Properties and applications of weakly convex functions and sets

G.E. Ivanov

Moscow Institute of Physics and Technology

The generalized approximation problem

Let E be a real Banach space.

A set $M \subset E$ is called a *quasiball* if M is convex closed and $0 \in \operatorname{int} M$.

The generalized approximation problem

Let E be a real Banach space.

A set $M \subset E$ is called a *quasiball* if M is convex closed and $0 \in \operatorname{int} M$.

Given $A \subset E$, $x \in E \setminus A$, we consider the problem

$$\text{to minimize } t > 0 \text{ such that } (x - tM) \cap A \neq \emptyset.$$

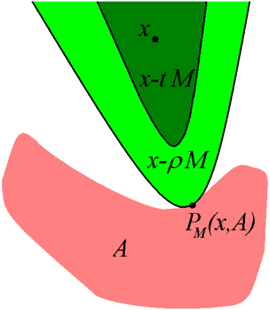
The generalized approximation problem

Let E be a real Banach space.

A set $M \subset E$ is called a *quasiball* if M is convex closed and $0 \in \operatorname{int} M$.

Given $A \subset E$, $x \in E \setminus A$, we consider the problem

$$\text{to minimize } t > 0 \text{ such that } (x - tM) \cap A \neq \emptyset.$$



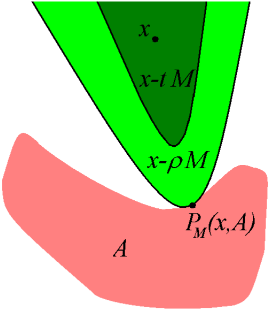
The generalized approximation problem

Let E be a real Banach space.

A set $M \subset E$ is called a *quasiball* if M is convex closed and $0 \in \operatorname{int} M$.

Given $A \subset E$, $x \in E \setminus A$, we consider the problem

$$\text{to minimize } t > 0 \text{ such that } (x - tM) \cap A \neq \emptyset.$$



The *M-distance*:
$\varrho_M(x, A) = \varrho = \inf\{t > 0 \mid (x - tM) \cap A \neq \emptyset\};$

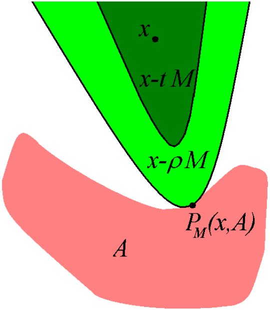
The generalized approximation problem

Let E be a real Banach space.

A set $M \subset E$ is called a *quasiball* if M is convex closed and $0 \in \operatorname{int} M$.

Given $A \subset E$, $x \in E \setminus A$, we consider the problem

$$\text{to minimize } t > 0 \text{ such that } (x - tM) \cap A \neq \emptyset.$$



The *M-distance*:
$\varrho_M(x, A) = \varrho = \inf\{t > 0 \mid (x - tM) \cap A \neq \emptyset\}$;

The *M-projection*:
$P_M(x, A) = A \cap (x - \varrho M)$.

The Minkowski functional

The Minkowski functional (or gauge functional) of the quasiball M:

$$\mu_M(x) = \inf\{t > 0 \mid x \in tM\}.$$

The Minkowski functional

The Minkowski functional (or gauge functional) of the quasiball M:

$$\mu_M(x) = \inf\{t > 0 \mid x \in tM\}.$$

We can rewrite the M-distance:

$$\begin{aligned} \varrho_M(x, A) &= \varrho = \inf\{t > 0 \mid (x - tM) \cap A \neq \emptyset\} \\ &= \inf\{t > 0 \mid \exists a \in A : \; x - a \in tM\} \\ &= \inf_{a \in A} \mu_M(x - a), \end{aligned}$$

The Minkowski functional

The Minkowski functional (or gauge functional) of the quasiball M:

$$\mu_M(x) = \inf\{t > 0 \mid x \in tM\}.$$

We can rewrite the M-distance:

$$\begin{aligned} \varrho_M(x, A) &= \varrho = \inf\{t > 0 \mid (x - tM) \cap A \neq \emptyset\} \\ &= \inf\{t > 0 \mid \exists a \in A : \; x - a \in tM\} \\ &= \inf_{a \in A} \mu_M(x - a), \end{aligned}$$

$$P_M(x, A) = A \cap (x - \varrho M) = \operatorname*{argmin}_{a \in A} \mu_M(x - a).$$

The first motivation: the ordinary approximation problem

If $M = B_1(0) = \{x \in E : \|x\| \leq 1\}$ is the unit ball, then

- $\mu_{B_1(0)}(x) = \|x\|$,
- $\varrho_M(x, A) = \inf\limits_{a \in A} \|x - a\|$ is the distance from x to A;
- $P_M(x, A) = \operatorname*{argmin}\limits_{a \in A} \|x - a\|$ is the metric projection of the point x onto the set A.

The first motivation: the ordinary approximation problem

If $M = B_1(0) = \{x \in E : \|x\| \le 1\}$ is the unit ball, then

- $\mu_{B_1(0)}(x) = \|x\|$,
- $\varrho_M(x, A) = \inf\limits_{a \in A} \|x - a\|$ is the distance from x to A;
- $P_M(x, A) = \operatorname*{argmin}\limits_{a \in A} \|x - a\|$ is the metric projection of the point x onto the set A.

Instead of the norm we consider the Minkowski functional, that is a *nonsymmetric seminorm*, since it is
positively homogeneous:

$$\mu_M(tx) = t\mu_M(x), \qquad \forall t \ge 0, \quad \forall x \in E$$

and *subadditive*:

$$\mu_M(x + y) \le \mu_M(x) + \mu_M(y), \qquad \forall x, y \in E.$$

So, we consider the approximation problem with respect to a nonsymmetric seminorm.

