatch, and a ruler

Prior to experiment read the paragraph concerning coupled pendulums in the introduction to this chapter.

The measurements are performed using the setup shown in Fig. 1.

One of the string ends is rigidly attached to the vertical support, while the other end runs over the sheave and is kept tight by the weight of mass M. Points A and B of the string are fixed. Points C and D divide the distance between A and B into three equal segments of length a each; identical gravity pendulums of mass m and length l are suspended at these points. Each pendulum is suspended on two threads (bifilarly) in the string plane, so that oscillations occur in the plane orthogonal to the string. String tension is much greater than the weight of the pendulums provided $M \gg m$. Vertical displacement of the string from equilibrium does not affect motion of the pendulums if oscillation amplitudes are small. Although horizontal displacement of the string is also rather small compared to the pendulum displacements, it provides weak coupling between the pendulums.

The displacements of points C and D of the string and both vertical (Fig. 2a) and horizontal (Fig. 2b) displacements of pendulums are shown in Fig. 2.

Assuming small displacements of the pendulums we obtain the following expression for tension T (see Fig. 2a)

$$mg \approx T. \tag{1}$$

Dynamic equations governing the horizontal components of pendulum displacements are (Fig. 2):

$$m\ddot{x}_1 = -T\sin\varphi_1 \approx -T\frac{x_1 - x_3}{l} \approx -mg\frac{x_1 - x_3}{l}, \tag{2}$$

$$m\ddot{x}_2 = -T\sin\varphi_2 \approx -T\frac{x_2 - x_4}{l} \approx -mg\frac{x_2 - x_4}{l}. \tag{3}$$

The relation between the string and suspension tensions can be obtained from Fig. 2:

$$T\frac{x_1 - x_3}{l} = F\frac{x_3}{a} + F\frac{x_3 - x_4}{a}, \tag{4}$$

$$T\frac{x_2 - x_4}{l} = F\frac{x_4}{a} + F\frac{x_4 - x_3}{a}. \tag{5}$$

Let us introduce a dimensionless parameter

$$\sigma = \frac{T}{F}\frac{a}{l} = \frac{m}{M}\frac{a}{l},$$

which is much less than unity in our case (weak coupling). Thus from Eqs. (4) and (5) we obtain

$$\sigma x_1 = (2+\sigma)x_3 - x_4, \qquad \sigma x_2 = (2+\sigma)x_4 - x_3. \tag{6}$$

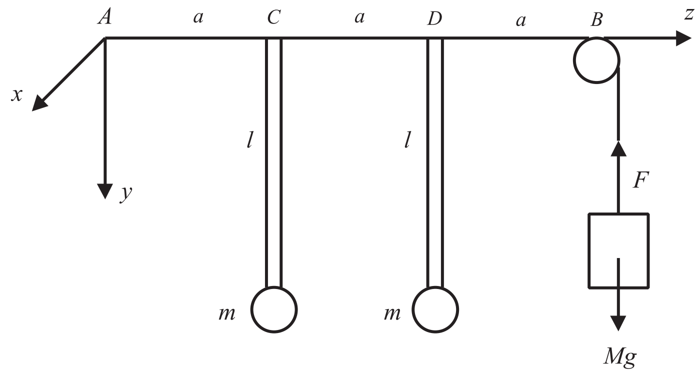

Fig. 1. Scheme of experimental setup

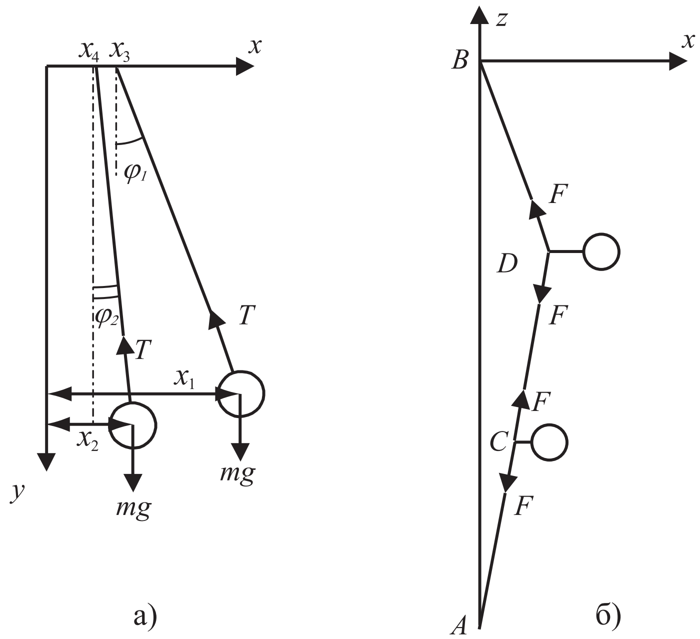

Fig. 2. Displacements of pendulums and string (a) view along the string, (b) top view

Neglecting σ compared to 2 we arrive at

$$x_3 = \sigma\frac{2x_1 + x_2}{3}, \qquad x_4 = \sigma\frac{x_1 + 2x_2}{3}. \tag{7}$$

Then the equations of pendulum motion become:

$$\ddot{x}_1 + \frac{g}{l}(1-\sigma)x_1 = \sigma\frac{g}{3l}(x_2 - x_1), \tag{8}$$

$$\ddot{x}_2 + \frac{g}{l}(1-\sigma)x_2 = \sigma\frac{g}{3l}(x_1 - x_2). \tag{9}$$

Notice that the system of equations (4.60)-(4.61) can be rewritten as

$$\ddot{\varphi}_1 + \frac{g}{l}\varphi_1 = \frac{g}{l}\varepsilon(\varphi_2 - \varphi_1),$$

$$\ddot{\varphi}_2 + \frac{g}{l}\varphi_2 = \frac{g}{l}\varepsilon(\varphi_1 - \varphi_2)$$

or

$$\ddot{\varphi}_1 + \omega_0^2\varphi_1 = \omega_0^2\varepsilon(\varphi_2 - \varphi_1), \tag{10}$$

$$\ddot{\varphi}_2 + \omega_0^2\varphi_2 = \omega_0^2\varepsilon(\varphi_1 - \varphi_2). \tag{11}$$

Equations (10) and (11) coincide with Eqs. (8) and (9) except for the notations. One can introduce the quantities

$$\frac{g}{l}(1-\sigma) = \omega_0^2, \qquad \sigma\frac{g}{3l} = \omega_0^2\varepsilon$$

and thus obtain

$$\frac{\sigma}{1-\sigma} = 3\varepsilon$$

or

$$\sigma(1+\sigma) \approx 3\varepsilon,$$

i.e.

$$\sigma \approx 3\varepsilon \qquad \text{(for weak coupling)}.$$

Now Eqs. (8) and (9) become

$$\ddot{x}_1 + \omega_0^2 x_1 = \omega_0^2\varepsilon(x_2 - x_1), \tag{12}$$

$$\ddot{x}_2 + \omega_0^2 x_2 = -\omega_0^2\varepsilon(x_2 - x_1). \tag{13}$$

Thus all theoretical results derived in the introduction to this chapter are valid for this experiment. In particular, energy transfer from one pendulum to another and vice versa takes the time (4.79):

$$\tau = \frac{2\pi}{\omega_0\varepsilon}. \tag{14}$$

One can see that the coupling parameter can be written as

$$\varepsilon = \frac{1}{3}\left(1 - \frac{\omega_0^2 l}{g}\right). \tag{15}$$

Using Eq. (15) one can rewrite the relation (14) as

$$\tau = \frac{6\pi}{\omega_0(1 - \omega_0^2 l/g)} \approx 6\pi\frac{Ml}{ma}\sqrt{\frac{l}{g}}. \tag{16}$$

Equation (16) can be experimentally verified by measuring the partial frequency of a pendulum, its length, and the time of energy transfer.

LABORATORY ASSIGNMENT

1. Measure the pendulum lengths, the distance between fixed points of the string and between pendulum suspension points. Write down the pendulum masses and the weight which keeps the string tight.
2. Measure the periods of normal oscillation modes. To measure the period of in-phase oscillations T_1 deflect the pendulums from the vertical by equal angles (about 30°) in the same direction and release them simultaneously. Time readouts should be taken when the pendulums pass through their equilibrium positions (about 10 oscillations). Repeat the measurement 2–3 times and average the results. To measure the period of antiphase oscillations T_2 the initial deflections should be in the opposite directions.
3. Measure the periods of partial oscillations. For this purpose one of the pendulums should be detached or put on a support.
4. Observe swinging of one pendulum by another. For this purpose deflect only one pendulum and measurethe period of beatings τ.
5. Check validity of the relation

$$\frac{1}{\tau} = \frac{1}{T_1} + \frac{1}{T_2}. \tag{17}$$

6. Repeat the previous measurements for different string tensions.
7. Plot the dependence of the beatings period on the string tension.

8. Compare the results obtained with the theoretical predictions given by Eq. (16).

Questions

1. Give some examples of oscillators with two degrees of freedom.
2. What are normal oscillations (normal modes)?
3. What are partial oscillations?
4. At which initial condition does the swinging of pendulums occur in turn?
5. Derive the equation (17).

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Lab 1.4.5

Study of string oscillations

Purpose of the lab: to study the dependence of the frequency of string oscillations on the tension; to study the formation of standing waves on the string.

Tools and instruments: bar with a fixed string, audio-frequency generator, constant magnet, weights

One of the main properties of a string is its flexibility which is due to a large ratio of the string length to its diameter. Even strings made of stiff materials almost do not resist a bending if the size of the bent section is much greater than the string diameter. This fact allows us to neglect the stress due to bending in this lab.

A horizontal string with fixed endpoints sags in a gravitational field when poorly tightened. Increasing the tension will straighten the string almost to a straight line; in this situation the tension is sufficiently greater than the weight of the string. For this reason we will neglect the gravity when considering straightly tightened strings.

A tight string with fixed ends is well suited for the study of oscillation processes since it makes possible a direct observation of the simplest types of oscillations and waves excited on the string. It is also possible to determine the parameters of the oscillations and compare the results with theoretical predictions.

