Consecutive integers divisible by the square of their largest prime factors

JEAN-MARIE DE KONINCK, NICOLAS DOYON AND FLORIAN LUCA

Abstract

Given fixed integers $k \geq 1$ and $\ell \geq 1$, let $E_{k,\ell}$ be the set of those positive integers n such that $P(n+i)^\ell \mid n+i$ for each $i = 0, 1, \ldots, k-1$, where $P(n)$ stands for the largest prime factor of n. We study the counting function given by $E(x) = \#\{n \leq x : n \in E_{2,2}\}$, showing in particular that $E(x) \gg x^{1/4}/\log x$ and that there exists a positive constant c such that $E(x) \ll x \exp\{-c\sqrt{\log x \log\log x}\}$. Then, given an integer $r \geq 2$, we consider the problem of searching for consecutive integers each of which is divisible by a power of its r-th largest prime factor.

Édition du 22 juin 2013, pm

1 Introduction

Let $P(n)$ stand for the largest prime factor of an integer $n \geq 2$. Set $P(1) = 1$. Given an arbitrary positive integer ℓ and a finite set of distinct primes, say $\{p_0, p_1, \ldots, p_{k-1}\}$, the Chinese Remainder Theorem guarantees the existence of infinitely many integers n such that $p_i^\ell \mid n+i$ for $i = 0, 1, \ldots, k-1$. However, this theorem does not guarantee that such integers n will also have the property that $P(n+i) = p_i$ for $i = 0, 1, \ldots, k-1$, although such is the case in some particular instances, for example when $\ell = 2$, $k = 3$ and $n = 1\,294\,298$, in which case we indeed have

$$\begin{aligned} 1\,294\,298 &= 2 \cdot 61 \cdot 103^2, \\ 1\,294\,299 &= 3^4 \cdot 19 \cdot 29^2, \\ 1\,294\,300 &= 2^2 \cdot 5^2 \cdot 7 \cdot 43^2. \end{aligned}$$

This motivates the following definition. Given fixed positive integers k and ℓ, set

$$E_{k,\ell} := \{n \in \mathbb{N} : P(n+i)^\ell \mid n+i \ \text{ for each } i = 0, 1, \ldots, k-1\}.$$

Many elements of $E_{2,2}$, $E_{2,3}$, $E_{2,4}$, $E_{2,5}$ and $E_{3,2}$ are given in the book of the first author [2]. However, no elements of $E_{3,3}$ and $E_{4,2}$ are known. In fact, if n belongs to any one of these last two sets, it can be shown that $n > 10^{30}$.

Nevertheless, it seems reasonable to conjecture that, given any fixed integers $k \geq 2$ and $\ell \geq 2$, then $\#E_{k,\ell} = \infty$.

This is certainly true in the particular case $k = \ell = 2$, as it is an immediate consequence of the fact that the Fermat-Pell equation $x^2 - 2y^2 = 1$ has infinitely many positive integer solutions (x, y).

Here we focus our attention on the size of

$$E(x) = E_{2,2}(x) := \#\{n \le x : n \in E_{2,2}\}.$$

Then, for a given integer $r \ge 2$, we consider the problem of searching for consecutive integers each of which is divisible by a power of its r-th largest prime factor.

2 Preliminary results

Theorem 1. *Let p and q be two distinct prime numbers. Then, there are only finitely many integers n for which $P(n)^2 \mid n$ and $P(n+1)^2 \mid (n+1)$ with $P(n) = p$ and $P(n+1) = q$.*

Proof. This follows immediately from the fact that, as n becomes large,

$$\max(P(n), P(n+1)) \gg \log\log n. \tag{2.1}$$

How does one obtain (2.1) ? There is a deep theorem in diophantine analysis which asserts that if $f(x) \in \mathbb{Z}[x]$ is a polynomial with at least two distinct roots, then there exists a positive constant $C := C(f)$ such that $P(f(n)) > C \log\log n$ if n is sufficiently large. This result can be found in the book of Shorey and Tijdeman [8] (see inequality (31) on Page 134). Thus, choosing $f(x) = (x+1)(x+2)$, we immediately obtain (2.1). □

3 Evaluating the size of $E(x)$

One can obtain the expected size of $E(x)$ as follows. Let us first recall the Ψ function defined as

$$\Psi(x, y) = \#\{n \le x : P(n) \le y\} \qquad (2 \le y \le x).$$

It is known that, setting $u = \log x / \log y$, then, keeping u fixed, we have

$$\Psi(x, y) = (1 + o(1))\rho(u)x \qquad (x \to \infty),$$

where $\rho(u)$ is the Dickman function, whose behavior is given by

$$\rho(u) = \exp\{-u(\log u + \log\log u - 1 + o(1))\} \quad \text{as} \quad (u \to \infty) \tag{3.1}$$

(see for instance Theorem 9.3 in the book of De Koninck and Luca [4]).

