Last update: Nov. 6, 2015.

A CONJECTURE ON UNIT FRACTIONS INVOLVING PRIMES

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Abstract. We present a conjecture on unit fractions involving primes, and provide numerical data supporting the conjecture.

Unit fractions have the form $1/n$ with $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. A sum of finitely many distinct unit fractions is called a *Egyptian fraction* as it was first studied by the ancient Egyptians around 1650 B.C. As

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)},$$

any positive rational number $r = m/n$ with $m, n \in \mathbb{Z}^+$ is an Egyptian fraction. (This easy fact was first proved by Fibonacci in 1202 and it implies that the series $\sum_{n=1}^{\infty} 1/n$ diverges.) For example,

$$1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \left(\frac{1}{2+1} + \frac{1}{2\times 3}\right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

See also Graham [Gr] and Guy [Gu, pp. 252–262] for known problems and results on Egyptian fractions.

Euclid proved that there are infinitely many primes. In 1737, Euler showed further that $\sum_p 1/p$ diverges, where p runs over all the primes. Equivalently, $\sum_p 1/(p-1)$ and $\sum_p 1/(p+1)$ diverge. By Dirichlet's theorem, for any $d = \pm 1$ and $n \in \mathbb{Z}^+$ there are infinitely many primes p with $p \equiv d \pmod{n}$. Motivated by this, we formulate the following conjecture.

Conjecture. (i) (Sept. 9, 2015) *For any positive rational number r, there is a finite set P_r^- of primes such that*

$$\sum_{p \in P_r^-} \frac{1}{p-1} = r. \tag{1}$$

2010 *Mathematics Subject Classification.* Primary 11D68; Secondary 11A41.
Keywords. Prime numbers, unit fractions, Egyptian fractions, representations of rational numbers.

(ii) (Sept. 10, 2015) *For any positive rational number r, there is a finite set P_r^+ of primes such that*

$$\sum_{p\in P_r^+}\frac{1}{p+1}=r. \tag{2}$$

The author made the conjecture public by adding comments (cf. [S1]) on the sequence A000040 of primes in OEIS. He also sent a message (cf. [S2]) to Number Theory Mailing List to report part (i) of the conjecture. The author would like to offer 500 US dollars as the first complete solution to the conjecture.

Recall that a positive integer n is called a *practical number* if each $m=1,\ldots,n$ can be written as the sum of some distinct (positive) divisors of n. 1 is the only odd practical numbers, and all powers of two are practical numbers. The distribution of practical numbers is quite similar to that of prime numbers. For $x>0$ let $P(x)$ denote the number of practical numbers not exceeding x. Similar to the Prime Number Theorem, we have

$$P(x)\sim c\frac{x}{\log x}\quad\text{for some constant } c>0,$$

which was conjectured by M. Margenstern [M] in 1991 and proved by A. Weingartner [W] in 2014. In view of the above conjecture on unit fractions involving primes, on Sept. 12, 2015 the author conjectured that any positive rational number r can be written as $\sum_{j=1}^k 1/q_j$, where $q_1,\ldots,q_k$ are distinct practical numbers. (See the author's comments (cf. [S3]) added to the sequence A005153 of practical numbers in OEIS.) For example,

$$\frac{10}{11}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{48}+\frac{1}{132}+\frac{1}{176}$$

with $2,4,8,48,132,176$ all practical numbers.

We have checked the conjecture for all those rational numbers $r\in(0,1]$ with denominators among $1,\ldots,30$. Below we provide 12 tables containing related data. Note that Tables 8 and 12 were produced by Prof. Qing-Hu Hou at Tianjin Univ. (Nov. 6, 2015) on the author's request.