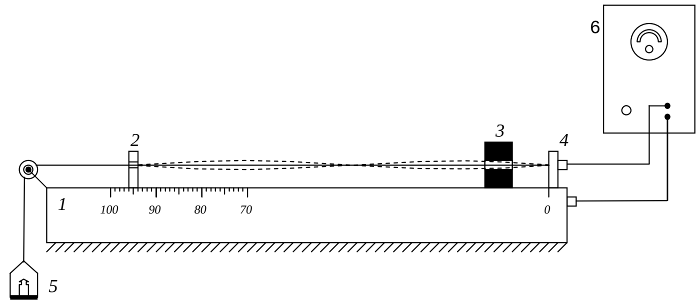

Fig. 1. Experimental setup

Motion of string segments can be caused by a perturbation of the string shape or by a transmission of momentum along the string. The string tension tends to restore its initial straight shape, which results in the motion of string segments. The perturbation propagates along the string.

From eq. (4.95) one obtains an expression for the speed of a transverse wave propagating along a string

$$u = \sqrt{\frac{F}{\rho_l}}, \tag{1}$$

where F is the tension, ρ_l is the mass of the string per unit length. For a given frequency ν the wavelength is

$$\lambda = \frac{u}{\nu}. \tag{2}$$

The frequencies of normal modes of the string are given by eq. (4.97):

$$\nu_n = n\frac{u}{2l}, \tag{3}$$

where l is the string length, n is the number of half-wavelengths.

Laboratory setup. The experimental setup is shown in Fig. 1. Bearings 2 and 4 and magnet 3 are placed on massive bar 1, the bearing 2 and the magnet 3 can be moved along the bar while the bearing 4 is fixed. One of the string ends is fixed in the bearing 4. Then the string is threaded between the poles of the magnet, the bearing 4 (which allows for horizontal string displacements), and the fixed block. Plate 5 is suspended on the loose end of the string; by placing different weights on the plate one can vary a string tension.

An alternating voltage generated by the audio-frequency generator 6 is applied between the massive bar 1 and the string end fixed in the bearing 4. An Ampere force due to the magnetic field acting on the current makes the string vibrate. The frequency of the force swinging the string is equal to the frequency of the current oscillations, i.e. the frequency of the generator.

The Ampere force results in string oscillations and wave propagation; the waves are reflected by the bearings 2 and 4 and interfere, which results in a standing wave provided the string length is an integer of half-wave-lengths.

In real experiments there always exist losses of energy due to air friction, transmission of energy to the bearings, irreversible processes in the string, etc. To maintain the oscillations one needs to supply energy to the string. In a stationary regime the amount of the supplied energy equals the amount of the dissipated energy. In the experimental setup the Ampere force not only excites the string oscillations but also maintains them.

In this situation the energy flux propagates along the string. But the energy propagation in a pure standing wave is prohibited (see the introduction to this chapter). Therefore a traveling wave must exist, actually this leads to the smearing of the standing-wave nodes. If the energy losses per period are much less than the energy stored in the string a traveling-wave factor is much less than unity:

$$\frac{A_1 - A_2}{A_2} \ll 1. \tag{4}$$

Here A_1 and A_2 are the incident and the reflected wave amplitudes, respectively. In this case one can use the equations obtained for a pure standing wave. It is worth mentioning that the quantity $A_1 - A_2$ can be estimated by observing the smearing of the nodes; it equals half of the smearing amplitude. The wave amplitude in an antinode is $2A_2$.

If inequality (4) is not well satisfied, one should decrease the output power of the generator. This would decrease the rate of energy loss compared to the energy stored in the wave.

One more fact should be mentioned. The Ampere force will excite polarized waves with the plane of oscillations orthogonal to the direction of the magnetic field. In real experiments it is not always possible to obtain the linearly polarized waves.

LABORATORY ASSIGNMENT

1. Examine the experimental setup. Place the bearing 2 (Fig. 1) so that the length L of the oscillating part of the string is longer than 80 cm.
2. Turn on the power supply of the audio-frequency generator.

3. Set the harmonic output signal of the generator and the minimal range of the output frequencies.
4. Put some weights on the plate.
5. Move the magnet and vary the generator frequency to obtain a pattern of standing waves. (Moving the magnet along the string changes a location of the point where the Ampere force is applied. The point must be close to a node although they should not coincide.)
6. Increase the generator frequency at a constant tension and obtain the patterns of standing waves corresponding to $n = 1, 2, 3, \ldots.$ up to not less than 6. For each pattern write down the corresponding frequency; repeat the measurement by increasing and decreasing the generator frequency. Carry out this procedure for different values (at least five) of the string tension.
7. While carrying out the experiment check if inequality (4) holds. For this purpose one should measure a node smearing and the amplitude of oscillations in an antinode. If (4) is not well satisfied the output power of the generator must be reduced.
8. For each value of the string tension F plot the resonant frequency ν_n vs n. Calculate the slope of the curves and determine the wave velocity u using (3) at a given value of the tension. Estimate the error of the results.
9. Plot the wave velocity squared u^2 vs the string tension F. Calculate the slope of the line and determine the linear density ρ_l of the string using (1). Estimate the error and compare the result with the value written on the experimental setup.

Questions

1. What are longitudinal and transverse waves? Write down the wave equation.
2. Derive the wave equation. Give a definition of node and antinode of a standing wave. Describe an energy propagation along an oscillating string.
3. Prove that the velocity of transverse wave on a string equals $u = \sqrt{F/\rho_l}$. Compare this value with the velocity obtained in the experiment.
4. Describe the reflection of a wave from the fixed end and from the end which moves freely in a plane orthogonal to the direction of the string tension. What is the value of a phase shift between the incident and reflected waves?
5. What condition must be satisfied for a traveling wave not to affect the oscillation pattern? How can one check the condition experimentally?

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Lab 1.4.6

Measurement of speed of ultrasound in liquid by means of ultrasound interferometer

Purpose of the lab: to measure wavelength of ultrasound in different liquids by means of ultrasound interferometer and to calculate speeds of ultrasound and adiabatic compressibility of the liquids.

Tools and instruments: an ultrasound interferometer, frequency generator Г4-42, and an ammeter.

Sound waves with a frequency greater than 20 kHz are called ultrasound. Unlike sound, ultrasound is not perceptible by human ear.

Ultrasound waves can propagate in solids and fluids just like ordinary sound waves. In solids ultrasound propagates in the form of longitudinal and transverse waves; in fluids there are only longitudinal waves. The speed of ultrasound depends on elastic properties and density of the medium in which ultrasound propagates. Therefore elastic properties of a medium can be determined if the speed of ultrasound and the medium mass density are known.

In the lab the speed of ultrasound in liquid is measured. There are several methods of measuring the speed. The method of ultrasound interferometry used in the lab is one of the most precise.

A standing wave is excited between an emitter and a rigid reflecting surface. (See the introduction to the chapter.) The distance between the emitter and the reflector must be an integer multiple of half wavelengths:

$$l = n\frac{\lambda}{2}, \qquad c_s = \lambda\nu_n, \tag{1}$$

where c_s is the speed of ultrasound and ν_n is the wave frequency.

The interferometer can be considered as a resonator tuned to the frequencies derived from (1):

$$\nu_n = n\frac{c_s}{2l}. \tag{2}$$

These frequencies correspond to standing waves of the resonator, they are called resonant frequencies. Two adjacent resonant frequencies correspond to distances l between the emitter and the reflector separated by

$$\Delta l = \frac{\lambda}{2}. \tag{3}$$

Equation (3) is more general than (1). Indeed, Eq. (1) is derived on the assumption that both ends of the column of liquid are closed by absolutely

elastic walls which completely reflect the sound. This assumption is never satisfied, so a phase shift between the incident and reflected waves never equals π.

Equation (3), which specifies the distance between two consecutive resonances, is independent of the details of reflection from the top and bottom of the container. As long as a resonance is detected, further increment of the column height by $\lambda/2$ increases the path of the wave between two consecutive reflections by λ, so the phase changes by 2π and the next resonance occurs.

Consider a method of exciting the ultrasound. Usually one employs a flat quartz crystal placed between the plates of a capacitor (the plates are glued or thermally sprayed on the crystal). The size of the crystal changes periodically due to electric field (piezo-effect) of a desired frequency. The oscillations are then transferred to liquid.

Usually the quartz crystal is placed in the liquid to avoid extra surfaces reflecting the sound. In our case the plate is rigidly fixed to the container bottom. The oscillations are transferred to the liquid through the bottom which in ideal case would coincide with a node.

However oscillations cannot be excited at the node because there is no motion and no work can be done. This looks like a contradiction since energy must be transferred from the emitter to the liquid to compensate losses on the reflecting surfaces and due to internal friction. The bottom coincides with an oscillation node only for an ideal liquid in which there are no losses. No losses means no compensation. In a real liquid the energy losses are imminent and the bottom needs not be immobile.

In resonance and in the absence of energy losses, the amplitudes of the waves propagating in opposite directions are equal and their sum is a standing wave. In reality the amplitude A_{up} of the wave propagating upward from the emitter somewhat exceeds the amplitude A_{down} of the downward wave. The sum of the waves is a standing wave with the amplitude of $2A_{down}$ and a propagating wave with the amplitude of $A_{up} - A_{down}$. The propagating wave transfers energy and «blurs» the wave pattern at the nodes.

Now let us discuss how to measure a sound wavelength. It should already be clear that the measurement is essentially the measurement of the distance between two consecutive positions of the reflector for which a resonance occurs. According to Eq. (3) one finds the wavelength by doubling the distance obtained.

The speed of sound c_s can then be found from Eq. (1). In addition to the wavelength one should know the frequency of oscillations of the quartz crystal which coincides with the signal frequency.

Using the value of c_s one could determine compressibility χ of liquid:

$$c_s = \sqrt{\frac{1}{\chi\rho}}, \qquad \chi = \frac{1}{\rho c_s^2}, \tag{4}$$

where ρ is the liquid density. Since propagation of sound is an adiabatic process, this equation defines adiabatic compressibility χ_{ad}. The adiabatic and thermal compressibility of liquid do not differ much, e.g. for water the difference is 1%, the difference between them can often be neglected.

A strong electrolyte dissolved in water dissociates into ions. The electric field of an ion aligns the nearby water molecules that drastically reduces the compressibility. Roughly speaking, each ion becomes the center of a sphere which compressibility is almost zero. As a result the compressibility of the solution decreases and the speed of ultrasound rises sharply.