The second motivation: the minimal time problem

(due to Vladimir Goncharov)

Consider a control system with constant dynamics, described by the differential inclusion

$$\dot{y}(\tau) \in -M$$

with initial position $y(0) = x$ and the target set A.

The second motivation: the minimal time problem

(due to Vladimir Goncharov)

Consider a control system with constant dynamics, described by the differential inclusion

$$\dot{y}(\tau) \in -M$$

with initial position $y(0) = x$ and the target set A.

Then

- $\varrho = \varrho_M(x, A)$ is the minimal time nedeed to attain the target set A from x by trajectory of the differential inclusion: $\varrho = \inf\{\tau > 0 \,:\, y(\tau) \in A\}$;
- $P_M(x, A)$ is the set of points $y(\varrho)$ where the optimal trajectories attain the target set.

The third motivation: the infimal convolution problem

The *infimal convolution* of the functions $f : E \to \mathbb{R} \cup \{+\infty\}$ and $g : E \to \mathbb{R} \cup \{+\infty\}$ is

$$(f \boxplus g)(x) = \inf_{u \in E} \Big(f(u) + g(x-u) \Big), \qquad x \in E.$$

The third motivation: the infimal convolution problem

The *infimal convolution* of the functions $f : E \to \mathbb{R} \cup \{+\infty\}$ and $g : E \to \mathbb{R} \cup \{+\infty\}$ is

$$(f \boxplus g)(x) = \inf_{u \in E} \Big(f(u) + g(x - u) \Big), \qquad x \in E.$$

In particular, if

$$f(u) = \begin{cases} 0, & u \in A, \\ +\infty, & u \notin A \end{cases}$$

is the *indicator* function of the set A, then

$$(f \boxplus g)(x) = \inf_{u \in A} g(x - u).$$

The third motivation: the infimal convolution problem

The *infimal convolution* of the functions $f : E \to \mathbb{R} \cup \{+\infty\}$ and $g : E \to \mathbb{R} \cup \{+\infty\}$ is

$$(f \boxplus g)(x) = \inf_{u \in E} \Big(f(u) + g(x-u) \Big), \qquad x \in E.$$

In particular, if

$$f(u) = \begin{cases} 0, & u \in A, \\ +\infty, & u \notin A \end{cases}$$

is the *indicator* function of the set A, then

$$(f \boxplus g)(x) = \inf_{u \in A} g(x-u).$$

If, moreover, $g(x) = \mu_M(x)$ is the Minkowski functional of a quasiball M, then

$$(f \boxplus g)(x) = \varrho_M(x, A), \qquad \operatorname*{argmin}_{u \in E} \Big(f(u) + g(x-u) \Big) = P_M(x, A).$$

The third motivation: the infimal convolution problem

The *effective domain* of a function $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is

$$\text{dom } f = \{x \in E \mid f(x) \in \mathbb{R}\}.$$

The *epigraph* of f is $\quad \text{epi } f = \{(x, y) \in E \times \mathbb{R} : y \geq f(x)\}.$

The third motivation: the infimal convolution problem

The *effective domain* of a function $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is

$$\mathrm{dom}\ f = \{x \in E \,|\, f(x) \in \mathbb{R}\}.$$

The *epigraph* of f is $\quad \mathrm{epi}\ f = \{(x, y) \in E \times \mathbb{R} : y \geq f(x)\}.$

Consider the infimal convolution problem for functions $f : E \to \mathbb{R} \cup \{+\infty\}$ and $g : E \to \mathbb{R} \cup \{+\infty\}$.
Assume that the function $g : E \to \mathbb{R}$ is convex, continuous, and $g(0) < 0$.
Then epi g is a quasiball (unbounded!).

Lemma 1.

Denote $M = \mathrm{epi}\ g$, $A = \mathrm{epi}\ f$. *For any* $x_0 \in \mathrm{dom}\,(f \boxplus g)$ *we have*

$$u_0 \in \operatorname*{argmin}_{u \in E} \Big(f(u) + g(x_0 - u) \Big) \quad \Leftrightarrow \quad \Big(u_0, f(u_0) \Big) \in P_M(z_0, A),$$

where $z_0 = \Big(x_0, (f \boxplus g)(x_0) \Big)$.

Well posedness

So, we consider the following minimization problems

$$\inf_{a\in A} \mu_M(x-a) \tag{1}$$

and

$$\inf_{u\in E}\Big(f(u)+g(x-u)\Big). \tag{2}$$

Well posedness

So, we consider the following minimization problems

$$\inf_{a \in A} \mu_M(x - a) \tag{1}$$

and

$$\inf_{u \in E} \Big(f(u) + g(x - u) \Big). \tag{2}$$

A minimization problem

$$\text{to minimize } F(x) \text{ with } x \in X$$

is called *well posed* if it has a unique solution x^* and any minimizing sequence $\{x_k\} \subset X$, i.e.

$$\lim_{k \to \infty} F(x_k) = \inf_{x \in X} F(x)$$

converges to x^*.

Well posedness

So, we consider the following minimization problems

$$\inf_{a\in A} \mu_M(x-a) \tag{1}$$

and

$$\inf_{u\in E}\Big(f(u)+g(x-u)\Big). \tag{2}$$

A minimization problem

$$\text{to minimize } F(x) \text{ with } x\in X$$

is called *well posed* if it has a unique solution x^* and any minimizing sequence $\{x_k\}\subset X$, i.e.

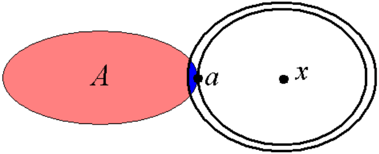
$$\lim_{k\to\infty} F(x_k) = \inf_{x\in X} F(x)$$

converges to x^*.