Table 1: P_r^- and P_r^+ for $r \in (0, 1]$ with denominators among $1, \dots, 8$

r	P_r^-	P_r^+
1	$\{2\}$, $\{3, 5, 7, 13\}$	$\{2, 3, 5, 7, 11, 23\}$
1/2	$\{3\}$	$\{2, 5\}$
1/3	$\{5, 11\}$	$\{2\}$
2/3	$\{3, 7\}$	$\{2, 3, 11\}$
1/4	$\{5\}$	$\{3\}$
3/4	$\{3, 5\}$	$\{2, 3, 5\}$
1/5	$\{7, 31\}$	$\{5, 29\}$
2/5	$\{5, 11, 29, 71\}$	$\{2, 17, 89\}$
3/5	$\{3, 11\}$	$\{2, 3, 59\}$
4/5	$\{3, 5, 29, 71\}$	$\{2, 3, 5, 19\}$
1/6	$\{7\}$	$\{5\}$
5/6	$\{3, 5, 13\}$	$\{2, 3, 5, 11\}$
1/7	$\{13, 29, 43\}$	$\{7, 71, 251\}$
2/7	$\{5, 29\}$	$\{3, 31, 223\}$
3/7	$\{5, 13, 17, 43, 113\}$	$\{2, 11, 83\}$
4/7	$\{3, 17, 113\}$	$\{2, 5, 13\}$
5/7	$\{3, 7, 31, 71\}$	$\{2, 3, 7, 167\}$
6/7	$\{3, 5, 13, 43\}$	$\{2, 3, 5, 11, 41\}$
1/8	$\{11, 41\}$	$\{7\}$
3/8	$\{5, 11, 41\}$	$\{2, 23\}$
5/8	$\{3, 11, 41\}$	$\{2, 3, 23\}$
7/8	$\{3, 5, 11, 41\}$	$\{2, 3, 5, 7\}$

Table 2: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among $9, \ldots, 12$

r	P_r^-	P_r^+
1/9	$\{13, 37\}$	$\{11, 41, 251\}$
2/9	$\{7, 19\}$	$\{5, 17\}$
4/9	$\{5, 7, 37\}$	$\{2, 11, 41, 251\}$
5/9	$\{3, 19\}$	$\{2, 5, 17\}$
7/9	$\{3, 5, 37\}$	$\{2, 3, 5, 41, 251\}$
1/10	$\{11\}$	$\{11, 59\}$
3/10	$\{5, 29, 71\}$	$\{3, 19\}$
7/10	$\{3, 7, 31\}$	$\{2, 3, 11, 29\}$
9/10	$\{3, 5, 11, 29, 71\}$	$\{2, 3, 5, 7, 47, 239\}$
1/11	$\{23, 67, 73, 89, 199\}$	$\{11, 131\}$
2/11	$\{7, 67\}$	$\{7, 43, 47, 109, 239\}$
3/11	$\{7, 19, 23, 199\}$	$\{3, 43\}$
4/11	$\{5, 13, 73, 89, 199\}$	$\{2, 43, 131\}$
5/11	$\{5, 13, 19, 23, 67, 199\}$	$\{2, 11, 47, 109, 239, 263\}$
6/11	$\{3, 23\}$	$\{2, 5, 23, 263\}$
7/11	$\{3, 11, 41, 89\}$	$\{2, 3, 23, 131, 263\}$
8/11	$\{3, 7, 19, 199\}$	$\{2, 3, 7, 71, 197\}$
9/11	$\{3, 5, 37, 67, 73, 89\}$	$\{2, 3, 5, 17, 131, 197\}$
10/11	$\{3, 5, 13, 19, 67, 199\}$	$\{2, 3, 5, 7, 43, 131, 263\}$
1/12	$\{13\}$	$\{11\}$
5/12	$\{5, 7\}$	$\{2, 11\}$
7/12	$\{3, 13\}$	$\{2, 3\}$
11/12	$\{3, 5, 7\}$	$\{2, 3, 5, 7, 23\}$