The probability Q that $P(n)^2 \mid n$ is

$$Q = \frac{1}{x} \sum_{\substack{n \le x \\ P^2(n) \mid n}} 1 = \frac{1}{x} \sum_{\substack{mp^2 \le x \\ P(m) \le p}} 1 = \frac{1}{x} \sum_{p \le x^{1/2}} \Psi\left(\frac{x}{p^2}, p\right)$$

$$
\begin{aligned}
&= (1+o(1)) \sum_{p \le x^{1/2}} \frac{1}{p^2} \rho\left(\frac{\log x}{\log p} - 2\right) \\
&= (1+o(1)) \int_2^{\sqrt{x}} \frac{1}{t^2 \log t} \rho\left(\frac{\log x}{\log t} - 2\right) dt \\
&= (1+o(1)) \int_{\log 2}^{\frac{1}{2}\log x} \frac{1}{v e^v} \rho\left(\frac{\log x}{v} - 2\right) dv \\
&= (1+o(1)) \int_{\log 2}^{\frac{1}{2}\log x} f(v)\, dv,
\end{aligned}
$$

say, as $x \to \infty$. Here,

$$f(v) = \frac{1}{v e^v} \rho\left(\frac{\log x}{v} - 2\right) \qquad (\log 2 \le v \le (\log x)/2).$$

Define

$$\eta(x) := \sqrt{\log x \log\log x}.$$

Setting

(3.2)

$$h(v) = \frac{1}{v} \exp\left\{-v - \left(\frac{\log x}{v} - 2\right)\left(\log\left(\frac{\log x}{v} - 2\right) + \log\log\left(\frac{\log x}{v} - 2\right)\right)\right\},$$

so that $f(v) = (1+o(1))h(v)$ as $x \to \infty$ in such a way that $v = o(\log x)$ (here, we used estimate (3.1)), we observe that the maximum of $h(v)$ is obtained when $v = (\sqrt{2}/2 + o(1))\eta(x)$ as $x \to \infty$. Substituting this value in (3.2), we obtain that

$$Q = e^{-(1+o(1))\sqrt{2}\eta(x)} \qquad (x \to \infty). \tag{3.3}$$

Hence, if we could assume that $P^2(n) \mid n$ and $P^2(n+1) \mid n+1$ are two independent events, the following conditional result would then follow from (3.3):

$$E(x) = x e^{-(2+o(1))\sqrt{2}\eta(x)} = x e^{-(2+o(1))\sqrt{2\log x \log\log x}} \qquad (x \to \infty). \tag{3.4}$$

Remark 1. *This method can be extended to obtain heuristic estimates for* $E_{k,\ell}(x)$ *for arbitrary integers* $k \ge 2$, $\ell \ge 2$. *Let* $\alpha(\ell)$ *be the real number uniquely defined by*

$$\#\{n \le x : P(n)^\ell | n\} = x \exp(-(1+o(1))\alpha(\ell)\eta(x)).$$

Ivić [7] has given an unconditional proof of the heuristic estimate (3.3) *showing in particular that* $\alpha(2)$ *exists and* $\alpha(2) = \sqrt{2}$. *We therefore conjecture that*

$$\#\{n \le x : P(n+i)^\ell | (n+i), i = 0, 1, \ldots, k-1\} = x \exp(-(1+o(1))\alpha(\ell)k\eta(x))$$

as $x \to \infty$.

4 The quest for a lower bound for $E(x)$

Theorem 2. *As x becomes large,*

$$E(x) \gg x^{1/4}/\log x. \tag{4.1}$$

Proof. For any prime p, we easily check that

$$(2p^2-1)^2-1 = 4p^2(p-1)(p+1),$$

implying that

$$E(x) \gg x^{1/4}/\log x.$$

Under the reasonable conjecture that the set of integers n for which $P(n) > P(n+1)$ and $P(n) > P(n-1)$ is of positive lower density, the denominator on the right hand side of (4.1) can be dropped, in which case we would get $E(x) \gg x^{1/4}$. □

Remark 2. *Another polynomial identity yielding to the same conclusion is*

$$n(4n+3)^2+1 = (n+1)(4n+1)^2.$$

The integers $n(4n+3)^2+1$ will be counted by $E(x)$ whenever the conditions $P(4n+3) > P(n)$ and $P(4n+1) > P(n+1)$ are simultaneously satisfied. The set of integers simultaneously satisfying these conditions is believed to be of density 1/4. If one could show that this set is indeed of positive lower density, then we would obtain $E(x) \gg x^{1/3}$.

Based on the above heuristics, we strongly believe $E(x)$ to be larger than $x^{1-\varepsilon}$ for any $\varepsilon > 0$ once x is large, which would at least support the more ambitious estimate (3.4). The problem of proving stronger lower bounds on $E(x)$ is however intrinsically difficult.