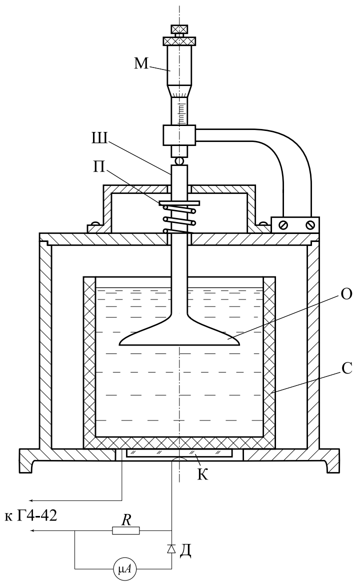

Fig. 1. Ultrasound interferometer

Laboratory setup. The interferometer used in the lab consists of cylinder C (see Fig. 1), quartz plate K is glued to its bottom. The plate is cut in a special way (the so called «X-cut») and possesses piezoelectric properties. Charges of opposite sign accumulate on the opposite crystal faces to which stress or compression is applied. The reverse effect is used in the interferometer: a periodic voltage applied to the horizontal crystal faces coated with silver makes the crystal oscillate. The alternating voltage is generated by the standard frequency generator Г4-42 which graduation scale has an error less than 1%. The generator has a resonance amplifier tuned to the eigenfrequency of the quartz plate (1 MHz). A voltage applied to the crystal is tens of volts.

The thickness of the container bottom is chosen so that resonance occurs in the working range of frequencies.

The container bottom excited by the quartz crystal transmits ultrasound to the bulk of liquid. This prevents a contact between the liquid and the quartz crystal and allows one to study even conducting liquids which otherwise would damage or short-circuit the crystal.

The current supplied to the crystal is controlled by an ammeter. The latter is connected in series with diode Д and in parallel with resistor

R which is included in the crystal supply circuit. The ammeter serves to detect resonance. A power consumed by the crystal rises sharply in resonance and so does a current through the resistor.

Disk O made of stainless steel serves as the interferometer reflector. Its lower surface is parallel to the container bottom. Micrometric screw M is used to move the disk up and down. Spring П lifts the rod Ш up thereby maintaining mechanical contact between the rod and the micrometric screw.

LABORATORY ASSIGNMENT

1. Turn on generator Г4-42 and let it warm up for several minutes. Empty the container by unclamping a hose if there is any liquid inside. Set the range of working frequencies of the generator, i.e. the range containing the eigenfrequency 1 MHz of the quartz crystal.

   Find the resonant frequency of the crystal by adjusting the frequency to achieve a maximum of the current. Using the knob «output level» choose the signal amplitude so that the ammeter readings are approximately 2/3 of the scale. Using the micrometric screw move the reflector down and watch the readings. If the readings exhibit periodic behavior make sure that it is due to resonance (e.g. consecutive maxima are separated by equal distances). It could happen that it is not possible to detect a resonance. This does not necessarily mean that the interferometer does not work properly since detecting resonance in air column requires more sensitive instruments than for liquids.

   Small deviations of the readings can be due to touching the micrometric screw. This changes the interferometer electrical capacitance and the output generator frequency as well. One could avoid such deviations by turning the screw carefully and keeping in touch with the screw knob.
2. Clamp the hose and fill the container with water using a funnel. Raise the reflector but keep its working surface under water. Make sure that the surface is free of air bubbles. Check the resonant frequency. Move the reflector down and watch the ammeter readings to determine how many half-wavelengths fit the distance traversed by the reflector.

   Plot the # of a maximum as abscissa and the maximum position as ordinate. Verify that the points lie on a straight line. Using Eq. (3) determine graphically the speed of ultrasound in water.

   Using Eq. (4) calculate the adiabatic compressibility χ_{ad} of water. Repeat the experiment 4–5 times. Estimate the error of c_s and χ_{ad}.
3. Repeat the experiment with `NaCl` water solutions with concentrations of 5, 10, 15, and 20%. Measure the solution density with a hydrometer. Plot c_s and χ_{ad} versus concentration. Using the plot determine the concentration

and χ_{ad} of a standard solution. Rinse the container with the standard solution before filling it.

At the end of the experiment the container must be rinsed with pure water.

Questions

1. Which mechanical oscillations are called ultrasonic?
2. What are longitudinal and transverse waves? In which media can the waves propagate?
3. Write down a mathematical expression for a plane wave.
4. What conditions should be met to make wave interference possible?
5. Derive an equation which specifies the condition of resonance in the interferometer. How does the equation depend on boundary conditions?
6. What conditions should be met to create a standing wave? Give definitions of node and anti-node. How is energy transferred in the wave?
7. Why is the speed of ultrasound greater in a solution of electrolyte than in the pure liquid?
8. Suppose the open surface of liquid is used instead of the metallic reflector. The height of the liquid column can be gradually varied by slowly emptying the container. What is the phase difference between the incident and reflected waves on the air-liquid boundary?
9. How should the interferometer be modified in order to do the same measurements with gases?

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Lab 1.4.7

Determination of elastic constants of liquids and solids via measurement of speed of ultrasound

Purpose of the lab: to measure the speed of sound in liquids and solids and to calculate elastic constants of the studied media using the results of measurements.

Tools and instruments: An ultrasound sensor, a gage post, a set of samples, a millimeter ruler, and prism probes.

Ultrasound is a mechanical oscillation with the frequency exceeding 20 kHz. Plane waves are the simplest type of ultrasound waves, they can be longitudinal and transverse. In longitudinal waves particle displacement coincides with the direction of wave propagation, in transverse waves it is perpendicular. Longitudinal ultrasound waves can propagate in any medium. Transverse waves propagate only in solids where shear stress is possible.

Under normal conditions, the speed of ultrasound is about 300 m/s in air, 1500 m/s in water, 5700 m/s in quartz, 6000 m/s in steel.

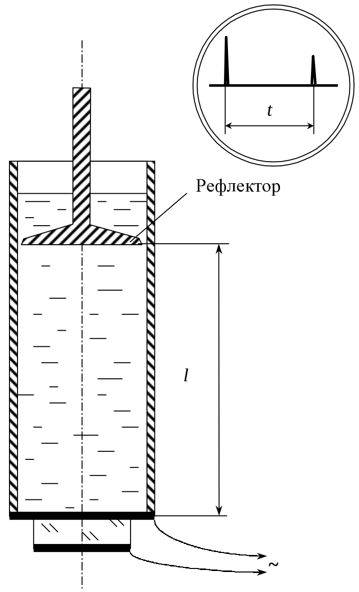

Fig. 1. Pulse method of ultrasound speed measurement

Generation and detection of ultrasound waves. Pulse method is one of popular methods of ultrasound speed measurement. A short pulse of ultrasound is sent to the tested medium and the time t of ultrasound propagation at some distance l is measured. The ultrasound speed is determined by the simple formula:

$$c_s = \frac{l}{t}. \tag{1}$$

An ultrasound pulse is generated by a piezoelectric transducer. The pulse is detected by a receiver, placed at some distance from the transducer. As an alternative the receiver can be replaced by a reflector (see Fig. 1). In this case the reflected pulse returns to the transducer, which not only generates but also detects ultrasound. When a scheme with the reflector is used the distance is passed twice, so the distance between the transducer and the reflector in Eq. (1) should be doubled.

To measure the time of pulse propagation it is convenient to use an oscilloscope which shows two pulses corresponding to the moment of signal emission and its return. The time t is determined from the distance between the pulses on the screen (the oscilloscope sweep is calibrated). The ultrasound speed measured by this method is the group velocity which is not the same as the phase velocity mentioned above. These two velocities are equal if there is no dispersion (dispersion is a dependence of the phase velocity on the wavelength).

Usually barium titanate piezoelectrical plates are used as transducers. ($BaTiO_3$). To excite both longitudinal and transverse waves in the body under study the so called prism probes are used. The transducer is located at some angle α to the working surface of the rectangular prism probe (see Fig. 2) which can be made of plexiglass. The transducer generates a longitudinal wave in plexiglass which is incident at the angle α onto the interface between the plexiglass and the studied body. At small angles of incidence the wave diffracted on the interface contains both longitudinal and transverse waves. As their speeds are different, two reflected pulses can be seen on the oscilloscope beside the initial one.

The probe should be glued to the sample to transmit transverse waves, a liquid lubricant will not do.

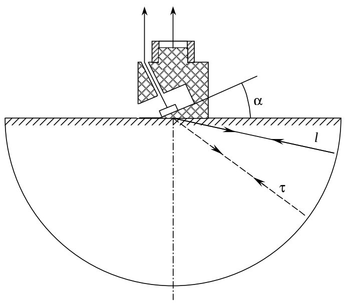

Fig. 2. Prism probe

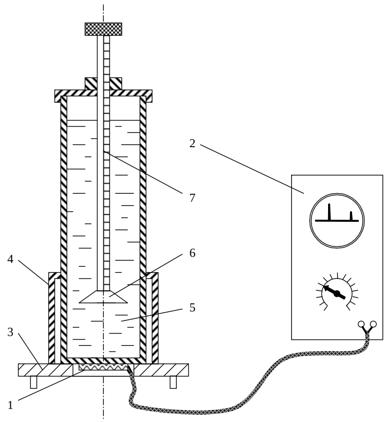

Fig. 3. Installation scheme

Laboratory setup. A standard ultrasound sensor is used to measure speed of ultrasound in liquids. (The instrument is designated for measuring the depth of defects under the surface of an object). A generator excites short high-frequency oscillation pulses in the transducer (made of barium titanate $BaTiO_3$). A pulse is transmitted into the sample through a thin layer of lubricant. After reflection from the opposite side of the sample the pulse returns to the transducer which converts it back to electric signal. Then the amplified signal is applied to the sensor CRT. Signals on the screen are seen as pulses: the transmitted one is at the beginning of the sweep and the reflected ones are located to its right. The distance between the pulses is proportional to the time t of ultrasound passage from the transducer to the reflective surface and backwards. This distance is

measured with the aid of a mark (a step on a sweep line) which can be moved along the line by the depth gauge control.

The installation setup is shown in Fig. 3. The transducer 1, connected to the ultrasound sensor 2 with a shielded cable is attached into the bottom 3 of the gage post. A studied rod or the cylindrical stainless steel vessel with liquid 5 is securely placed on the post in the support 4. A contact between the transducer and the sample is maintained by a layer of lubricant which transfers only longitudinal waves into the sample. In solid samples, the pulse is reflected from the top free end; in liquids the piston 6 made of stainless steel serves as the reflective surface, its height above the bottom is measured by the scale on the rod 7. Water is used to calibrate the depth gauge scale (the propagation speed is $c_s = 1497$ m/s at 25 °C and the temperature coefficient $dc_s/dt = 2.5$ m/(s·K)).