Question. What properties of sets M and A in problem (1) and of functions f and g in problem (2) are needed for well possedness of these problems?

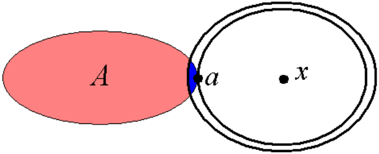
Well posedness of the ordinary approximation problem

If A is a convex closed set in a Hilbert space H, then the ordinary approximation problem is well posed for any $x \in H$.

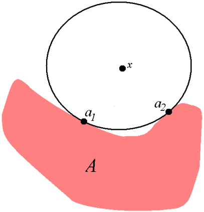


Well posedness of the ordinary approximation problem

If A is a convex closed set in a Hilbert space H, then the ordinary approximation problem is well posed for any $x \in H$.

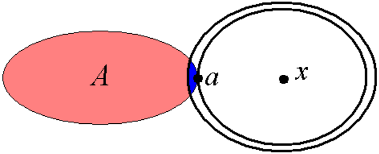


If $A \subset E$ is unconvex, then the solution of the ordinary approximation problem may be not unique even if $E = \mathbb{R}^n$.

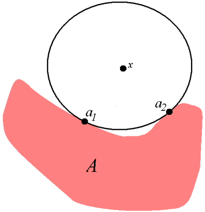


Well posedness of the ordinary approximation problem

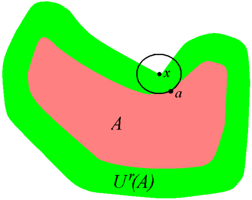
If A is a convex closed set in a Hilbert space H, then the ordinary approximation problem is well posed for any $x \in H$.



If $A \subset E$ is unconvex, then the solution of the ordinary approximation problem may be not unique even if $E = \mathbb{R}^n$.



However, if the boundary of a closed unconvex set A is smooth, then there is a positive number $r > 0$ such that the ordinary approximation problem is well posed for any x in r-*tube* around A



$$U^r(A) = \{x \in E \mid 0 < \varrho_{B_1(0)}(x, A) < r\}.$$

Historical review

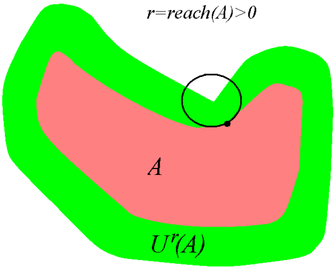
Federer (1959) for a set $A \subset \mathbb{R}^n$ defined

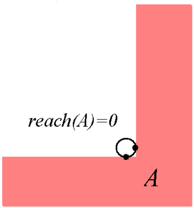
$$\operatorname{reach}(A) = \sup\{r > 0 \,|\, P_{B_1(0)}(x, A) \text{ is a singleton } \forall x \in U^r(A)\}.$$

Historical review

Federer (1959) for a set $A \subset \mathbb{R}^n$ defined

$$\operatorname{reach}(A) = \sup\{r > 0 \,|\, P_{B_1(0)}(x, A) \text{ is a singleton } \forall x \in U^r(A)\}.$$

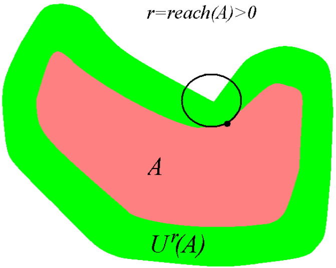


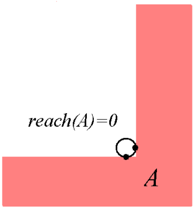


Historical review

Federer (1959) for a set $A \subset \mathbb{R}^n$ defined

$$\operatorname{reach}(A) = \sup\{r > 0 \mid P_{B_1(0)}(x, A) \text{ is a singleton } \forall x \in U^r(A)\}.$$





Federer proved that the distance function $\varrho_{B_1(0)}(\cdot, A)$ is continuously differentiable on the set $U^r(A)$ with $r = \operatorname{reach}(A)$.

Historical review

Clarke, Stern and Wolenski (1995) introduced and studied the *proximally smooth sets* in a Hilbert space H. A set $A \subset H$ is said to be r-proximally smooth if the distance function $\varrho_{B_1(0)}(\cdot, A)$ is continuously differentiable on $U^r(A)$.

Historical review

Clarke, Stern and Wolenski (1995) introduced and studied the *proximally smooth sets* in a Hilbert space H. A set $A \subset H$ is said to be r-proximally smooth if the distance function $\varrho_{B_1(0)}(\cdot, A)$ is continuously differentiable on $U^r(A)$.

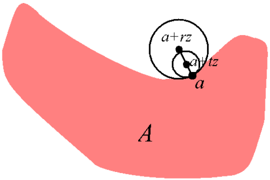
Poliquin and Rockafellar (1996) introduced the notion of prox-regularity. A set A is called *uniformly r-prox-regular* if

$$P_{B_1(0)}(a + rz, A) = \{a\}, \qquad \forall a \in A, \quad \forall z \in N^P(a, A) : \; \|z\| < 1,$$

where

$$N^P(a, A) = \{z \in E \mid \exists t > 0 : \; a \in P_{B_1(0)}(a + tz, A)\}.$$

is the *proximal normal cone* to a set $A \subset E$ at a point $a \in A$.



Poliquin, Rockafellar and Thibault (2000) showed that in a Hilbert space the class of r-proximally smooth sets coincides with the class of uniformly r-prox-regular sets.

The moduli of convexity and smoothness

The *modulus of convexity* of a Banach space E is

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \;\middle|\; x, y \in \partial B_1(0),\ \|x-y\| \geq \varepsilon \right\}, \quad \varepsilon \in (0,2].$$

The space E is called *uniformly convex* if $\delta_E(\varepsilon) > 0\ \forall \varepsilon \in (0,2]$.
The modulus of convexity is of *power type* q if for some $C > 0$ one has $\delta_E(\varepsilon) \geq C\varepsilon^q\ \forall \varepsilon \in (0,2]$.