Remark 3. *Assuming that for some function f such that $\lim_{x\to\infty} f(x) = \infty$,*

$$E(x) \gg x/f(x), \tag{4.2}$$

one can show that

$$\#\{n \le x : P(n(n+1)) \le f(x)^{1+\varepsilon}\} \gg x/f(x). \tag{4.3}$$

This observation follows directly from the fact that

$$\#\{n \le x : P(n) > f(x)^{1+\varepsilon}, P(n)^2|n\} \ll \frac{x}{f(x)^{1+2\varepsilon}}.$$

The distribution of $P(n(n+1))$ has been the topic of several studies and has proven to be a very tough nut to crack. In order to hope to improve significantly our lower bound (say to obtain $E(x) \gg x^{1+o(1)}$), one would have to show that inequality (4.3) holds for some function $f(x)$ satisfying $f(x) = x^{o(1)}$ as $x \to \infty$; however, no tool seems currently available to achieve this. Obtaining a lower bound with the right order of magnitude would imply that inequality (4.3) holds with $f(x) \ll \exp(\eta(x))$, which seems a remote achievement.

5 An upper bound for $E(x)$

Let

$$\begin{aligned}\Psi(x,y;q,a) &= \#\{n \le x : P(n) \le y,\ n \equiv a \pmod q\},\\ \Psi_q(x,y) &= \#\{n \le x : P(n) \le y,\ (n,q)=1\}.\end{aligned}$$

Starting from a trivial estimate in the initial range, Granville proved (see formulas (1.2) and (1.3) in [5]) that, for any fixed positive number A and uniformly in the range $x \ge y \ge 2$, $q \le \min(x, y^A)$, and $(a,q)=1$, the estimate

$$\Psi(x,y;q,a) = \frac{1}{\phi(q)}\Psi_q(x,y)\left\{1 + O_A\left(\frac{\log q}{\log y}\right)\right\} \tag{5.1}$$

holds. By a more delicate argument, in the same paper, Granville proved the following stronger result.

Theorem 3 (Granville)**.** *For any fixed $\varepsilon > 0$ and uniformly in the range $x \ge y \ge 2$, $1 \le q \le y^{1-\varepsilon}$, and $(a,q)=1$, we have*

$$\Psi(x,y;q,a) = \frac{1}{\phi(q)}\Psi_q(x,y)\left\{1 + O_A\left(\frac{\log q}{u^c \log y} + \frac{1}{\log y}\right)\right\}, \tag{5.2}$$

where c is some positive constant.

Note that (5.2) implies the lower estimate

$$\Psi(x,y;q,a) \gg \frac{1}{\phi(q)}\Psi_q(x,y)$$

provided q is less than a sufficiently small power of y, while (5.1) shows that the corresponding upper bound holds whenever q does not exceed x and is bounded by a fixed, but arbitrarily large power of y.

For one, we have

$$E(x) < E_{1,2}(x) = xe^{-(\sqrt{2}+o(1))\eta(x)} \qquad (x \to \infty).$$

On the other hand, recall that from the previous section, we expect to have

$$E(x) = xe^{-(2\sqrt{2}+o(1))\eta(x)} \qquad (x \to \infty).$$

We will now prove an intermediate result.

Theorem 4. *The inequality*

$$E(x) \ll xe^{-c\eta(x)}$$

holds for large x with $c = (25/24)\sqrt{2} \in (\sqrt{2}, 2\sqrt{2})$.

Proof. Let $0 < a < \sqrt{2}/2$ be a constant whose exact value will be determined later, and consider the interval

$$I_a(x) := \left[\exp\left((\sqrt{2}/2 - a)\eta(x)\right), \exp\left((\sqrt{2}/2 + a)\eta(x)\right)\right].$$

We split the positive integers $n \le x$ counted by $E(x)$ in two categories.

Category 1. Numbers $n \le x$ for which both $P(n) \in I_a(x)$ and $P(n+1) \in I_a(x)$ hold.

Category 2. Numbers $n \le x$ for which $P(n) \notin I_a(x)$ or $P(n+1) \notin I_a(x)$.

Let $C_1(x)$ (resp. $C_2(x)$) be the number of integers $n \le x$ which belong to Category 1 (resp. Category 2).

In order to count the number of integers $n \le x$ falling into Category 1 we first consider those integers n for which the corresponding largest primes $p = P(n)$ and $q = P(n+1)$ satisfy $p > q$ and let $C_1'(x)$ be their counting number. Let $C_1''(x)$ stand for the other $n \le x$ counted by $C_1(x)$.

