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Strong and weak convexity: application to the gradient projection algorithm

mini SYMPOSIUM
Some aspects of Variational Analysis and Applications

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Strongly convex sets of radius R

Let X be a Banach space.

Definition. (A. Pliś, Ch. Olech, H. Frankowska, E. Polovinkin...) A nonempty set $A \subset X$ is strongly convex of radius $R > 0$ if

$$A = \bigcap_{x \in B} B_R(x), \qquad B \subset X.$$

The subset of centers $B \subset X$ is arbitrary.

Define $\varrho_A(x) = \inf_{a \in A} \|x - a\|$.

Let $R > 0$, $\quad U_A(R) = \{x \in E \mid 0 < \varrho_A(x) < R\}$.

Let $\pi(\mathfrak{u}, A)$ be the metric projection of the point $\mathfrak{u}$ on the set A.

Proximally smooth set with constant R

Definition. (J. Borwein, F. Clarke,...). A set $A \subset E$ is called *proximally smooth* with constant R, if the distance function $\mathfrak{u} \mapsto \varrho_A(\mathfrak{u})$ continuously differentiable on the set $U_A(R)$.

Definition. (F. Clarke) We shall say that the set $A \subset E$ satisfies the *supporting condition of weak convexity* with constant $R > 0$, if from the inclusions $\mathfrak{u} \in U_A(R)$ and $x \in \pi(\mathfrak{u}, A)$ we have the following inequality

$$\varrho_A\left(x + \frac{R}{\|\mathfrak{u} - x\|}(\mathfrak{u} - x)\right) \geq R.$$

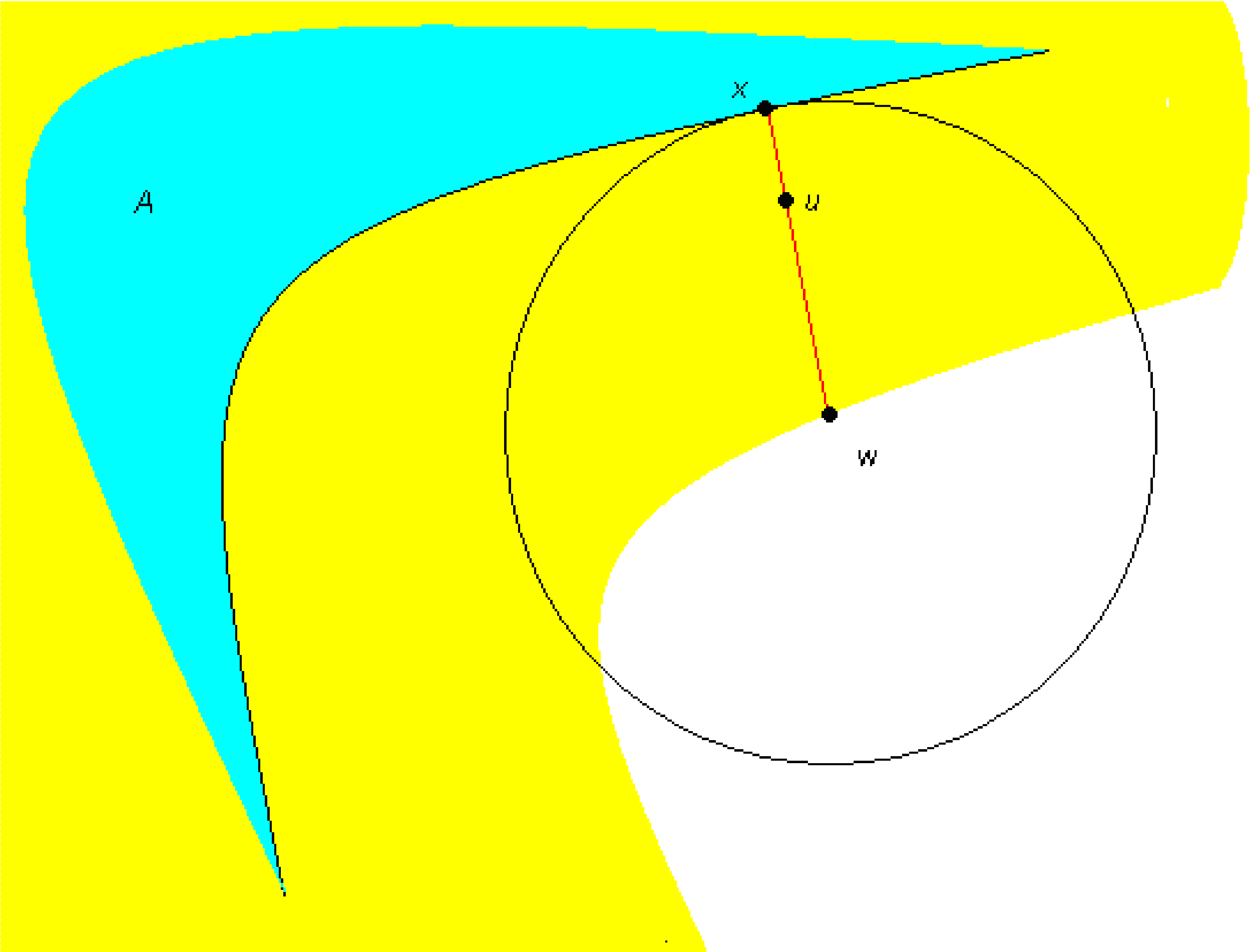
The following theorem gives characterization of proximally smooth sets.