The moduli of convexity and smoothness

The *modulus of convexity* of a Banach space E is

$$\delta_E(\varepsilon) = \inf\left\{ 1 - \frac{\|x+y\|}{2} \,\middle|\, x, y \in \partial B_1(0),\ \|x-y\| \geq \varepsilon \right\}, \quad \varepsilon \in (0, 2].$$

The space E is called *uniformly convex* if $\delta_E(\varepsilon) > 0\ \forall \varepsilon \in (0, 2]$.
The modulus of convexity is of *power type* q if for some $C > 0$ one has $\delta_E(\varepsilon) \geq C\varepsilon^q\ \forall \varepsilon \in (0, 2]$.

The *modulus of smoothness* of a Banach space E is

$$\beta_E(\tau) = \sup\left\{ \frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 \,\middle|\, x, y \in \partial B_1(0) \right\}, \qquad \tau \geq 0.$$

The space E is called *uniformly smooth* if $\lim_{\tau \to +0} \frac{\beta_E(\tau)}{\tau} = 0$.
The modulus of smoothness is of *power type* s with $s > 1$ if for some $c > 0$ one has $\beta_E(\tau) \leq c\tau^s\ \forall \tau \geq 0$.

Historical review

Proposition 1. (Bernard, Thibault and Zlateva (2006).)

Assume that the moduli of uniform convexity and uniform smoothness of a Banach space E are of power types. Then for a closed set $A \subset E$ the following statements are equivalent:

(i) A *is uniformly* r*-prox-regular;*

(ii) $P_{B_1(0)}(\cdot, A)$ *is single-valued and continuous on* $U^r(A)$*;*

(iii) $\varrho_{B_1(0)}(\cdot, A)$ *is continuously differentiable on* $U^r(A)$*.*

Historical review

Proposition 1. (Bernard, Thibault and Zlateva (2006).)

Assume that the moduli of uniform convexity and uniform smoothness of a Banach space E are of power types. Then for a closed set $A \subset E$ the following statements are equivalent:

(i) *A is uniformly r-prox-regular;*

(ii) *$P_{B_1(0)}(\cdot, A)$ is single-valued and continuous on $U^r(A)$;*

(iii) *$\varrho_{B_1(0)}(\cdot, A)$ is continuously differentiable on $U^r(A)$.*

We have proved that statements (i) and (ii) of Proposition 1 are equivalent provided that E is a uniformly convex Banach space without any assumption about smoothness of E. The assumption of Proposition 1 about power type of the moduli may be omitted. Moreover, we have extended Proposition 1 for nonsymmetric seminorm (or a quasiball).

Weakly convex sets

Let $M \subset E$ be a quasiball.

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N_M^1(a, A) = \{z \in \partial M \mid \exists t > 0 : \ a \in P_M(a + tz, A)\}.$$

Weakly convex sets

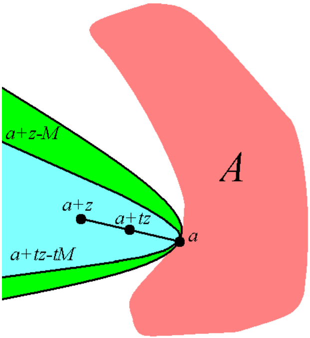
Let $M \subset E$ be a quasiball.

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N_M^1(a, A) = \{z \in \partial M \mid \exists t > 0 : \ a \in P_M(a + tz, A)\}.$$

A set $A \subset E$ is called *weakly convex* w.r.t. the quasiball M if $N_M^1(a_0, A) \neq \emptyset$ for some $a_0 \in A$ and

$$a \in P_M(a + z, A), \qquad \forall a \in A, \quad \forall z \in N_M^1(a, A).$$



Weakly convex sets

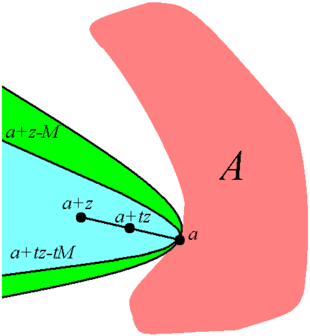
Let $M \subset E$ be a quasiball.

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1_M(a, A) = \{z \in \partial M \mid \exists t > 0 : \ a \in P_M(a + tz, A)\}.$$

A set $A \subset E$ is called *weakly convex* w.r.t. the quasiball M if $N^1_M(a_0, A) \neq \emptyset$ for some $a_0 \in A$ and

$$a \in P_M(a + z, A), \qquad \forall a \in A, \quad \forall z \in N^1_M(a, A).$$

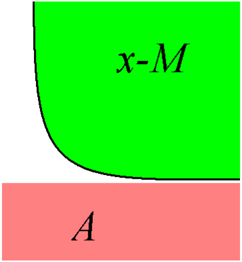


Note:

- In the case of uniformly convex space and $M = B_r(0)$, $r > 0$ the family of weakly convex sets is exactly the family of r-prox-regular sets.
- Any convex set $A \subset E$ is weakly convex w.r.t. any quasiball M.

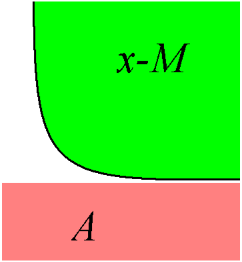
Parabolic sets

If the quasiball is unbounded the M-projection may be empty even for a convex closed set A in a finite dimensional space.