Writing $n = mp^2$ with $P(m) \le p$ and $mp^2 + 1 \equiv 0 \pmod{q^2}$, we then get

$$C_1'(x) \le \sum_{p \in I_a(x)} \sum_{q \in I_a(x)} \Psi\left(\frac{x}{p^2}, p; q^2, r\right), \tag{5.3}$$

where r stands for the inverse of $-p^2$ modulo q^2. In order to be able to use the Granville estimate (5.2), we choose a small $\varepsilon > 0$ and relax (5.3) to

$$C_1'(x) \le \sum_{p \in I_a(x)} \sum_{q \in I_a(x)} \Psi\left(\frac{x}{p^2}, p^{2+\varepsilon}; q^2, r\right). \tag{5.4}$$

Now, using (5.2), we obtain from (5.4) that

$$C_1'(x) \ll \sum_{p \in I_a(x)} \sum_{q \in I_a(x)} \frac{1}{q^2} \Psi\left(\frac{x}{p^2}, p^{2+\varepsilon}\right). \tag{5.5}$$

Since

$$\sum_{q \in I_a(x)} \frac{1}{q^2} \ll \sum_{n > \exp((\sqrt{2}/2 - a)\eta(x))} \frac{1}{n^2} \ll \exp\left(-(\sqrt{2}/2 - a)\eta(x)\right),$$

it follows from (5.5) that

$$C_1'(x) \ll \exp\left(-(\sqrt{2}/2 - a)\eta(x)\right) \sum_{p \in I_a(x)} \Psi\left(\frac{x}{p^2}, p^{2+\varepsilon}\right). \tag{5.6}$$

Setting $e^v = p$ and proceeding as in Section 3, we easily obtain that

$$\sum_{p \in I_a(x)} \Psi\left(\frac{x}{p^2}, p^{2+\varepsilon}\right) \ll x \int_{(\sqrt{2}/2 - a)\eta(x)}^{(\sqrt{2}/2 + a)\eta(x)} \frac{1}{ve^v} \rho\left(\frac{\log x}{(2+\varepsilon)v} - \frac{2}{2+\varepsilon}\right) dv. \tag{5.7}$$

Bringing together (5.6) and (5.7), recalling the known estimate

$$\rho(u) \ll \exp(-u \log u) \tag{5.8}$$

(see for instance Corollary 9.18 in the book of De Koninck and Luca [4]), we obtain

$$\begin{aligned} C_1'(x) &\ll x \exp(-(\sqrt{2}/2 - a)\eta(x)) \\ &\quad \times \max_{e^v \in I_a(x)} \exp\left(\left(-v - \frac{\log x}{(2+\varepsilon)v} \log\left(\frac{\log x}{(2+\varepsilon)v}\right)\right)(1+o(1))\right). \end{aligned} \tag{5.9}$$

Setting $v = c\eta(x)$, we get that

$$\begin{aligned} &\max_{e^v \in I_a(x)} \exp\left(-v - \frac{\log x}{(2+\varepsilon)v} \log\left(\frac{\log x}{(2+\varepsilon)v}\right)(1+o(1))\right) \\ &\quad = \max_{\sqrt{2}/2 - a \le c \le \sqrt{2}/2 + a} \exp\left(\left(-c - \frac{1}{2(2+\varepsilon)c} + o(1)\right)\eta(x)\right) \\ &\quad = \exp\left(\left(-(\sqrt{2}/2 - a) - \frac{1}{2(2+\varepsilon)(\sqrt{2}/2 - a)} + o(1)\right)\eta(x)\right) \\ &\quad = \exp\left(\left(-(\sqrt{2}/2 - a) - \frac{1}{(2+\varepsilon)(\sqrt{2} - 2a)} + o(1)\right)\eta(x)\right). \end{aligned} \tag{5.10}$$

Gathering (5.9) and (5.10), we obtain that

$$C_1'(x) \le x \exp\left(\left(-\sqrt{2} + 2a - \frac{1}{(2+\varepsilon)(\sqrt{2} - 2a)} + o(1)\right)\eta(x)\right). \tag{5.11}$$

Since ε can be taken arbitrarily small (it particular, it can be made to tend to zero), (5.11) can be replaced by the simpler estimate

$$C_1'(x) \le x \exp\left(\left(-\sqrt{2} + 2a - \frac{1}{2(\sqrt{2} - 2a)} + o(1)\right)\eta(x)\right). \tag{5.12}$$

The case where $q > p$ can be treated in a similar way, this time by setting $mq^2 = n+1$ where $m \le x/q^2$ must satisfy a congruence condition modulo p^2, in which case we obtain an upper bound for $C_1''(x)$ similar to the one in (5.12), implying that in the end we have that

$$C_1(x) \ll x \exp\left(\left(-\sqrt{2} + 2a - \frac{1}{2(\sqrt{2} - 2a)} + o(1)\right)\eta(x)\right). \tag{5.13}$$

As we did in the case of $C_1(x)$, we split the counting function $C_2(x)$ in two. Let $C_2'(x)$ (resp. $C_2''(x)$) be the cardinality of those numbers $n \le x$ such that $P(n) \notin I_a(x)$ (resp. $P(n+1) \notin I_a(x)$), so that $C_2(x) \le C_2'(x) + C_2''(x)$.