By measuring the ultrasound speed (and calibrating the device) one can measure the time interval between the transmitted and reflected pulses or between two sequentially reflected pulses. The latter method is preferable because the result does not include the error due to passage of the ultrasound through the bottom of the vessel.

To measure the speed of transverse ultrasound waves (as well as longitudinal ones), an installation with a prism probe should be connected to the ultrasound sensor instead of the gage post. The sample has a shape of a semicylinder. The probe is located on its axis (Fig. 2) so that the distances passed by longitudinal and transverse waves in the sample are the same (they are equal to the double radius of the semicylinder) and do not depend on the angle at which the waves enter the sample. An acoustic contact between the probe and the sample is achieved by means of a thin layer of mineral wax or BPh-2 adhesive. These substances can transmit tangential stress to the sample.

LABORATORY ASSIGNMENT

1. Plug the ultrasound sensor in the AC supply. Switch it on by turning the «Intensity» knob clockwise.
2. Warm up the sensor for 1–2 minutes, then obtain a clean and sharp image of the sweep line by turning the «Intensity» and «Focus» knobs. Set the beginning of the sweep at the left side of the screen using the «Shift X» knob. Set the «Frequency» switch to 5 MHz which corresponds to the resonant frequency of the transducer. Set other switches to the following positions: «Electronic magnifier» to «Off», «Measurement type» to «Smooth», «Automatic control area» to the outmost right position, «Sensitivity» to the middle position, «Time corrected gain» to the outmost right position, «Pulse power» to the outmost right position, «Cutoff» to the middle position, and I and I + II switches to I position.

3. Calibrate the scale of the depth gauge. For this purpose place a vessel filled with water into the measurement gage. Before placing the vessel or a sample do not forget to grease the emitter surface with light oil! Set the «Measurement type» switch to the «Д. Пр.» position. Using the «Sonic range» switch set the necessary range (in accordance with the distance t from the emitter to the surface of the reflective piston). Calibrate the scale using several (5–6) distances between the emitter and the piston. Plot the calibration curve in the coordinates of the depth scale marks and the calculated time of the pulse passage. The distance l is measured by means of the scale on the piston rod. The speed of ultrasound in water is given in the introductory section.
4. Measure the speed c_l of longitudinal ultrasound waves in the samples made of different materials (steel, aluminum, brass, organic glass, and so on) and liquids (tetrachloromethane and oil). Measure the length l of the solid samples using the millimeter ruler and the distance between the vessel bottom and the reflector using marks on the piston rod. The time of pulse passage can be determined by means of the depth gauge scale and the calibration curve. Calculate the speed of ultrasound in the materials under consideration.

   Hint. When carrying out the experiment make sure that the pulses chosen for the measurement correspond to two sequentially reflected pulses. Various ghost pulses can appear on the oscilloscope screen, e.g. those due to direct reflection from the sample bottom.

   For the shortest samples there is a minor difference in the amplitudes of reflected pulses, while for the longest samples the difference in the amplitudes of two sequentially reflected pulses can be quite large. Sometimes it is necessary to increase the sensitivity of the amplifier («Sensitivity») to be able to see the second reflected signal.
5. Measure the speed of longitudinal c_l and transverse c_τ waves in different materials (steel, aluminum, brass and so on) by using the prism probe with the angle of incidence α which ensures transmission of both types of the waves into the studied medium (the value of the angle is indicated on the probe prism). The time of passage of each pulse can be determined by means of the depth gauge scale and the calibration curve. The ultrasound path should be measured with the millimeter ruler.
6. Calculate the Poisson ratio μ, Young's modulus E, and shear modulus G for the studied solids by using the following formulae

$$c_\tau = \sqrt{\frac{G}{\rho}},$$

$$c_l = \sqrt{\frac{E(1-\mu)}{\rho(1+\mu)(1-2\mu)}},$$

$$G = \frac{E}{2(1+\mu)}.$$

   The material density ρ can be taken from tables.
7. Calculate the adiabatic compressibility for the liquids under study using the formulae

$$\chi = \frac{1}{\rho c_l^2}.$$

8. Evaluate the errors of the results obtained and compare the results with the tabulated values.

Questions

1. When measuring the speed of ultrasound by means of the ultrasound sensor one can see ghost pulses on the screen in addition to sequentially reflected pulses. Why are these pulses seen? How can one get rid of them?
2. When measuring the ultrasound speed using the prism probe, a systematic error is introduced because there is a wedge-shaped part of the plexiglass probe between the emitter and the material under study. Evaluate this error for given sizes of the probe and the sample.
3. Show that the reflection coefficient of the ultrasound wave on the interface between two media does not depend on the direction of wave propagation.

Literature

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TABLES

Table 1

Physical constants

Quantity	Symbol or equation	Value
Speed of light in vacuum	c	299 792 458 m/s (exact)
Planck constant	h $\hbar = h/2\pi$	$6.62606876(52) \cdot 10^{-34}$ J·s $1.054571596(82) \cdot 10^{-34}$ J·s
Boltzmann constant	k	$1.3806503(24) \cdot 10^{-23}$ J/K
Avogadro constant	N_A	$6.02214199(47) \cdot 10^{23}\ mol^{-1}$
Atomic mass unit	1 u	$1.66053873(13) \cdot 10^{-27}$ kg
Gas constant	$R = kN_A$	8.314472(15) J/(mol·K)
molar volume, ideal gas at STP ($T_0 = 273.15$ K, $P_0 = 101325$ Pa)	$V_0 = \dfrac{RT_0}{P_0}$	$22.413996(39) \cdot 10^{-3}\ \dfrac{m^3}{mol}$
Gravitational constant	G	$6.673(10) \cdot 10^{-11}\ N \cdot m^2/kg^2$
Electron charge (magnitude)	e	$1.602176462(63) \cdot 10^{-19}$ C $4.8032042 \cdot 10^{-10}$ esu
Electron charge-to-mass ratio	e/m_e	$1.758820174(71) \cdot 10^{11}$ C/kg
Electron mass	m_e	$0.910938188(72) \cdot 10^{-30}$ kg
Proton mass	m_p	$1.67262158(13) \cdot 10^{-27}$ kg
Neutron mass	m_n	$1.67492716(13) \cdot 10^{-27}$ kg

Table 1 (cont'd)

Quantity	Symbol or equation	Value
Electron rest energy	m_ec^2	0.510998902(21) MeV
Proton rest energy	m_pc^2	938.271998(38) MeV
Neutron rest energy	m_nc^2	939.565330(38) MeV

Uncertainty of the last digits is shown in parenthesis.

Table 2

Conversion of units

Length:

Angstrom
$1\ \text{Å} = 10^{-10}\ m = 10^{-8}\ cm = 0.1\ nm$

Astronomical unit
$1\ AU = 1.5 \cdot 10^{11}\ m = 1.5 \cdot 10^{13}\ cm$

Light year
$1\ lyr = 9.5 \cdot 10^{15}\ m = 9.5 \cdot 10^{17}\ cm$

Parsec
$1\ pc = 3.1 \cdot 10^{16}\ m = 3.1 \cdot 10^{18}\ cm$

Pressure:

Atmosphere (standard)
$1\ atm = 760\ mm\ Hg = 101325\ Pa\ (exact)$

Energy:

Erg
$1\ erg = 10^{-7}\ J$

Calorie
$1\ cal = 4.1868\ J\ (exact)$

Electron-volt
$1\ eV = 1.6021765 \cdot 10^{-19}\ J = 1.6021765 \cdot 10^{-12}\ erg$

Temperature corresponding to 1 eV,
$11605\ K$

T a b l e 3

Astrophysical constants

Solar mass
$M_{\rm C} = 1.99 \cdot 10^{30}$ кг $= 1{,}99 \cdot 10^{33}\ g$

Solar luminosity
$L_{\rm C} = 3.86 \cdot 10^{26}\ W = 3.86 \cdot 10^{33}\ erg/s$

Solar constant
$E_{\rm C} = 1.35 \cdot 10^{3}\ W/m^2 = 1.35 \cdot 10^{6}\ erg/(s \cdot cm^2)$

Solar radius
$R_{\rm C} = 6.96 \cdot 10^{5}\ km = 6.96 \cdot 10^{8}\ m$

Solar angular diameter as viewed from Earth
$\alpha_{\rm C} = 0.92 \cdot 10^{-2}\ rad$

Solar surface temperature
$T_{\rm C} = 5.9 \cdot 10^{3}\ K$

Earth mass
$M_{\text{З}} = 5.98 \cdot 10^{24}\ kg = 5.98 \cdot 10^{27}\ g$

Earth mean density
$\rho_E = 5.52 \cdot 10^{3}\ kg/m^3 = 5.52\ g/cm^3$

Earth equatorial (a) and polar (b) radius
$a = 6378\ km,\ \ b = 6357\ km$

Mean radius of equivalent sphere
$R = 6371\ km$

Standard gravitational acceleration
$g_n = 9.80665\ m/s^2$

Average distance between Sun and Earth
$L_E = 1\ AU = 1.5 \cdot 10^{8}\ km = 1.5 \cdot 10^{11}\ m$

Average temperature of Earth surface
$T_E = 300$ K

Earth average orbital velocity
$v_E = 30\ km/s = 3 \cdot 10^{4}\ m/s$

Angular velocity of Earth rotation
$\omega_E = 0.727 \cdot 10^{-4}\ rad/s$

Earth escape velocities (1-st and 2-nd)
$v_1 = \sqrt{GM_{\text{З}}/R_E} = 7.9\ km/s = 7.9 \cdot 10^{3}\ m/s,$
$v_2 = v_1\sqrt{2} = 11.2\ km/s = 11.2 \cdot 10^{3}\ m/s$

Venus mass
$M_V = 0.82 M_E = 4.87 \cdot 10^{24}\ kg = 4.87 \cdot 10^{27}\ g$

Average distance between Venus and Sun
$L_V = 1.08 \cdot 10^{8}\ km = 1.08 \cdot 10^{11}\ m$

