The Partition Function and Ramanujan's $5k + 4$ Congruence

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The unrestricted partition function, $p(n)$, is a much-studied function in additive number theory which also has uses in many other areas, including the golden ratio (see [1]). The function serves as a counter for the number of ways a positive integer can be split up into addends. For example, $p(4) = 5$ since the number 4 can be split up in 5 ways, called partitions: 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. Even though the order of the addends in any particular partition is of no concern when counting partitions, customarily one writes the addends from largest to smallest. Partitions may be viewed geometrically using Ferrers diagrams, a way of producing these functions graphically with points.

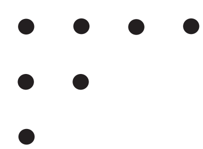

Figure 1

Figure 1 shows a Ferrers diagram of the number 7. Top to bottom, it shows $4+2+1$, but from left to right it reveals $3+2+1+1$. This is the best way of representing the function graphically, and also produces 2 partitions with one picture, if the picture is not symmetric with respect to the diagonal, as shown in Figure 2.

The main part of my thesis deals with the so-called $5k+4$ congruence, discovered by the famous Indian mathematician Ramanujan. Ramanujan was inspired by staring at a table of partitions constructed by MacMahon, who found and listed the values for $p(n)$ up to $p(200)$ (see [2]). The congruence is as follows: If a natural number n has the form $5k+4$, where k is a natural number, then $p(n)$ is divisible by 5 (that is, $p(n) \equiv 0 \pmod 5$). Many proofs of this identity involve modular functions, but Kruyswijk has proved the $5k+4$ congruence and other similar identities involving $p(n)$ without any modular functions. I am essentially following Kruyswijk's steps as a basis for my thesis, as is partially outlined in Apostol's book (see [3] and [2], respectively).

There are two essential tools used in proving the $5k+4$ identity: *Euler's generating function for* $p(n)$ and *Euler's Pentagonal Number Theorem.* Euler's Generating Function, shown below,

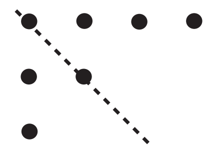

Figure 2

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} \frac{1}{1-x^m} \tag{1}$$

valid for $|x| < 1$, is very influential because it helps us compute specific values of $p(n)$ without explicitly trying to figure out every combination. (Here, we define $p(0) = 1$.) In fact, MacMahon, a mathematician known for his lists and tables of values, used this function to help construct his list of values of $p(n)$. To see why the product given above generates values of the partition function, one first expresses each term of the product as a geometric series, then multiplies these series together, and finally gathers like powers of x. In the process, via the additive laws of exponents, one sees that gathering these powers of x is tantamount to computing values of $p(n)$.

On the other hand, pentagonal numbers are numbers formed from graphs such as in figure 3, where the number of dots in the nth figure represent the nth pentagonal number $\omega(n)$. One notices from the figures that a formula for $\omega(n)$ may be obtained by summing the terms of an arithmetic progression, yielding $\omega(n) = \frac{3n^2-n}{2}$. In the sequel, we use this formula to extend $\omega(n)$ to a function on the entire set of integers.

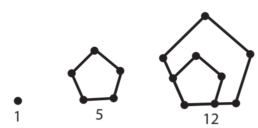

Figure 3

The Pentagonal Number Theorem relates the pentagonal numbers to the reciprocal of Euler's generating function. It states, for $|x| < 1$,

$$\sum_{m=1}^{\infty} (1-x^m) = 1 + \sum_{n=1}^{\infty} (-1)^n (x^{\omega(n)} + x^{\omega(-n)}) \tag{2}$$

There is a beautiful combinatorial proof based on trying to establish a one-to-one correspondence between odd and even unequal partitions. This proof, done by Franklin, is called by George Andrews "one of the truly remarkable achievements of nineteenth-century American mathematicians."

To conclude, I provide an outline of a proof of the $5k+4$ congruence. We will need some notation: $\phi(x) = \sum_{n=1}^{\infty}(1-x^n)$ and $\alpha = e^{2\pi i/5}$. Throughout, x represents a complex number with $|x| < 1$. To begin, using elementary facts about integer divisibility and complex n-th roots of unity, one may show that

$$\prod_{h=1}^{k} (1 - x^n e^{2\pi i n h/k}) = \begin{cases} 1 - x^{nk} & \text{if } (n,k) = 1 \\ (1-x^n)^k & \text{if } k|n \end{cases}$$

Using this equation in the case $k = 5$, one deduces

$$\prod_{n=1}^{\infty} \prod_{h=1}^{5} (1 - x^n \alpha^h) = \frac{\phi(x^5)^6}{\phi(x^{25})}$$

and we can manipulate the last equation to obtain

$$\sum_{m=0}^{\infty} p(m)x^m = \frac{\phi(x^5)^6}{\phi(x^{25})} \prod_{h=1}^{4} \prod_{n=1}^{\infty} (1 - x^n \alpha^h)$$

by peeling off the portion of the product corresponding to $h = 5$ in the case $k = 5$ and applying the Generating Function for $p(n)$.

To finish the proof, one applies the Pentagonal Number Theorem to the product in the right hand side of the last equation and then discards all but the terms type 4 (that is, terms where the power on x has remainder 4 upon division by 5), to obtain:

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = V_4 \frac{\phi(x^{25})}{\phi(x^5)^6}$$

where V_4 represents the type 4 terms from $\prod_{h=1}^{4} \prod_{n=1}^{\infty} (1 - x^n \alpha^h)$. With some more work we discover that $V_4 = 5x^4\, \phi(x^{25})^4$, and by replacing x^5 by x, we obtain

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5 \frac{\phi(x^5)^5}{\phi(x)^6}$$

Since the coefficients of the series in the right-hand side of the previous equation are all divisible by 5, we conclude that $p(5m+4)$ is divisible by 5 for all numbers m.

References

[1] G. Andrews, Number theory, Saunders College Publishing (1971).

[2] T. Apostol, Introduction to analytic number theory, Springer-Verlag (1986).

[3] D. Kruyswijk, *On some well-known properties of the partition function* $p(n)$, Nieuw. Arch. Wisk. **23** (2) (1950) 97–107.