We first deal with $C_2'(x)$.

We clearly have

$$C_2'(x) \le \#\{n \le x : P(n)^2 | n, P(n) \notin I_a(x)\}. \tag{5.14}$$

The right hand side of (5.14) can be estimated as we did in Section 3 so that

$$C_2'(x) \le I_1(x) + I_2(x),$$

where

$$I_1(x) := x \int_{\log 2}^{(\sqrt{2}/2 - a)\eta(x)} \frac{1}{ve^v} \rho\left(\frac{\log x}{v} - 2\right) dv$$

and

$$I_2(x) := x \int_{(\sqrt{2}/2 + a)\eta(x)}^{\frac{1}{2}\log x} \frac{1}{ve^v} \rho\left(\frac{\log x}{v} - 2\right) dv.$$

Again using (5.8), we easily obtain that

$$I_1 \ll x \exp\left(\left(-\left(\frac{\sqrt{2}}{2} - a\right) - \frac{1}{\sqrt{2} - 2a}\right)\eta(x)\right)$$

and

$$I_2 \ll x \exp\left(\left(-\left(\frac{\sqrt{2}}{2} + a\right) - \frac{1}{\sqrt{2} + 2a}\right)\eta(x)\right).$$

Combining these upper bounds, we obtain that

$$C_2'(x) \ll x \exp\left(\left(-\left(\frac{\sqrt{2}}{2} + a\right) - \frac{1}{\sqrt{2} + 2a}\right)\eta(x)\right). \tag{5.15}$$

The same reasoning leads to an upper bound for $C_2''(x)$ similar to the one in (5.15), thus implying that

$$C_2(x) \ll x \exp\left(\left(-\left(\frac{\sqrt{2}}{2} + a\right) - \frac{1}{\sqrt{2} + 2a}\right)\eta(x)\right). \tag{5.16}$$

The choice of a is optimal when the bounds in (5.13) and (5.16) coincide, that is when

$$-\sqrt{2} + 2a - \frac{1}{2\sqrt{2} - 4a} = -\frac{\sqrt{2}}{2} - a - \frac{1}{\sqrt{2} + 2a}.$$

The above equation simplifies to

$$-\frac{\sqrt{2}}{2} + 3a = \frac{-\sqrt{2} + 6a}{4 - 8a^2}.$$

Thus, in the end, a is given by the solution to the third degree equation

$$24a^3 - 4\sqrt{2}a^2 - 6a + \sqrt{2} = 0.$$

Setting a^* as the solution of this last equation, we find that $a^* = \sqrt{2}/6$, so that

$$c := \sqrt{2} - 2a^* + \frac{1}{2\sqrt{2} - 4a^*} = \frac{25}{24}\sqrt{2},$$

which completes the proof of the theorem. □

Remark 4. *This method can be extended to show that*

$$E_{k,\ell}(x) \ll x \exp(-\beta(k,\ell)\eta(x)),$$

where the constants $\beta(k,\ell)$ *satisfy the two properties*

$$\alpha(\ell) < \beta(k,\ell) < k\alpha(\ell)$$

and

$$\beta(k,\ell) \asymp \sqrt{k} \qquad (k \to \infty)$$

for any fixed integer ℓ.

6 Consecutive integers divisible by a power of their r-th largest prime factor

In this section, we will use the *prime k-tuples Conjecture*, namely:

Conjecture 1. *(*PRIME k-TUPLES CONJECTURE *(weak form)). Let* $k \geq 2$ *be an integer. Let* $a_1, \ldots, a_k$ *and* $b_1, \ldots, b_k$ *be integers such that each* $a_i > 00$ *and for which there exist no fixed prime number dividing* $(a_1n + b_1)\cdots(a_kn + b_k)$ *for all positive integers* n. *Then there exist infinitely many integers* n *for which the numbers* $a_in + b_i$ *for* $i = 1, \ldots, k$ *are simultaneously primes.*

This conjecture is also believed to hold in the following quantitative form:

Conjecture 2. *(*PRIME k-TUPLES CONJECTURE *(strong form)). Let* $k \geq 2$ *be an integer. Let* $a_1, \ldots, a_k$ *and* $b_1, \ldots, b_k$ *be integers such that each* $a_i > 00$ *and for which there exist no fixed prime number dividing* $(a_1n + b_1)\cdots(a_kn + b_k)$ *for all positive integers* n. *Define* $S(x)$ *as the set of integers* $n \leq x$ *for which* $a_in + b_i$ *for* $i = 1, \ldots, k$, *are simultaneously primes. Then there exists a positive constant* C *depending on* $a_1, \ldots, a_k, b_1, \ldots, b_k$ *such that*

$$\#S(x) = (1 + o(1))C\frac{x}{(\log x)^k} \qquad (x \to \infty). \tag{6.1}$$

The above conjectures can be found in Chapter 2 of our book [4].