Parabolic sets

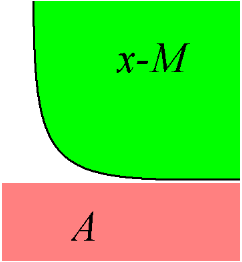
If the quasiball is unbounded the M-projection may be empty even for a convex closed set A in a finite dimensional space.



To avoid this unwanted effect we introduce the notion of parabolic set.
A set $M \subset E$ is said to be *parabolic* if it is closed convex and for every $b \in E$ the set $M \setminus (2M - b)$ is bounded.

Parabolic sets

If the quasiball is unbounded the M-projection may be empty even for a convex closed set A in a finite dimensional space.



To avoid this unwanted effect we introduce the notion of parabolic set.
A set $M \subset E$ is said to be *parabolic* if it is closed convex and for every $b \in E$ the set $M \setminus (2M - b)$ is bounded.

Note:

- Any bounded set is parabolic.
- The epigraph of the parabola $y = x^2$ is parabolic while the epigraph of the hyperbola $y = \frac{1}{x}$, $x > 0$ is not parabolic.
- The epigraph of any convex coercive function $f : E \to \mathbb{R}$ is parabolic. (The function f is called coercive if $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty$.)

Bounded uniform convexity

A quasiball $M \subset E$ is called *uniformly convex* if $\delta_M(\varepsilon) > 0 \ \forall \varepsilon > 0$, where

$$\delta_M(\varepsilon) = \inf \left\{ 1 - \mu_M \left(\frac{x+y}{2} \right) : \ x, y \in M, \ \|x - y\| \geq \varepsilon \right\}.$$

The uniform convexity of the quasiball is essential for the metric projection to exist and to be unique. But unbounded quasiball can't be uniformly convex. That's why we introduce the following weakened modification of the uniform convexity.

Bounded uniform convexity

A quasiball $M \subset E$ is called *uniformly convex* if $\delta_M(\varepsilon) > 0 \ \forall \varepsilon > 0$, where

$$\delta_M(\varepsilon) = \inf \left\{ 1 - \mu_M \left(\frac{x+y}{2} \right) : \ x, y \in M, \ \|x - y\| \geq \varepsilon \right\}.$$

The uniform convexity of the quasiball is essential for the metric projection to exist and to be unique. But unbounded quasiball can't be uniformly convex. That's why we introduce the following weakened modification of the uniform convexity.

A quasiball $M \subset E$ is called *boundedly uniformly convex* if $\delta_M(\varepsilon, R) > 0 \ \forall \varepsilon > 0, \forall R > 0$, where

$$\delta_M(\varepsilon, R) = \inf \left\{ 1 - \mu_M \left(\frac{x+y}{2} \right) : \ x, y \in M \cap B_R(0), \ \|x - y\| \geq \varepsilon \right\}.$$

Characterization of weakly convex sets

Theorem 1.

Let $M \subset E$ be a boundedly uniformly convex and parabolic quasiball, $A \subset E$ be a closed set, $U_M(A) = \{x \in E \mid 0 < \varrho_M(x, A) < 1\} \neq \emptyset$. Then the assertions (i)–(iii) are equivalent:

- (i) *A is weakly convex w.r.t. M;*
- (ii) *for any $x_0 \in U_M(A)$ the generilized approximation problem*

$$\min_{a \in A} \mu_M(x - a)$$

  is well posed;
- (iii) *the M-projection mapping $x \mapsto P_M(x, A)$ is single-valued and continuous on $U_M(A)$.*

Characterization of weakly convex sets

Theorem 1.

Let $M \subset E$ be a boundedly uniformly convex and parabolic quasiball, $A \subset E$ be a closed set, $U_M(A) = \{x \in E \mid 0 < \varrho_M(x, A) < 1\} \neq \emptyset$. Then the assertions (i)–(iii) are equivalent:

(i) *A is weakly convex w.r.t. M;*

(ii) *for any $x_0 \in U_M(A)$ the generilized approximation problem*

$$\min_{a \in A} \mu_M(x - a)$$

is well posed;

(iii) *the M-projection mapping $x \mapsto P_M(x, A)$ is single-valued and continuous on $U_M(A)$.*

If additionally the Minkowski functional of M is Fréchet differentiable on $E \setminus \{0\}$, then each statement (i)–(iii) is equivalent to

(iv) *the function $\varrho_M(\cdot, A)$ is Fréchet differentiable on $U_M(A)$;*

Weakly convex functions

Given a function $g : E \to \mathbb{R} \cup \{+\infty\}$ and a number $t > 0$ we consider the function

$$g_t(x) = t \cdot g\left(\frac{x}{t}\right), \qquad \forall x \in E.$$

Note: epi $g_t = t \cdot$ epi g.

Weakly convex functions

Given a function $g : E \to \mathbb{R} \cup \{+\infty\}$ and a number $t > 0$ we consider the function

$$g_t(x) = t \cdot g\left(\frac{x}{t}\right), \qquad \forall x \in E.$$

Note: $\text{epi } g_t = t \cdot \text{epi } g$.

The *g-predifferential* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ at a point $x_0 \in \text{dom } f$ is defined by

$$\pi_g f(x_0) = \{u \in \text{dom } g \,|\, \exists t > 0 : \ (f \boxplus g_t)(x_0 + tu) = f(x) + g_t(tu)\}.$$

Weakly convex functions

Given a function $g : E \to \mathbb{R} \cup \{+\infty\}$ and a number $t > 0$ we consider the function

$$g_t(x) = t \cdot g\left(\frac{x}{t}\right), \qquad \forall x \in E.$$

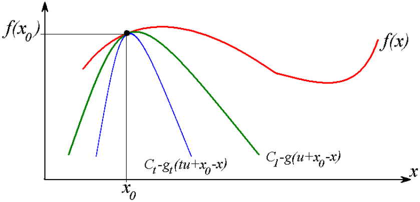
Note: epi $g_t = t \cdot$ epi g.