Theorem 1. (F. Clarke, R. Stern, P. Wolenski, F. Bernard, L. Thibault, N. Zlateva...) *Let E be a uniformly convex and uniformly smooth Banach space; $A \subset E$ be a closed subset, $R > 0$. Then the following conditions are equivalent :*

(1). *The set A satisfies the supporting condition of weak convexity with the constant R.*

(2). *The set A is proximally smooth with constant R.*

(3). *The projection mapping $\mathfrak{u} \mapsto \pi(\mathfrak{u}, A)$ is single valued and continuous on the set $U_A(R)$.*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Metric projection: strongly convex and proximally smooth sets

Let X be a real Hilbert space.

1. (M. Balashov, M. Golubev). Let $A \subset X$ be a strongly convex set of radius R, $\varrho_A(x_i) \geq r \geq 0$, $i = 1, 2$. Then

$$\|\pi(x_1, A) - \pi(x_2, A)\| \leq \frac{R}{R+r}\|x_1 - x_2\|. \tag{1}$$

2. (F. Clarke, R. Stern, P. Wolenski). Let $A \subset X$ be a proximally smooth set of radius R, $\varrho_A(x_i) \leq r < R$, $i = 1, 2$. Then

$$\|\pi(x_1, A) - \pi(x_2, A)\| \leq \frac{R}{R-r}\|x_1 - x_2\|. \tag{2}$$

Level sets of functions

Next two results were obtained by J.-Ph. Vial.

Let $f : X \to \mathbb{R}$ be a strongly convex function with constant $\varkappa$, i.e. the function $f(x) - \frac{\varkappa}{2}\|x\|^2$ is convex. Let $\alpha \in \mathbb{R}$, $\mathcal{L}_f(\alpha) = \{x \in X \mid f(x) \leq \alpha\}$ and $M = \sup\limits_{x \in \mathcal{L}_f(\alpha),\, f'(x) \in \partial f(x)} \|f'(x)\|$. Then the set $\mathcal{L}_f(\alpha)$ is strongly convex of radius $R = \frac{M}{\varkappa}$.

Let $f : X \to \mathbb{R}$ be a function with the Lipschitz continuous gradient with the Lipschitz constant L on the lower level set $\mathcal{L}_f(\alpha) = \{x \in X \mid f(x) \leq \alpha\}$. Let $m = \inf\limits_{x \in \partial \mathcal{L}_f(\alpha)} \|f'(x)\|$. Then the upper Lebesgue set of the form $\mathcal{U}_f(\alpha) = \{x \in X \mid f(x) \geq \alpha\}$ is proximally smooth with constant $R = \frac{m}{L}$.

Strongly convex set, f′∈Lip

We shall consider the next problem in a real Hilbert space X:

$$\max_{x\in A} f(x). \tag{1}$$

(i) The set A is strongly convex of radius $r > 0$,

(ii) The function $f : X \to \mathbb{R}$ has the Lipschitz continuous gradient on the convex set A with constant $L > 0$,

(iii) $r < \frac{m}{L}$, where $m = \inf_{x\in\partial A} \|f'(x)\|$.

Strongly convex set, f´∈Lip

Theorem 1.

Suppose that conditions (i) — (iii) take place in the problem (1). Then the iteration process $x_0 \in \partial A$,

$$x_{k+1} = \arg\max_{x \in A}(f'(x_k), x), \qquad k = 0, 1, 2, 3, \ldots$$

converges to the unique solution z_0 of the problem with the rate of geometric progression with common ratio $q = \frac{Lr}{m} < 1$, i.e.

$$\|x_{k+1} - z_0\| \leq q\|x_k - z_0\|.$$

Proximally smooth set, strongly convex function

Now

$$\min_{x \in A} f(x). \tag{2}$$

(i) The set A is a closed proximally smooth with constant of proximal smoothness $R > 0$,

(ii) The function $f : X \to \mathbb{R}$ is a strongly convex with constant of strong convexity $\varkappa > 0$,

(iii) Let $\alpha \in \mathbb{R}$, $\mathcal{L}_f(\alpha) \cap A \neq \emptyset$ and the function f has the Lipshcitz continuous gradient with constant $L > 0$ on the set $\mathcal{L}_f(\alpha)$,

(iv) Suppose that $\frac{M}{\varkappa} < R$, where $M = \sup_{x \in \mathcal{L}_f(\alpha)} \|f'(x)\|$.

Proximally smooth set, strongly convex function

Theorem 2.

Suppose that conditions (i) — (iv) take place in the problem (2). Then for any initial point $x_0 \in A \cap \mathcal{L}_f(\alpha)$ the iteration process

$$x_{k+1} = P_A\left(x_k - tf'(x_k)\right), \quad t = \frac{\varkappa - \frac{M}{R}}{L^2 - \frac{\varkappa M}{R}},$$

converges to the unique solution $z_0 \in A$ of the problem (2) with the rate of geometric progression with common ratio

$$q(t) = \frac{R}{R - tM}\sqrt{1 - 2t\varkappa + t^2L^2} \in (0,1), \quad t = \frac{\varkappa - \frac{M}{R}}{L^2 - \frac{\varkappa M}{R}},$$

namely,

$$\|x_{k+1} - z_0\| \leq q(t)\|x_k - z_0\|.$$

Thank you!