The upper bound implied in the above conjecture has been proved unconditionally using sieve methods (see Chapter 2 of [6]). We state it as follows.

Theorem 5. *Let $k \geq 2$ be an integer. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be integers such that each $a_i > 0$ and such that there exists no fixed prime number which divides $(a_1 n + b_1) \cdots (a_k n + b_k)$ for all positive integers n. Define $S(x)$ as the set of integers $n \leq x$ for which $a_i n + b_i$ for $i = 1, \ldots, k$, are simultaneously primes. Then*

$$\#S(x) \ll \frac{x}{(\log x)^k}. \tag{6.2}$$

Given an integer $n \geq 2$, let $\omega(n)$ stand for its number of distinct prime factors and let $p(n)$ be its smallest prime factor. Set $\omega(1) = 0$ and $p(1) = 1$. Given an integer $n \geq 2$ written as $n = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$, where $q_1 < \cdots < q_s$ are primes, and given a positive integer $r \leq s := \omega(n)$, we let $P_r(n)$ stand for the r-th largest prime factor of n, that is $P_r(n) = q_{s-r+1}$. In particular, $P_1(n) = P(n)$.

Finally, given positive integers k, ℓ, r, let

$$\begin{aligned} E_{k,\ell}^{(r)} &= \{n \in \mathbb{N} : P_r^\ell(n+i) \mid n+i \quad \text{for} \quad i = 0, 1, \ldots, k-1\}, \\ E_{k,\ell}^{(r)}(x) &= \#\{n \leq x : n \in E_{k,\ell}^{(r)}\}. \end{aligned}$$

We will now prove two conditional results and a third unconditional one.

Theorem 6. *Assume that the Prime k-tuples Conjecture (weak form) is true. Given integers $r \geq 2$, $k \geq 1$ and $\ell \geq 1$, then $\#E_{k,\ell}^{(r)} = +\infty$.*

Theorem 7. *Assume that the Prime k-tuples Conjecture (strong form) is true. Given integers $k \geq 1$ and $\ell \geq 1$, then*

$$E_{k,\ell}^{(2)}(x) = \frac{x}{(\log x)^{k(1+o(1))}} \qquad (x \to \infty).$$

Theorem 8. *Given integers $k \geq 1$ and $\ell \geq 1$, then*

$$E_{k,\ell}^{(2)}(x) \ll \frac{x}{(\log x)^k}.$$

Proof of Theorem 6. In fact, we will prove more, namely that given an arbitrary integer ℓ and any k primes $p_0 < p_1 < \cdots < p_{k-1}$ with $p_0 > k$, there exist infinitely many positive integers n such that $P_r^\ell(n+i) \mid n+i$ and $P_r(n+i) = p_i$ for all $i = 0, 1, \ldots, k-1$.

We will apply a technique used by the first author [3] to prove the existence of infinitely many integers n such that $\beta(n) = \beta(n+1) = \cdots = \beta(n+k-1)$, where $\beta(n) = \sum_{p|n,\ p<P(n)} p$, assuming the *Prime k-tuples Conjecture*.

Let us first consider the case $r = 2$. Set

$$Q_i = p_i^\ell \qquad (i = 0, 1, \ldots, k-1).$$

Then let $Q = \prod_{i=0}^{k-1} Q_i$ and consider the system of congruences

$$(6.3) \qquad \begin{cases} n & \equiv \; Q_0 \pmod{Q_0^2}, \\ n+1 & \equiv \; Q_1 \pmod{Q_1^2}, \\ & \vdots \\ n+k-1 & \equiv \; Q_{k-1} \pmod{Q_{k-1}^2}. \end{cases}$$

By the Chinese Remainder Theorem, this system of congruences has a positive solution $n = n_0 < Q^2$, all other solutions being given by

$$n = n_0 + mQ^2, \qquad m = 0, 1, 2, \dots \ .$$

Observe that

$$(6.4) \qquad \gcd\left(\frac{n_0 + i}{Q_i}, Q\right) = 1 \qquad \text{for} \quad i = 0, 1, \dots, k-1.$$

Then, for each $i = 0, 1, \dots, k-1$, we have

$$n + i = n_0 + i + mQ^2 = Q_i \left(\frac{n_0 + i}{Q_i} + m\frac{Q^2}{Q_i}\right) = Q_i p_i(m),$$