The *g-predifferential* of a function $f : E \to \mathbb{R} \cup \{+\infty\}$ at a point $x_0 \in \operatorname{dom} f$ is defined by

$$\pi_g f(x_0) = \{u \in \operatorname{dom} g \,|\, \exists t > 0 : \; (f \boxplus g_t)(x_0 + tu) = f(x) + g_t(tu)\}.$$

A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be *weakly convex* with respect to $g : E \to \mathbb{R} \cup \{+\infty\}$ if $\operatorname{dom}(f \boxplus g) \neq \emptyset$ and

$$(f \boxplus g)(x_0 + u) = f(x_0) + g(u), \qquad \forall x \in \operatorname{dom} f, \quad \forall u \in \pi_g f(x_0).$$



Weakly convex functions

Weakly convex functions

Theorem 2.

Let $E = H$ be a Hilbert space. Assume that a function $g : H \to \mathbb{R}$ be defined by $g(x) = \frac{\|x\|^2}{2}$. Assume that a function $f : H \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and $\operatorname{dom} f \neq \emptyset$. Then the following statements are equivalent:

(i) f is weakly convex w.r.t. g;

(ii) the function $x \mapsto f(x) + g(x)$ is convex.

Weakly convex functions

Theorem 2.

Let $E = H$ be a Hilbert space. Assume that a function $g : H \to \mathbb{R}$ be defined by $g(x) = \frac{\|x\|^2}{2}$. Assume that a function $f : H \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and dom $f \neq \emptyset$. Then the following statements are equivalent:

(i) f is weakly convex w.r.t. g;

(ii) the function $x \mapsto f(x) + g(x)$ is convex.

Theorem 2 implies that in a Hilbert space the weak convexity w.r.t. the function $g(x) = \frac{\|x\|^2}{2}$ is equivalent to weak convexity by the terminology of Vial and lower-C^2 property due to Rockafellar.

Relations between the weak convexity of functions and sets

Theorem 3.

Let $g : E \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, $g(0) < 0$, and $0 \in \operatorname{int} \operatorname{dom} g$. Then for any function $f : E \to \mathbb{R} \cup \{+\infty\}$ the following statements are equivalent:

(i) the function f is weakly convex w.r.t. the function g;

(ii) the set $\operatorname{epi} f$ is weakly convex w.r.t. the quasiball $\operatorname{epi} g$.

Well posedness of the infimal convolution problem

Theorem 4.

Let $g : E \to \mathbb{R}$ be a coercive function, bounded on any bounded set, and uniformly convex on any convex bounded set. Suppose that a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous. The following statements are equivalent:

(i) the function f is weakly convex w.r.t. the function g;

(ii) for any $t \in (0,1)$ and $x_0 \in E$ the problem

$$\min_{u \in E} \Big(f(u) + g_t(x_0 - u) \Big)$$

is well posed.

Continuity modulus of the metric projection

Proposition 2. (Bernard, Thibault and Zlateva (2011).)

Assume that the moduli of uniform convexity and smoothness of the space E *are of power types* q *and* $s \geq 1$*, respectively. Let* $0 < r' < \frac{r}{2}$ *and let the set* $A \subset E$ *be uniformly* r*-prox-regular. Then for any* $R > 0$ *the metric projection* $x \mapsto P_{B_1(0)}(x, A)$ *is Hölder continuous with the exponent* $\frac{1}{q}$ *on* $U^{r'}(A) \cap B_R(0)$*.*

Continuity modulus of the metric projection

Proposition 2. (Bernard, Thibault and Zlateva (2011).)

Assume that the moduli of uniform convexity and smoothness of the space E are of power types q and $s \geq 1$, respectively. Let $0 < r' < \frac{r}{2}$ and let the set $A \subset E$ be uniformly r-prox-regular. Then for any $R > 0$ the metric projection $x \mapsto P_{B_1(0)}(x, A)$ is Hölder continuous with the exponent $\frac{1}{q}$ on $U^{r'}(A) \cap B_R(0)$.

Theorem 5.

Assume that the moduli of uniform convexity and smoothness of the quasiball M are of power types q and $s \geq 1$, respectively. Let $0 < r' < r$ and let the set $A \subset E$ be weakly convex w.r.t. the quasiball M. Then the metric projection $x \mapsto P_M(x, A)$ is Hölder continuous with the exponent $\frac{s}{q}$ on $U^{r'}(A)$.

Continuity modulus of the metric projection

Theorem 5 is the direct consequence of the following one.

Theorem 6.

Let the set $A \subset E$ *be weakly convex w.r.t. the quasiball* M*,* $B_\sigma(0) \subset M \subset B_{\varkappa\sigma}(0)$ *for some* $\sigma, \varkappa > 0$*. Assume that*

$$x_1, x_2 \in E, \quad \varrho_M(x_1, A) = \varrho \in (0,1), \quad a_1 \in P_M(x_1, A),\ a_2 \in P_M(x_2, A).$$

Then

$$\|a_1 - a_2\| \leq 4\varrho\delta_M^{-1}\left(\beta_M\left(\frac{(1+\varkappa)\|x_1 - x_2\|}{\min\{\varrho, 1-\varrho\}}\right)\right),$$

where $\delta_M^{-1}(\cdot)$ *is the inverse function of the modulus of convexity* $\delta_M(\cdot)$ *and*

$$\beta_M(\tau) = \sup\left\{ \frac{\mu_M(x+\tau y) + \mu_M(x - \tau y)}{2} - 1 \,\middle|\, x \in \partial M,\ y \in \partial B_1(0)\right\}$$

is the modulus of smoothnes of the quasiball M*.*

Hausdorff continuity and selections of the intersection of multifunctions

The Pompeiu–Hausdorff distance between $A \subset E$ and $C \subset E$ is

$$h(A, C) = \max \left\{ \sup_{a \in A} \varrho_{B_1(0)}(a, C),\ \sup_{c \in C} \varrho_{B_1(0)}(c, A) \right\}.$$

Let (T, ϱ_T) be a metric space. A multifunction $F : T \to 2^E$ is called *Hausdorff continuous* if for all $t_0 \in T$ we have $h(F(t), F(t_0)) \to 0$ as $t \to t_0$.