Venus year
$T_V = 225\ days$

Venus radius
$R_V = 0{,}99 R_E = 6.3 \cdot 10^{3}\ km = 6.3 \cdot 10^{6}\ m$

Venus mean density
$\rho_V = 4.7 \cdot 10^{3}\ kg/m^3 = 4.7\ g/cm^3$

Gravitational acceleration on Venus surface
$g_V = 0.84 g_E = 8.2\ m/s^2$

Mars mass
$M_{\rm M} = 0.11 M_E = 0.66 \cdot 10^{24}\ kg = 0.66 \cdot 10^{27}\ g$

Average distance between Mars and Sun
$L_M = (2.06 - 2.49) \cdot 10^{8}\ km$

Distance between Mars and Earth
$L_{ME} = (0.55 - 4.0) \cdot 10^{8}\ km$

Mars average density
$\rho_M = 4 \cdot 10^{3}\ kg/m^3 = 4\ g/cm^3$

Gravitational acceleration on Mars surface
$g_M = 0.37 g_E = 3.6\ m/s^2$

Moon mass
$M_L = 7.4 \cdot 10^{22}\ kg = 7.4 \cdot 10^{25}\ g$

Moon diameter
$D_L = 3.48 \cdot 10^{3}\ km = 3.48 \cdot 10^{6}\ m$

Average distance between Moon and Earth
$L_L = 3.84 \cdot 10^{5}\ km = 3.84 \cdot 10^{8}\ m$

Moon mean density
$\rho_L = 3.3 \cdot 10^{3}\ kg/m^3 = 3.3\ g/cm^3$

Gravitational acceleration on Moon surface
$g_M = 1.64\ m/s^2$

T a b l e 4

Gravitational acceleration at various latitudes

θ, deg	g, cm/s^2	θ, deg	g, cm/s^2	θ, deg	g, cm/s^2
0	978.0300	35	979.7299	70	982.6061
5	978.0692	40	980.1659	75	982.8665
10	978.1855	45	980.6159	80	983.0584
15	978.3756	50	981.0663	85	983.1759
20	978.6337	55	981.5034	90	983.2155
25	978.9521	60	981.9141		
30	979.3213	65	982.2853		

Table 5

Properties of elements at 760 mm Hg

ρ — density (at 20 °C); C_P — molar heat capacity (at 25 °C); t_m and t_{vap} — melting and vaporization points; q — molar enthalpy of fusion; r — molar enthalpy of vaporization; λ — thermal conductivity (at temperatures shown in parenthesis); α — linear coefficient of thermal expansion of isotropic substances at 0 °C.

Element	Symbol	ρ, $\frac{g}{cm^3}$	C_P, $\frac{J}{mol \cdot K}$	t_m,°C	t_{vap},°C	q, $\frac{kJ}{mol}$	r, $\frac{kJ}{mol}$	λ, $\frac{W}{m \cdot K}$	α, 10^{-6} K^{-1}
Aluminum	Al	2.70	24.35	660	2447	10.7	293.7	207 (27)	22.58
Barium	Ba	3.78	26.36	710	1637	7.66	150.9	—	19.45
Beryllium	Be	1.84	16.44	1283	2477	12.5	294	182 (27)	10.5
Boron (cryst.)	B	3.33	11.09	2030	3900	22.2	540	1.5 (27)	8
Bromine	Br	3.12	75.71	−7.3	58.2	10.58	30.0	—	8.3
Vanadium	V	5.96	24.7	1730	3380	17.5	458	33.2 (20)	—
Bismuth	Bi	9.75	25.52	271.3	1559	10.9	151.5	8 (20)	16.6[2]
Wolfram	W	18.6–19.1	24.8	3380	5530	35.2	799	130 (27)	4.3
Germanium	Ge	5.46	28.8	937.2	2830	29.8	334	60.3 (0)	5.8
Iron	Fe	7.87	25.02–26.74	1535	—	15.5	—	75(0)	12.1
Gold	Au	19.3	25.23	1063	2700	12.77	324.4	310(0)	14.0[2]
Indium	In	7.28	26.7	156.01	2075	3.27	226	88(20)	30.5[2]
Iodine	I	4.94	26.02	113.6	182.8	15.77	41.71	0.44(30)	93.0
Iridium	Ir	22.42	25.02	2443	4350	—	—	138(20)	6.5
Cadmium	Cd	8.65	26.32	321.03	765	6.40	99.81	93(20)	29.0
Potassium	K	0.87	29.96	63.4	753	2.33	77.5	100 (7)	84
Calcium	Ca	1.55	26.28	850	1487	8.66	150	98 (0)	22(0)
Cobalt	Co	8.71	24.6	1492	2255	15.3	383	70.9 (17)	12.0
Silicon (cryst.)	Si	2.42	—	1423	2355	46.5	394.5	167 (0)	2.3
Lithium	Li	0.534	24.65	180.5	1317	3.01	148.1	71 (0-100)	—
Magnesium	Mg	1.74	24.6	649	1120	8.95	131.8	165 (0)	—
Manganese	Mn	7.42	26.32	1244	2095	141.6	224.7	—	22.6
Copper	Cu	8.93	24.52	1083	2595	130.1	304	395–402 (20)	16.6[2]
Molybdenum	Mo	9.01	23.8	2625	4800	27.6	594	162 (27)	5.19

Table 5 (cont'd)

Element	Symbol	ρ, $\frac{g}{cm^3}$	C_P, $\frac{J}{mol \cdot K}$	t_m,°C	t_{vap},°C	q, $\frac{kJ}{mol}$	r, $\frac{kJ}{mol}$	λ, $\frac{W}{m \cdot K}$	α, 10^{-6} K^{-1}
Sodium	Na	0.971	28.12	97.82	890	2.602	89.04	133 (27)	72
Neodymium	Nd	6.96	27.49	1019	3110	14.6	—	—	8.6
Nickel	Ni	8.6–8.9	25.77	1453	2800	17.8	380.6	92 (20)	14.0
Tin (gray)	Sn	5.8	25.77	231.9	2687	7.07	290.4	65 (20)	—
Palladium	Pd	12.16	25.52	1552	3560	17.2	—	76.2 (20)	12.4[2]
Platinum	Pl	21.37	25.69	1769	4310	21.7	447	74.1 (20)	9
Rhodium	Rh	12.44	25.52	1960	3960	—	—	—	8.7
Mercury (liquid)	Hg	13.546	27.98	−38.86	356.73	2.295	59.11	8.45 (20)	—
Rubidium	Rb	1.53	30.88	38.7	701	2.20	69.20	35.5 (20)	90
Lead	Pb	11.34	26.44	327.3	1751	4.772	179.5	34.89 (20)	28.3
Selenium (cryst.)	Se	4.5	25.36	217.4	657	5.42	—	0.13 (25)	20.3
Sulphur (octa.)	S	2.1	22.60	115.18	444.6	1.718	90.75	0.2 (0)	74
Silver	Ag	10.42–10.59	25.49	960.8	2212	11.27	254.0	418 (27)	19.0[2]
Strontium	Sr	2.54	25.11	770	1367	9.2	138	—	20.6
Antimony	Sb	6.62	25.2	630.5	1637	20.41	128.2	23 (20)	9.2
Tantalum	Ta	16.6	25.4	2996	5400	31.4	75.3	63 (27)	6.2
Tellurium (cryst.)	Te	6.25	25.7	449.5	989.8	17.5	114.06	—	17.0
Titanium	Ti	4.5	25.02	1668	3280	15.5	430	15.5 (20)	7.7
Thorium	Th	11.1–11.3	27.32	1695	4200	15.65	544	35.6 (27)	9.8
Carbon (diamond)	C	3.52	6.12	—	—	—	—	—	1.2
Carbon (graphite)[1]	C	2.25	8.53	3500	3900	—	—	114 (20)	—
Uranium (13 °C)	U	18.7	27.8	1133	3900	19.7	412	22.5 (27)	10.7
Phosphorus (white)	P	1.83	24.69	44.2	—	2.51	—	—	125
Chromium	Cr	7.1	23.22	1903	2642	14.6	349	67 (27)	7.78
Cesium	Cs	1.87	31.4	28.64	685	2.18	65.9	23.8 (20)	97
Zinc	Zn	6.97	25.40	419.5	907	7.28	114.7	111 (20)	32
Zirconium	Zr	6.44	25.15	1855	4380	20	582	21.4 (20)	5.1

[1] Reactor graphite, $\rho = 1.65 - 1.72$ g/cm^3; the given value corresponds to $\lambda_\perp$ perpendicular to pressing direction, $\lambda_\perp/\lambda_\parallel = 1{,}5$.

[2] At 20 °C.

Table 6

Properties of solids (at 20 °C)

ρ — density; α — linear coefficient of thermal expansion; λ — thermal conductivity.

Substance	ρ, g/cm^3	α, 10^{-6} K^{-1}	λ, W/(m·K)
Alloys			
Bronze (Cu, Zn, Sn, Al)	8.7–8.9	16–20	200
Duralumin (Al, Cu)	2.8	27	186
Invar (Fe, Ni, C)	8.0	~1	11
Constantan (Cu, Ni)	8.8	15–17	21–22
Brass (Cu, Zn)	8.4–8.7	17–20	80–180
Manganin (Cu, Mn, Ni)	8.5	16	—
Platinum-Iridium alloy (Pl, Ir)	21–22	8.7	—
Steel	7.5–7.9	10–13	~40
Wood (dried)[1]			
Balsa (cork)	0.11–0.14	—	0.04
Bamboo	0.31–0.40	—	0.14–0.17
Beech	0.7–0.9	2.57	—
Birch	0.5–0.7	—	0.117
Oak	0.6–0.9	4.92	0.171
Cedar	0.49–0.57	—	0.08–0.09
Maple	0.62–0.75	6.38	0.12–0.13
Pine	0.37–0.60	5.41	0.08–0.11
Poplar	0.35–0.5	—	0.1
Ash	0.65–0.85	9.51	0.12–0.14
Minerals			
Diamond	3.01–3.52	1.5	628
Asbestos	2.0–2.8	—	0.1
Basalt	2.4–3.1	—	2.177
Plaster	1–2.3	—	0.18–1.05
Clay	1.8–2.6	8.1	1.05–1.26
Granite	2.34–2.76	8.3	2.7–3.3
Quartz (fused)	2.65	1.46	—
Lime	1.9–2.8	—	1.1
Marble	2.6–2.84	3–15	2.7–3
Mica	2.6–3.2	—	—

[1] Thermal conductivity of wood is given for directions perpendicular to fibers; thermal conductivity along fibers is greater by the factor of 2-3.