say. Observe that each $p_i(m)$ is a linear polynomial in m of the form $p_i(m) = a_i + b_i m$, where $\gcd(a_i, b_i) = 1$, in light of (6.4). For each $i \in \{0, 1, \dots, k-1\}$, write $a_i = a_i' a_i''$, where a_i' is the largest divisor of a_i with $P(a_i') \le k$. Let also $A = \mathrm{lcm}[a_i' : 0 \le i \le k-1]$. Note that A is a multiple of all primes $q \le k$. Indeed, for each $q \le k$, there exists $i \in \{0, 1, \dots, k-1\}$ such that $q \mid n_0 + i = Q_i a_i$, and since q does not divide Q_i, it follows that $q \mid a_i$. Now take $m = A^2 \ell$. Then $p_i(m) = a_i'(a_i'' + (A^2/a_i')\ell) = a_i' q_i(\ell)$ for $i = 0, 1, \dots, k-1$. If we could find an infinite sequence of positive integers $r_1 < r_2 < \cdots$ such that, for each positive integer j, the corresponding number $q_i(r_j)$ is a prime number larger than p_{k-1} for $i = 0, 1, \dots, k-1$, then our result would be proved for the case $r = 2$. Indeed, in this case, we would have that, for each positive integer j and each $i \in \{0, 1, \dots, k-1\}$,

$$P_2(n + i) = P_2(Q_i a_i' \, q_i(r_j)) = p_i,$$

and since by construction, we already have $p_i^\ell | n + i$, the result would follow immediately.

But, assuming the *Prime k-tuples Conjecture*, there exist infinitely many integers ℓ such that $q_0(\ell), \dots, q_{k-1}(\ell)$ are simultaneously primes, provided that we show that the condition that $q_0(\ell) \cdots q_{k-1}(\ell)$ is not a multiple of a fixed prime q for all positive integers ℓ. Well, if $q \le k$, then $q_i(\ell) \equiv a_i'' \pmod{q}$ for all $i = 0, \dots, k-1$, so in fact $q_0(\ell) \cdots q_{k-1}(\ell)$ is never a multiple of such a q for any ℓ. If $q > k$, then either q is not one of $p_0, \dots, p_{k-1}$, in which case choosing ℓ to be any residue class different from the residue classes $a_i''(A^2/a_i')^{-1} \pmod{q}$ (in total, at most k of them) will make

$q_0(\ell)\cdots q_{k-1}(\ell)$ not a multiple of q, while if $q = p_j$ for some $j = 0, \ldots, k-1$, then b_i is a multiple of q for all $i = 0, \ldots, k-1$, so that $q_i \equiv a_i'' \pmod q$ for $i = 0, \ldots, k-1$, and this is nonzero, for if not, then $p_j \mid a_i$, which is false because $p_j \mid b_i$ and a_i and b_i are coprime. This takes care of the case $r = 2$.

It remains to consider the case $r \geq 3$. This time, we let $m_0, m_1, \ldots, m_{k-1}$ be distinct positive integers satisfying the following conditions:

(i) $\gcd(m_i, m_j) = 1$ if $i \neq j$;

(ii) $p(m_i) > p_i$ for $i = 0, 1, \ldots, k-1$;

(iii) $\omega(m_i) = r - 2$ for $i = 0, 1, \ldots, k-1$;

(iv) $\gcd(m_i, p_j) = 1$ for all $i, j \in \{0, 1, \ldots, k-1\}$.

Then, set

$$Q_i = p_i^\ell m_i \qquad (i = 0, 1, \ldots, k-1)$$

and let $Q = \prod_{i=0}^{k-1} Q_i$. Now consider the corresponding system of congruences (6.3). Again, by the Chinese Remainder Theorem, this system has a solution $n = n_0 < Q^2$ and all solutions n are of the form $n = n_0 + mQ^2$ for some integer $m \geq 0$. Now, proceeding along the same lines as for the case $r = 2$, we obtain that, for each $i = 0, 1, \ldots, k-1$,

$$n + i = Q_i \left(\frac{n_0 + i}{Q_i} + m \frac{Q^2}{Q_i} \right) = Q_i \, q_i(m),$$

where each $p_i(m)$ is a linear polynomial in m of the form $p_i(m) = a_i + b_i m$, where $\gcd(a_i, b_i) = 1$. And again, taking $m = A^2 \ell$, writing

$$p_i(m) = a_i'(a_i'' + b_i(A^2/a_i')\ell) = a_i' q_i(\ell)$$

and applying the Prime k-tuples Conjecture to the polynomials $q_0(\ell), \cdots q_{k-1}(\ell)$, we can say that, for some positive integer ℓ, the numbers $q_0(\ell), q_1(\ell), \ldots, q_{k-1}(\ell)$ are simultaneously primes and in fact that this phenomenon occurs for infinitely many positive integers ℓ, thus completing the proof of Theorem 6. □

Proof of Theorem 7. The proof is similar to the one of Theorem 6. Indeed assume that $P_2(n) = p_1, \ P_2(n+1) = p_2, \ldots, \ P_2(n+k-1) = p_k$. Let Q be defined as above. Then, according to the strong from of the *Prime k-tuples Conjecture*, we obtain that

$$\#E_{k,\ell}^{(2)}(x) = \sum_{p_1, p_2, \ldots, p_k} \frac{x}{(p_1 p_2 \cdots p_k)^\ell} \frac{1}{(\log x)^k} = \frac{x}{(\log x)^{k(1+o(1))}} \qquad (x \to \infty),$$

thus establishing our claim. □

Proof of Theorem 8. We can proceed exactly as we did with the proof of Theorem 7 except that we replace the conditional equality (6.1) by the unconditional upper bound (6.2). □