Hausdorff continuity and selections of the intersection of multifunctions

The Pompeiu–Hausdorff distance between $A \subset E$ and $C \subset E$ is

$$h(A, C) = \max\left\{\sup_{a\in A} \varrho_{B_1(0)}(a, C),\ \sup_{c\in C} \varrho_{B_1(0)}(c, A)\right\}.$$

Let (T, ϱ_T) be a metric space. A multifunction $F : T \to 2^E$ is called *Hausdorff continuous* if for all $t_0 \in T$ we have $h(F(t), F(t_0)) \to 0$ as $t \to t_0$.

Consider two Hausdorff continuous multifunctions $A : T \to 2^E$ and $C : T \to 2^E$.

Hausdorff continuity and selections of the intersection of multifunctions

The Pompeiu–Hausdorff distance between $A \subset E$ and $C \subset E$ is

$$h(A, C) = \max\left\{\sup_{a\in A} \varrho_{B_1(0)}(a, C),\ \sup_{c\in C} \varrho_{B_1(0)}(c, A)\right\}.$$

Let (T, ϱ_T) be a metric space. A multifunction $F : T \to 2^E$ is called *Hausdorff continuous* if for all $t_0 \in T$ we have $h(F(t), F(t_0)) \to 0$ as $t \to t_0$.

Consider two Hausdorff continuous multifunctions $A : T \to 2^E$ and $C : T \to 2^E$.
What properties of the multifunctions are sufficient for the multifunction $F(t) = A(t) \cap C(t)$ to be Hausdorff continuous and to have a continuous selection on T?

Hausdorff continuity and selections of the intersection of multifunctions

The Pompeiu–Hausdorff distance between $A \subset E$ and $C \subset E$ is

$$h(A, C) = \max \left\{ \sup_{a \in A} \varrho_{B_1(0)}(a, C), \ \sup_{c \in C} \varrho_{B_1(0)}(c, A) \right\}.$$

Let (T, ϱ_T) be a metric space. A multifunction $F : T \to 2^E$ is called *Hausdorff continuous* if for all $t_0 \in T$ we have $h(F(t), F(t_0)) \to 0$ as $t \to t_0$.

Consider two Hausdorff continuous multifunctions $A : T \to 2^E$ and $C : T \to 2^E$.
What properties of the multifunctions are sufficient for the multifunction $F(t) = A(t) \cap C(t)$ to be Hausdorff continuous and to have a continuous selection on T?
Balashov and Repovš (2010) showed that to obtain the desired properties of $F(\cdot)$ it suffices to assume that $C(t)$ is closed and uniformly convex and $A(t)$ is closed and convex or satisfy some condition in terms of of the modulus of nonconvexity. The latter condition for unconvex sets may be satisfied only if the convexity modulus of the Banach space is of the second order.

Hausdorff continuity and selections of the intersection of multifunctions

The following theorem in terms of weak convexity states some sufficient conditions for $F(\cdot)$ to be continuous and to have a continuous selection.

Hausdorff continuity and selections of the intersection of multifunctions

The following theorem in terms of weak convexity states some sufficient conditions for $F(\cdot)$ to be continuous and to have a continuous selection.

Theorem 7.

Suppose that the multifunctions $A : T \to 2^E$ and $C : T \to 2^E$ are Hausdorff continuous. Assume that for any $t \in T$ the set $C(t)$ is a quasiball and the family $\{C(t)\}_{t\in T}$ is equi uniformly convex, i.e.

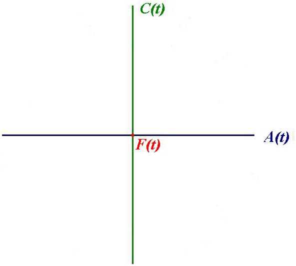
$$\inf_{t\in T} \delta_{C(t)}(\varepsilon) > 0 \qquad \forall \varepsilon > 0.$$

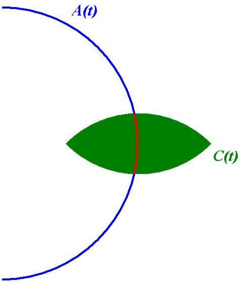
Suppose that there exists a constant $r \in (0, 1)$ such that for any $t \in T$ the set $rA(t)$ is weakly convex w.r.t. the quasiball $C(t)$. Assume that

$$F(t) = A(t) \cap C(t) \neq \emptyset \qquad \forall t \in T.$$

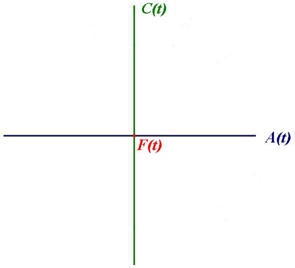
Then the multifunction $F(\cdot)$ is Hausdorff continuous and has a continuous selection on T.