Table 6 (cont'd)

Substance	ρ, g/cm^3	α, 10^{-6} K^{-1}	λ, W/(m·K)
Miscellaneous substances			
Cardboard	0.69	—	0.21
Brick	1.4–2.2	3-9	1–1.3
Ice	0.913	—	—
Paraffine	0.87–0.91	—	2.5
Plexiglas	1.16–1.20	92–130	0.17–0.18
Cork	0.22–0.26	—	—
Rubber	1.1	220	0.146
Glass	2.4–2.8	6	0.7–1.13
Flint glass	3.9–5.9	7–8	0.84
Porcelain	2.3–2.5	2.5–6	1.05
Ebonite	1.15	84.2	0.17
Amber	1.1	57	—

Table 7

Properties of liquids (at 760 mm Hg)

σ — surface tension at the temperature in the left column (a — liquid-air surface, v — liquid-vapor surface); η — viscosity at 20 °C; λ — thermal conductivity at 0 °C.

Liquid	t, °C	σ, 10^{-3} $\frac{N}{m}$	η, 10^{-3} $\frac{kg}{m \cdot s}$	λ, $\frac{W}{m \cdot K}$
Aniline	19.5	40.8 (v)	4.40	0.181
Acetone	16.8	23.3 (v)	0.324	0.170
Benzoyl	17.5	29.2 (a)	0.647	0.153
Water	20	72.75 (a)	1.0019	0.596
Glycerin	20	63.4 (a)	1495.0	0.290
Dichloroethane		—	—	0.146
Nitric acid 70%	20	59.4 (a)	—	—
Sulphuric acid 85%	18	57.6 (a)	27	—
Castor oil	18	33.1 (a)	986	—
Nitrobenzene	13.6	42.7 (v)	2.01	0.166
Tin	232	526.1 (CO_2)	—	34.3
Mercury	20	487 (v)	1.552	8.45
Turpentine	20	26.7 (a)	—	—
Methanol	20	23.0 (v)	0.578	0.222
Ethanol	20	22.75 (v)	1.200	0.184
Carbon tetrachloride	20	27 (v)	0.972	0.112
Diethyl ether	20	16.96 (v)	0.242	—

Table 8

Properties of liquids

ρ — density at 20 °C; t_m and t_{vap} — melting and vaporization points at standard pressure; t_{cr} — critical temperature; P_{cr} — critical pressure; c — specific heat capacity at 20 °C; q and r — specific latent heat of fusion and vaporization; β — bulk coefficient of thermal expansion at 20 °C.

Liquid	Formula	ρ, $\frac{kg}{m^3}$	t_m, °C	t_{vap}, °C	t_{cr}, °C	P_{cr}, atm	c, $\frac{J}{g\cdot K}$	q, $\frac{J}{g}$	r, $\frac{J}{g}$	β, 10^{-5} K^{-1}
Aniline	C_6H_7N	1026[1]	−6	184	426	52.4	2.156	87.5	458.9	85
Acetone	C_3H_6O	792	−95	56.5	235	47.0	2.18	82.0	521.2	143
Benzoyl	C_6H_6	897	+5.5	80.1	290.5	50.1	1.72	126	394.4	122
Water	H_2O	998.2	0.0	100.00	374	218	4.14	334	2259	18
Glycerin	$C_3H_8O_3$	1260	+20	290	—	—	2.43	176	—	51
Methanol	CH_4O	792.8	−93.9	61.1	240	78.7	2.39	68.7	1102	119
Nitrobenzene	$C_6H_5O_2N$	1173.2[2]	+5.9	210.9	—	—	1.419	—	—	—
Carbon disulfide	CS_2	1293	−111	46.3	275.0	77.0	1.00	—	356	—
Ethanol	C_2H_6O	789.3	−117	78.5	243.5	63.1	2.51	108	855	112
Toluene	C_7H_8	867	−95.0	110.6	320.6	41.6	1.616[3]	—	364	114
Carbon tetrachloride	CCl_4	1595	−23	76.7	283.1	45.0	—	16.2	195.1	122
Acetic acid	$C_2H_4O_2$	1049	+16.7	118	321.6	57.2	2.6[4]	187	405.3	107
Phenol	C_6H_6O	1073	+40.1	181.7	419	60.5	—	123	495.3	—
Chloroform	$CHCl_3$	1498.5[1]	−63.5	61	260	54.9	0.96	197	243	—
Diethyl ether	$C_4H_{10}O$	714	−116	34.5	193.8	35.5	2.34	98.4	355	163

[1] at 15 °C; [2] at 25 °C; [3] at 0 °C; [4] at 1–8 °C.

Table 9

Speed of sound at different media

Gases (at 0 °C)

Gas	c, m/s	dc/dt, $\frac{m}{s\cdot K}$	Gas	c, m/s	dc/dt, $\frac{m}{s\cdot K}$
Nitrogen	333.64	0.85	Oxygen	314.84	0.57
Ammonia	415.0	0.73	Methane	430	0.62
Argon	319.0	—	Neon	435	0.78
Hydrogen	1286.0	2.0	Water vapor (100 °C)	405	—
Air (dry)	331.46	0.607			
Helium	970	1.55	Carbon dioxide	260.3	0.87

Liquids

Liquid	t, °C	c, m/s	dc/dt, $\frac{m}{s\cdot K}$	Liquid	t, °C	c, m/s	dc/dt, $\frac{m}{s\cdot K}$
Nitrogen	−199.0	962	−10	Carbon disulfide	25	1149	−3.3
Aniline	20	1659	−4.0	Turpentine	25	1225	—
Acetone	25	1170	−5.5	Ethanol	20	1177	−3.6
Benzoyl	25	1295	−5.2				
Water	25	1497	+2.5	Toluene	25	1300	−4.3
Glicerine	26	1930	−1.8	Carbon tetra- chloride	25	930	−3.0
Kerosene	25	1315	−3.6				
Mercury	20	1451	−0.46				

Solids

$c_{\parallel}$ — speed of longitudinal waves, $c_{\perp}$ — speed of transversal waves, c — speed of longitudinal waves in thin rod.

Solids	$c_{\parallel}$, m/s	$c_{\perp}$, m/s	c, m/s
Aluminum	6400	3130	5240
Concrete	4250–5250	—	—
Wolfram	5174	2842	—
Granite	5400	—	—
Wood (oak, along fiber)	—	—	4100
Wood (pine, along fiber)	—	—	3600
Duralumin	6400	3120	—
Iron	5930	—	5170
Quartz crystal (X-cut)	5720	—	5440
Quartz fused	5980	3760	5760
Brass	4280–4700	2020–2110	3130–3450
Copper (oxidized)	4720	—	3790
Marble	—	—	3810
Nickel (oxidized, non-magnetic)	—	—	4810
Tin	3320	—	2730
Polystyrene	2350	1120	—
Polyethylene	2000	—	—
Silver	3700	1694	2802
Glass: crown	5260–6120	3050–3550	4710–5300
flint	3760–4800	—	3490–4550
Tool steel	5900–6100	—	5150
Stainless steel	5740	3092	—
Zinc	4170	—	3810
Ebonite	2500	—	—

Table 10

Elastic properties of materials (at 18 °C)

E and G — Young and shear modulus; μ — Poisson ratio; K — compressibility.

Material	E, 10^{10} $\frac{N}{m^2}$	G, 10^{10} $\frac{N}{2}$	μ	K, 10^{10} $\frac{N}{m^2}$
Metals				
Aluminum	7.05	2.63	0.345	7.58
Bronze (66% Cu)	9.7–10.2	3.3–3.7	0.34–0.40	11.2
Bismuth	3.19	1.20	0.33	3.13
Iron	19–20	7.7–8.3	0.29	16.9
Gold	7.8	2.7	0.44	21.7
Cadmium	4.9	1.92	0.30	4.16
Constantan	16.3	6.11	0.32	15.5
Brass	9.7–10.2	3.5	0.34–0.40	10.65
Copper	10.5–13.0	3.5–4.9	0.34	13.76
Nickel	20.4	7.9	0.28	16.1
Tin	5.43	2.04	0.33	5.29
Platinum	16.8	6.1	0.37	22.8
Lead	1.62	0.56	0.44	4.6
Silver	8.27	3.03	0.37	10.4
Steel	20–21	7.9–8.9	0.25–0.33	16.8
Titanium	11.6	4.38	0.32	10.7
Zinc	9.0	3.6	0.25	6.0
Miscellaneous materials				
Bamboo	3.3	—	—	—
Oak	1.3	—	—	—
Quartz fiber	7.3	—	—	—
Redwood	0.88	—	—	—
Rubber soft	0.00015–0.0005	0.00005–0.00015	0.46–0.49	16.8
Pine	0.9	—	—	—
Glass	5.1–7.1	3.1	0.17–0.32	3.75

Table 11

Surface tension of water and aniline at various temperatures

Interface: water — air, aniline — air.

t, °C	σ, 10^{-3} N/m Water	Aniline	t, °C	σ, 10^{-3} N/m Water
0	75.64	—	60	66.18
10	74.22	44.10	70	64.42
20	72.75	42.7	80	62.61
30	71.18	—	90	60.75
40	69.56	—	100	58.85
50	67.91	39.4		

Table 12

Viscosity of liquids at various temperatures

(η, 10^{-3} $N \cdot s/m^2$)

t, °C	Water	Glycerin	Sugar solution in water 20%	60%	t, °C	Castor oil	t, °C	Mercury
0	1.788	12100	3.804	238	5	3760	−20	1.86
10	1.306	3950	2.652	109.8	10	2418	0	1.69
15	1.140	—	2.267	74.6	15	1514	20	1.55
20	1.004	1480	1.960	56.5	20	950	30	1.50
25	0.894	—	1.704	43.86	25	621	50	1.41
30	0.801	600	1.504	33.78	30	451	100	1.24
40	0.653	330	1.193	21.28	35	312	200	1.05
50	0.549	180	0.970	14.01	40	231	300	0.95
60	0.470	102	0.808	9.83	100	16.9		
70	0.406	59	0.685	7.15				
80	0.356	35	0.590	5.40				
90	0.316	21						
100	0.283	13						