Remark 5. *If we relax our definition of* $E_{k,\ell}^{(r)}$ *to*

$$E_{k,\ell}^{(r),*} = \{n \in \mathbb{N} : P_j^\ell(n+i) \mid n+i \text{ for some } j \le r \text{ and for } i = 0, 1, \ldots, k-1\},$$

then the argument developed in the proof of Theorem 6 yields unconditionally that the sets $E_{k,\ell}^{(r),*}$ *are infinite as long as* $r \ge k(\ell-1)+1$.

7 Numerical data

Let $\rho = \rho(x)$ be the unique positive real number satisfying $E(x) = x^\rho$, then we have the following table:

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}	10^{11}
$E(x)$	1	2	5	13	28	79	204	549	1509	4231	12072
$\rho(x)$	0	0.15	0.233	0.278	0.289	0.316	0.330	0.342	0.353	0.363	0.371

8 Conjectures and related problems

We conjecture that for each pair of positive integers k, ℓ, the set $E_{k,\ell}$ is non empty. In particular, $\#E_{k,\ell} = \infty$.

We also conjecture that, given any two positive integers k and ℓ, there exists a constant $C = C_{k,\ell}$ such that any k-tuples of primes $(p_0, p_1, \ldots, p_{k-1})$ satisfying $\min_{0 \le i \le k-1} p_i > C$, there exists $n \in E_{k,\ell}$ for which

$$P(n+i) = p_i \quad \text{for } 0 \le i \le k-1. \tag{8.1}$$

However, observe that given fixed positive integers k and ℓ and a particular k-tuples of primes $(p_0, p_1, \ldots, p_{k-1})$ satisfying condition (8.1) for some $n \in E_{k,\ell}$, then, by Theorem 1, the number of such n's is finite for $k \ge 2$ and $\ell \ge 2$.

While the problem of determining how often consecutive integers are divisible by a given power of their largest prime factor is very hard, it is easy to determine how often consecutive integers are divisible by a power of their smallest prime factors. Indeed, one could easily show that, if $p(n)$ stands for the smallest prime factor of $n \ge 2$, then, as $x \to \infty$,

$$\#\{n \le x : p(n+i)^\ell | (n+i), i = 0, 1, \ldots, k-1\} = (1+o(1))xF_{k,\ell}$$

for some positive constant $F_{k,\ell}$ that can be numerically computed for any given values of k and ℓ. This simple observation leads naturally to the two related following problems:

1. What is the asymptotic behavior of $F_{k,\ell}$ as k and/or ℓ tends to infinity?

2. What can one say about the quantity

$$\#\{n \le x : p_s(n+i)^\ell | (n+i), i = 0, 1, \ldots, k-1\},$$

where $p_s(n)$ stands for the s-th smaller prime factor of n, if s is a function of x and/or of n? At first glance the problem becomes increasingly difficult when s is a rapidly growing function of n or of x. For instance, setting $s = \lfloor 0.5\omega(n) \rfloor$, one could try to examine how often consecutive integers are divisible by a power of their middle prime factor.

References

[1] A. Balog, *On triplets with descending largest prime factors*, Studia Sci. Math. Hungar. **38** (2001), 45–50.

[2] J.M. De Koninck, *Those Fascinating Numbers*, American Mathematical Society, Providence, RI, 426 pages, 2009.

[3] J.M. De Koninck, *Computational results and queries in number theory*, Annales Univ. Sci. Budapest **23** (2004), 149–161.

[4] J.M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.

[5] A. Granville, *On integers, without large prime factors, in arithmetic progressions II*, Philosophical Transactions of the Royal Society **345** (1993), 349–362.

[6] H. Halbertstam and H..-E. Richert, *Sieve methods*, Academic Press, 1974.

[7] A. Ivić, *On certain large additive functions*, Paul Erdős and his Mathematics I, Budapest, 2002, 319–331.

[8] T.N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Mathemnatics **87**, Cambridge University Press, Cambridge, 1986.

Jean-Marie De Koninck
Département de mathématiques et de statistique
Université Laval
Québec, Québec G1V 0A6

Canada
jmdk@mat.ulaval.ca

Nicolas Doyon
Département de mathématiques et de statistique
Université Laval
Québec, Québec G1V 0A6
Canada
nicodoyon77@hotmail.com

Florian Luca
Centro de Ciencias Matemáticas, UNAM
Apdo. Postal 61-3 (Xangari)
CP 58089, Morelia
Mexico
fluca@matmor.unam.mx

JMDK, le 22 juin 2013; fichier: DoyonDeKoLucaJCNTJune22pm.tex