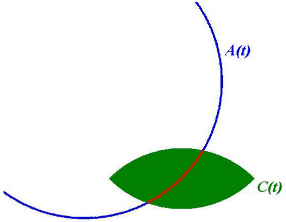
Hausdorff continuity of the intersection of multifunctions



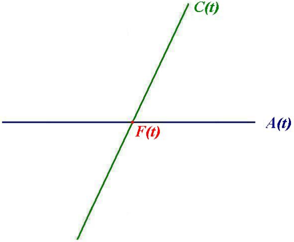


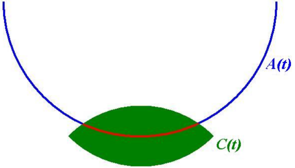
Hausdorff continuity of the intersection of multifunctions



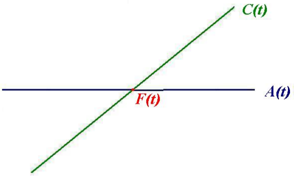


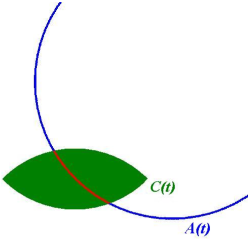
Hausdorff continuity of the intersection of multifunctions



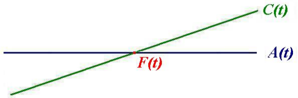


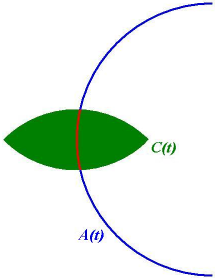
Hausdorff continuity of the intersection of multifunctions



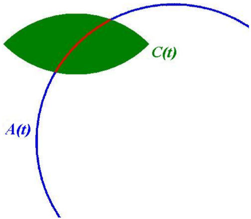


Hausdorff continuity of the intersection of multifunctions

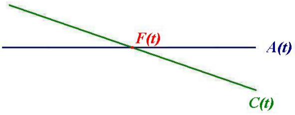


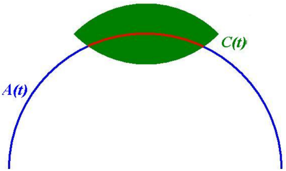


Hausdorff continuity of the intersection of multifunctions

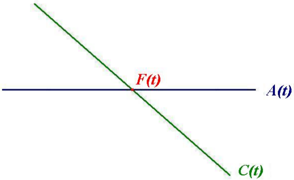


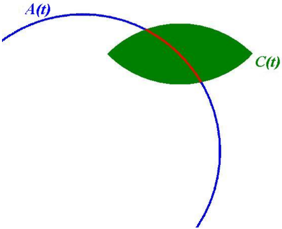
Hausdorff continuity of the intersection of multifunctions



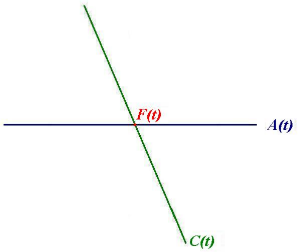


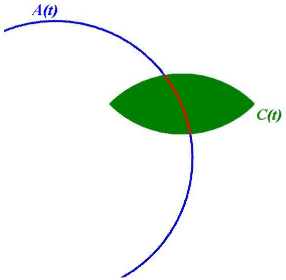
Hausdorff continuity of the intersection of multifunctions



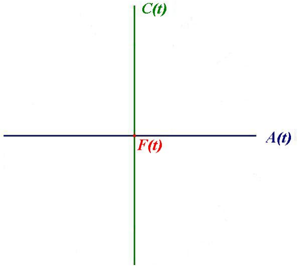


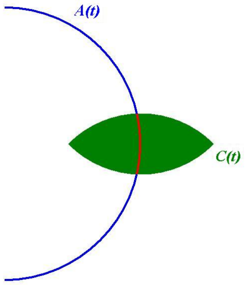
Hausdorff continuity of the intersection of multifunctions





Hausdorff continuity of the intersection of multifunctions





Hausdorff continuity of the intersection of multifunctions

The bibliography

[1] *H. Federer*: Curvature measures, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[2] *F. H. Clarke,R. J. Stern,P. R. Wolenski*: Proximal Smoothness and Lower–C^2 Property, J. Convex Analysis 2 (1995) 117–144.

[3] *R. A. Poliquin, R. T. Rockafellar, L. Thibault*: Local differentiability of distance functions, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[4] *F. Bernard, L. Thibault, N. Zlateva*: Characterization of proximal regular sets in super reflexive Banach spaces, J. Convex Analysis 13 (2006) 525–559.

[5] *F. Bernard, L. Thibault, N. Zlateva*: Prox-regular sets and epigraphs in uniformly convex Banach spaces: Various regularities and other properties, Trans. Amer. Math. Soc. 363 (2011) 2211–2247.

The bibliography

[6] *G. Colombo, V. V. Goncharov, B. S. Mordukhovich*: Well-Posedness of Minimal Time Problems with Constant Dynamics in Banach Spaces, Set-Valued and Var. Anal. 18:3-4 (2010) 349-372.

[7] *V. V. Goncharov, F. F. Pereira*: Neighbourhood retractions of nonconvex sets in a Hilbert space via sublinear functionals, J. Convex Analysis 18 (2011) 1–36.

[8] *V. V. Goncharov, F. F. Pereira*: Geometric Conditions for Regularity in a Time-Minimum Problem with Constant Dynamics, J. Convex Analysis 19 (2012) 631–669.

[9] *M. V. Balashov, Repovš D.*: Weakly convex sets and modulus of nonconvexity, J. Math. Anal. Appl. 371 (2010) 113—127.

For details see

1. *G. E. Ivanov*: Weak Convexity of Sets and Functions in a Banach Space, J. Convex Analysis. 22:2 (2015) 365–398.
2. *G. E. Ivanov*: Sharp Estimates for the Moduli of Continuity of Metric Projections onto Weakly Convex Sets, Izv. RAN. Ser. Mat. (2015).
3. *G. E. Ivanov*: Continuity and Selections of the Intersection Operator Applied to Nonconvex Sets, J. Convex Analysis. 22:4 (2015).

Thank you!