Table 13

Viscosity of glycerin-water solution

(glycerine mass ratio is shown)

t	10%	25%	50%	80%	95%	96%	97%	98%	99%	100%
20	1,31	2,09	6,03	61,8	544	659	802	971	1194	1495
25	1,15	1,81	5,02	45,7	365	434	522	627	772	942
30	1,02	1,59	4,23	34,8	248	296	353	423	510	662

Table 14

Compressibility of liquids

$$\varkappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

Liquid	Formula	Pressure P, atm	t, °C	$\varkappa$, 10^{-6} atm^{-1}
Aniline	$C_6H_5NH_2$	85.5	25	43.2
Acetone	$(CH_3)_2CO$	0–500	0	82
Benzoyl	C_6H_6	1–4	15.4	87
Water	H_2O	0–100	20	46.8
Glycerin	$C_3H_8O_3$	1–10	14.8	22.1
Kerosene	—	1–100	16.5	69.6
Sulphuric acid	H_2SO_4	1–16	0	302.5
Nitrobenzene	$C_6H_5NO_2$	86.5	25	46.1
Sulphur dioxide	CS_2	1–2	20	80.95
Methanol	CH_3OH	1–500	0	79.4
Ethanol	CH_3CH_2OH	1–50	0	96
Carbon tetrachloride	CCl_4	0–98.7	20	91.6
Carbon dioxide	CO_2	60	13	1740
Chloroform	$CHCl_3$	1–2	0	87.27
Bromoethane	C_2H_5Br	1–500	10.1	80

Table 15

Specific heat capacity of water and speed of sound in water at various temperatures

t, °C	c, J/(g·K)	v, m/s	t, °C	c, J/(g·K)	v, m/s
0	4.2174	1407	60	4.1841	1556
10	4.1919	1445	70	4.1893	1561
20	4.1816	1484	80	4.1961	1557
30	4.1782	1510	90	4.2048	
40	4.1783	1528	99	4.2145	
50	4.1804	1544			

Table 16

Boiling point of water at various pressures

P, torr	t, °C	P, torr	t, °C	P, torr	t, °C
680	96.9138	725	96.6846	770	100.3666
685	96.1153	730	98.8757	775	100.5484
690	97.3156	735	99.0657	780	100.7293
695	97.5146	740	99.2547	785	100.9092
700	97.7125	745	99.4426	790	101.0881
705	97.9092	750	99.6294	795	101.2661
710	98.1048	755	99.8152	799	101.4079
715	98.2992	760	100.000		
720	98.4925	765	100.1838		

Table 17

Water density at various pressures

t, °C	ρ, g/cm^3	t, °C	ρ, g/cm^3	t, °C	ρ, g/cm^3
0	0.99987	12	0.99952	24	0.99732
1	0.99993	13	0.99940	25	0.99707
2	0.99997	14	0.99927	26	0.99681
3	0.99999	15	0.99913	27	0.99654
4	1.00000	16	0.99897	28	0.99626
5	0.99999	17	0.99880	29	0.99597
6	0.99997	18	0.99862	30	0.99567
7	0.99993	19	0.99843	31	0.99537
8	0.99988	20	0.99823	32	0.99505
9	0.99981	21	0.99802	33	0.99472
10	0.99973	22	0.99780	34	0.99440
11	0.99963	23	0.99757	35	0.99406

Table 18

Diffusion coefficient of saline (at 18 °C)

Concentration of NaCl,mol/l	D, 10^{-5} cm^2/s
0.05	1.26
0.40	1.2
1.00	1.24
2.0	1.29
3.0	1.36
4.0	1.43
5.0	1.49

Table 19

Diffusion coefficients of inorganic substance in water solution

Solute	Concentration, mol/l	t, °C	D, 10^{-5} cm^2/s
Br_2	0.0050	25	1.18
CO_2	0[1]	18	1,46
$CaCl_2$	1.5	9	0.84
$CdSO_4$	1.0	16.8	0.33
Cl_2	0.1	16.3	1.3
$CoCl_2$	0.0127	11	0.73
$CuCl_2$	1.5	10	0.5
$CuSO_4$	0.1	17	0.45
H_2	0[1]	18	3.6
HCl	0.2	25	3.0
HNO_3	3.0	6	1.8
KBr	1.0	10	1.2
KCl	0.1	25	1.89
KNO_3	0.2	18	1.39
KOH	0.1	13.5	2.0
K_2SO_4	0.02	19.6	1.27
LiCl	1.0	18	1.06
$MgSO_2$	1.0	15.5	0.53
N_2	0[1]	18	1.63
NH_3	0.683	4	1.23
NaBr	2.9	10	1.0
Na_2CO_3	2.4	10	0.45
NaCl	1.0	18.5	1.24
$NaNO_3$	0.6	13	1.04
O_2	0[1]	25	2.60
NaOH	0.1	12	1.29

[1] Low concentration.

Table 20

Diffusion coefficients of gases

Coefficients of self-diffusion (at $t = 0$ °C, $P = 1$ atm)

Gas	D, cm^2/s	Gas	D, cm^2/s
Nitrogen N_2	0.17	Xenon Xe	0.048
Argon Ar	0.156	Krypton Kr	0.08
Hydrogen H_2	1.28	Methane CH_4	0.206
Water vapor	0.277	Neon Ne	1.62
Helium He	1.62	Carbon oxide CO	0.175
Oxygen O_2	0.18	Carbon dioxide CO_2	0.097

Coefficients of inter-diffusion (at $t = 0$ °C)

System	D, cm^2/s	System	D, cm^2/s
He — CH_4	0.57	H_2 — air	0.66
He — O_2	0.45	H_2 — CH_4	0.62
He — air	0.62	H_2 — O_2	0.69
Ne — H_2	0.99	CH_4 — N_2	0.2
Ne — N_2	0.28	CH_4 — O_2	0.22
Ar — CH_4	0.172	CH_4 — air	0.186
Ar — O_2	0.167	N_2 — H_2O	0.204
Ar — air	0.165	N_2 — CO_2	0.208
Ar — CO_2	0.177	CO — O_2	0.175
Kr — N_2	0.13	CO — air	0.182
Kr — CO	0.13	O_2 — CO_2	0.174
Xe — H_2	0.54	air — CO_2	0.207
Xe — N_2	0.106	H_2O — CO_2	0.41

Table 21

Thermal conductivity of air at various temperatures

(at $P = 1$ atm)

t, °C	λ, 10^{-2} $\frac{W}{m \cdot K}$	t, °C	λ, 10^{-2} $\frac{W}{m \cdot K}$	t, °C	λ, 10^{-2} $\frac{W}{m \cdot K}$
−173	0.922	−23	2.207	27	2.553
−143	1.204	−3	2.348	37	2.621
−113	1.404	0.1	2.370	67	2.836
−83	1.741	7	2.417	97	3.026
−53	1.983	17	2.485		

Table 22

The Joule-Thomson coefficients

($\mu_{J-T} = \Delta T / \Delta P$; in units of K/atm)

t, °C	Carbon oxide (CO)			
	P, atm			
	1	50	100	200
0	0.295	0.240	0.190	0.093
25	0.251	0.206	0.162	0.084
50	0.213	0.175	0.137	0.072
100	0.150	0.122	0.095	0.049

T, K	Hydrogen (H_2)			
	P, atm			
	≈ 0	20	100	180
60	0.391	0.287	0.035	—
70	0.287	0.234	0.059	−0.039
80	0.220	0.192	0.061	−0.037

t °C	Methane (CH_4)			
	P, atm			
	≈ 0	17	51	102.1
21.1	0.405	0.425	0.410	0.332
37.8	0.359	0.375	0.365	0.294
71.1	0.283	0.298	0.290	0.229
104.4	0.227	0.239	0.233	0.180

t °C	Ethane (C_2H_8)			
	P, atm			
	≈ 0	17	51	102.1
21.1	0.939	1.217	—	—
37.8	0.833	1.037	—	—
71.1	0.657	0.760	0.890	0.353
104.4	0.498	0.572	0.586	0.399

T a b l e 22 (cont'd)

t, °C	Argon (Ar)				Helium (He)
	P, atm				P, atm
	1	20	100	200	200
−150	1.81	—	−0.025	−0.056	−0.052
−100	0.860	0.800	0.285	0.040	−0.058
0	0.431	0.406	0.305	0.192	−0.0616
25	0.371	0.350	0.264	0.175	—
100	0.242	0.224	0.175	0.127	−0.0638
200	0.137	0.126	0.095	0.068	−0.0641

t, °C	Nitrogen (N_2), Oxygen (O_2)			
	P, atm			
	1	20	100	200
−150	1.265	1.128	0.020	−0.027
−100	0.649	0.594	0.274	0.058
0	0.267	0.250	0.169	0.087
25	0.222	0.206	0.140	0.078
100	0.129	0.119	0.077	0.042
200	0.056	0.048	0.026	0.006

t, °C	Carbon dioxide (CO_2)			
	P, atm			
	1	20	100	200
−25	1.650	0.000	−0.005	−0.012
0	1.290	1.402	0.022	0.005
20	1.105	1.136	0.070	0.027
40	0.958	0.966	0.262	0.066
60	0.838	0.833	0.625	0.125
80	0.735	0.724	0.597	0.196
100	0.649	0.638	0.541	0.256
200	0.373	0.358	0.315	0.246

t, °C	Air			
	P, atm			
	1	20	100	200
−100	0.5895	0.5700	0.2775	0.0655
−50	0.3910	0.3690	0.2505	0.1270
−25	0.3225	0.3010	0.2130	0.1240
0	0.2746	0.2577	0.1446	0.1097
25	0.2320	0.2173	0.1550	0.0959
50	0.1956	0.1830	0.1310	0.0829
75	0.1614	0.1508	0.1087	0.0707
100	0.1355	0.1258	0.0884	0.0580

T a b l e 23

Critical properties and parameters a and b in Van der Waals equation

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT;\ a = \frac{27}{8} RT_{cr} b,\ b = \frac{RT_{cr}}{8P_{cr}}$$

Substance	T_{cr} K	P_{cr} MPa	ρ_{cr} $\frac{g}{cm^3}$	a $\frac{N \cdot m^4}{mol^2}$	b $\frac{cm^3}{mol}$
Nitrogen (N_2)	126.25	3.399	0.304	0.1368	38.607
Argon (Ar)	150.65	4.86	0.531	0.1361	32.191
Water (vapor) (H_2O)	647.30	22.12	0.32	0.5524	30.413
Hydrogen (H_2)	33.24	1.297	0.0310	0.02484	26.635
Air	132.45	3.77	0.35	0.1357	36.51
Helium (He)	5.20	0.229	0.0693	0.00344	23.599
Nitrous oxide (N_2O)	309.58	7.255	0.453	0.3852	44.347
Oxygen (O_2)	154.78	5.081	0.41	0.1375	31.662
Neon (Ne)	44.45	2.72	0.484	0.0211	16.948
Nitric oxide (NO)	180	6.54	0.52	0.1444	28.579
Carbon oxide (CO)	132.92	3.499	0.301	0.1473	39.482
Methane (CH_4)	190.60	4.63	0.160	0.2288	42.777
Methanol (CH_4O)	513.15	7.95	0.272	0.9654	67.047
Ethanol (C_2H_6O)	516	6.4	0.276	1.2164	84.006
Sulphur dioxide (CS_2)	552	7.90	0.44	1.1243	72.585
Carbon dioxide (CO_2)	304.15	7.387	0.468	0.3652	42.792
Chlorine (Cl_2)	417	7.71	0.573	0.6576	56.202
Carbon tetrachloride (CCl_4)	556.25	4.56	0.558	1.9789	126.78
Ethane (C_2H_6)	305.45	4.87	0.203	0.5571	64.997

T a b l e 24

Temperature dependence of parameters a and b of argon

Temperature, °C	a, 10^6 atm·cm^6/mol^2	b, cm^3/mol
151	1.90	61
157	1.87	59,5
163	1.84	58
173	1.785	55,5
183	1.735	53
193	1.69	51
213	1.60	48
233	1.53	45
253	1.47	43
273	1.42	41

Table 25

Properties of gases

M — molecular mass; ρ — density (at $t = 0$ °C, $P = 1$ atm); t_{cr} — critical temperature; $P_{\text{кр}}$ — critical pressure; $\rho_{\text{кр}}$ — critical density; t_m — melting point (at $P = 1$ atm); t_{vap} — boiling point (at $P = 1$ atm).

Substance	Formula	M	ρ, kg/m^3	t_{cr}, °C	P_{cr}, atm	ρ_{cr}, kg/m^3	t_m, °C	t_{vap}, °C
Nitrogen	N_2	28.016	1.2505	147.1	33.5	311	−210.02	−195.81
Ammonium	NH_3	17.031	0.7714	132.4	112.0	234	−77.7	−33.4
Argon	Ar	39.944	1.7839	122.4	48.0	531	−189.3	−185.9
Hydrogen	H_2	2.0158	0.08988	239.9	12.80	31,0	−259.20	−252.78
Water vapor	H_2O	18.0156	0.768	374.2	218.5	324	0,00	100.00
Dry air [1]	—	28.96	1.2928	140.7	37.2	310	−213	−193
Helium	He	4.002	0.1785	267.9	2.26	69,3	−272.2	−268.93
Nitrogen dioxide	N_2O	44.013	1.9775	36.5	71.7	450	−90	−88.6
Oxygen	O_2	32.000	1.42896	118.8	49.7	430	−218.83	−182.97
Methane	CH_4	16.04	0.7168	82.5	45.7	162	−182.5	−116.7
Neon	Ne	20.183	0.8999	228.7	26.9	484	−248.60	−246.1
Nitric oxide	NO	30.006	1.3402	92,9	64.6	520	−167	−150
Carbon oxide	CO	28.01	1.2500	140.2	34.5	301	−205	−191.5
Carbon dioxide	CO_2	44.01	1.9768	31.0	73	460	−56.6[2]	−78.48[3]
Chlorine	Cl_2	70.914	3.22	144	76.1	573	−100.5	−33.95

[1] Air composition (volume fraction): 78.03% N_2, 20.99% O_2, 0.933% Ar, 0.03% CO_2, 0.01% H_2, 0.0018% Ne etc..
[2] At $P = 5.12$ atm (triple point).
[3] Sublimation temperature.

Table 26

Thermal properties of gases

c_p и C_p — specific and molar heat capacity (for given temperature ranges); $\gamma = c_p/c_v$ at 20 °C; η — dynamic viscosity at 20 °C; λ — thermal conductivity at 0 °C; $\beta = (1/V)(\partial V/\partial T)_P$ — coefficient of thermal expansion

Gas	Formula	t, °C	c_p, $\frac{J}{g \cdot K}$	C_p, $\frac{J}{mol \cdot K}$	γ	λ, $10^{-2}\,\frac{W}{m \cdot K}$	η, $10^{-7}\,\frac{kg}{m \cdot c}$	t, °C	β, 10^{-3} K^{-1}
Nitrogen	N_2	0–20	1.038	29.1	1.404	2.43	174	0–100	3.671
Ammonia (vapor)	NH_3	24–200	2.244	38.1	1.34	2.18	97,0	—	—
Argon	Ar	15	0.523	20.9	1.67	1.62	222	100	3.676
Acetone (vapor)	C_3H_6O	26–110	1.566	90.9	1.26	1.70	73,5	—	—
Hydrogen	H_2	10–200	14.273	28.8	1.41	16.84	88	100	3.679
Water vapor[1]	H_2O	100	1.867	34.5	1.324	2.35	128	1–120	4.187
Dry air	—	0–100	0.992	29.3	1.40	2.41	181	—	—
Helium	He	−180	5.238	21.0	1.66	14.15	194	100	3.659
Nitrous oxide	N_2O	16–200	0.946	41.7	1.32	1.51	146	0	3.761
Oxygen	O_2	13–207	0.909	29.1	1.40	2.44	200	0–100	3.67
Methane	CH_4	18–208	2.483	39.8	1.31	3.02	109	−50 ÷ +50	3.580
Nitrogen oxide	NO_2	13–172	0.967	29.0	1.40	2.38	188	0	3.677
Carbon oxide	CO	26–198	1.038	28.5	1.40	2.32	177	0–100	3.671
Sulfur dioxide	SO_2	16–202	0,561	36,0	1,29	0,77	126	—	—
Carbon oxide	CO_2	15	0.846	37.1	1.30	1.45	144,8	0–100	3.723
Chlorine	Cl_2	13–202	0.519	36.8	1.36	0.72	132	0–100	3.830
Ethylene	C_2H_4	15–100	1.670	46.8	1.25	1.64	103	—	—

[1] λ is measured at 100 °C.

Table 27

Viscosity of gases and vapors at various temperatures

η, 10^{-8} kg/(m·s)

t, °C	Nitrogen N_2	Argon Ar	Hydrogen H_2	Water vapor	Air	Helium He	Oxygen O_2	Carbon-dioxide CO_2
−75	1285	1585	677	—	1312	1526	1452	1007
−50	1419	1760	733	—	1445	1640	1612	1126
−25	1542	1930	788	—	1582	1750	1753	1247
0	1665	2085	840	883	1708	1860	1910	1367
20	1766	2215	880	—	1812	1946	2026	1463
25	1778	2248	890	975	1840	1968	2052	1486
50	1883	2400	938	1065	1954	2065	2182	1607
75	1986	2550	985	1157	2068	2175	2310	1716
100	2086	2695	1033	1250	2180	2281	2437	1827

Table 28

Pressure and density of saturated water vapor at various temperatures

t, °C	P, torr	ρ, g/m³	t, °C	P, torr	ρ, g/m³	t, °C	P, torr	ρ, g/m³
−30	0.28	0.33	−2	3.88	4.13	26	25.21	24.4
−28	0.35	0.41	0	4.58	4.84	28	28.35	27.2
−26	0.43	0.51	2	5.29	5.60	30	31.82	30.3
−24	0.52	0.60	4	6.10	6.40	32	35.66	33.9
−22	0.64	0.73	6	7.01	7.3	34	39.90	37.6
−20	0.77	0.88	8	8.05	8.3	36	44.56	41.8
−18	0.94	1.05	10	9.21	9.4	38	49.69	46.3
−16	1.13	1.27	12	10.52	10.7	40	55.32	51.2
−14	1.36	1.51	14	11.99	12.1	50	92.5	83.0
−12	1.63	1.80	16	13.63	13.6	60	149.4	130
−10	1.95	2.14	18	15.48	15.4	70	233.7	198
−8	2.32	2.54	20	17.54	17.3	80	355.1	293
−6	2.76	2.99	22	19.83	19.4	90	525.8	424
−4	3.28	3.51	24	22.38	21.8	100	760.0	598

Table 29

Emf of thermocouples at various temperatures

t, °C	emf, mV			
	Platinum — platinum+10% Rhodium	Chromel — Alumel	Iron — Constantan	Copper — Constantan
100	0.64	4.1	5	4
200	1.44	8.1	11	9
300	2.31	12.2	16	15
400	3.25	16.4	22	21
500	4.22	20.6	27	
600	5.23	24.9	33	
700	6.26	29.1	39	
800	7.34	33.3	45	
900	8.45	37.4	52	
1000	9.59	41.3	58	
1200	11.95	48.9		
1400	14.37	55.9		
1600	16.77			

Table 30

Specific resistance and temperature coefficient of resistivity of metal wires (at 18 °C)

Metal	ρ, 10^{-6} Ohm·cm	$\alpha \cdot 10^4$, K^{-1}
Aluminum	3,21	38
Wolfram	5.5	51
Iron (0.1% C)	12.0	62
Gold	2.42	40
Brass	6–9	10
Manganin (3% Ni, 12% Mn, 85% Cu)	44.5	0.02–0.5
Copper	1.78	42,8
Nickel	11.8	27
Constantan (40% Ni, 1.2% Mn, 58.8% Cu)	49.0	$-0.4 \div 0.1$
Nichrome (67.5% Ni, 1.5% Mn, 16% Fe, 15% Cr)	110	1,7
Tin	11.3	45
Platinum	11.0	38
Lead	20.8	43
Silver	1.66	40
Zinc	6.1	37

Table 31

Work function

Metal	A, eV
Aluminum	4.25
Barium	2.49
Wolfram	4.54
Iron	4.31
Copper	4.40
Nickel	4.50
Barium oxide (thin film on wolfram)	1.1
Tin	4.38
Platinum	5.32
Mercury	4.52
Silver	4.3
Cesium	1.81
Zinc	4.24

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