Table 3: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among $13, 14, 15$

r	P_r^-	P_r^+
1/13	$\{29, 53, 71, 131\}$	$\{23, 71, 83, 181, 251\}$
2/13	$\{11, 31, 79, 131\}$	$\{11, 23, 79, 103, 239, 389\}$
3/13	$\{7, 29, 71, 131, 157\}$	$\{7, 13, 59, 103, 181, 389\}$
4/13	$\{5, 29, 71, 131\}$	$\{3, 29, 79, 179, 239, 467\}$
5/13	$\{5, 11, 41, 157, 313\}$	$\{2, 31, 103, 223, 251, 503\}$
6/13	$\{5, 7, 37, 73, 313\}$	$\{2, 11, 29, 151, 311, 569\}$
7/13	$\{3, 53, 79, 157\}$	$\{2, 5, 41, 167, 181, 311\}$
8/13	$\{3, 13, 53, 79\}$	$\{2, 3, 53, 179, 233, 269\}$
9/13	$\{3, 7, 53, 157\}$	$\{2, 3, 11, 71, 103, 467\}$
10/13	$\{3, 5, 53\}$	$\{2, 3, 5, 71, 311, 467\}$
11/13	$\{3, 5, 13, 79\}$	$\{2, 3, 5, 11, 103, 311\}$
12/13	$\{3, 5, 7, 157\}$	$\{2, 3, 5, 7, 23, 233, 467\}$
1/14	$\{17, 113\}$	$\{13\}$
3/14	$\{7, 31, 71\}$	$\{5, 31, 83, 223\}$
5/14	$\{5, 13, 43\}$	$\{2, 41\}$
9/14	$\{3, 13, 29, 43\}$	$\{2, 3, 19, 179, 251\}$
11/14	$\{3, 5, 29\}$	$\{2, 3, 5, 31, 223\}$
13/14	$\{3, 5, 7, 109, 379\}$	$\{2, 3, 5, 7, 23, 83\}$
1/15	$\{17, 241\}$	$\{19, 59\}$
2/15	$\{11, 31\}$	$\{11, 19\}$
4/15	$\{5, 61\}$	$\{3, 59\}$
7/15	$\{5, 7, 29, 71\}$	$\{2, 11, 19\}$
8/15	$\{3, 31\}$	$\{2, 5, 29\}$
11/15	$\{3, 7, 17, 241\}$	$\{2, 3, 7, 47, 239\}$
13/15	$\{3, 5, 11, 61\}$	$\{2, 3, 5, 11, 29\}$
14/15	$\{3, 5, 7, 61\}$	$\{2, 3, 5, 7, 17, 359\}$

Table 4: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among 16 and 17

r	P_r^-	P_r^+
1/16	$\{17\}$	$\{23, 47\}$
3/16	$\{7, 61, 241\}$	$\{5, 47\}$
5/16	$\{5, 17\}$	$\{3, 19, 79\}$
7/16	$\{5, 7, 61, 241\}$	$\{2, 11, 47\}$
9/16	$\{3, 17\}$	$\{2, 5, 19, 79\}$
11/16	$\{3, 7, 61, 241\}$	$\{2, 3, 11, 47\}$
13/16	$\{3, 5, 17\}$	$\{2, 3, 5, 19, 79\}$
15/16	$\{3, 5, 7, 61, 241\}$	$\{2, 3, 5, 7, 19, 79\}$
1/17	$\{19, 307\}$	$\{17, 467, 883\}$ (Qing-Hu Hou)
2/17	$\{13, 73, 103, 137, 307\}$	$\{11, 101, 107, 179, 269, 271, 431\}$
3/17	$\{7, 103\}$	$\{7, 31, 167, 223, 239, 271, 509\}$
4/17	$\{7, 19, 103, 307\}$	$\{5, 23, 79, 179, 239, 359, 509\}$
5/17	$\{5, 37, 73, 409\}$	$\{3, 67, 101, 109, 239, 271, 373\}$
6/17	$\{5, 13, 73, 307, 409\}$	$\{3, 13, 83, 101, 239, 271, 509\}$
7/17	$\{5, 13, 19, 103, 137, 307, 409\}$	$\{2, 17, 103, 197, 263, 373, 571\}$
8/17	$\{5, 7, 31, 103, 211, 281, 409\}$	$\{3, 7, 19, 31, 107, 431, 647, 1699, 2591, 4049\}$ (Qing-Hu Hou)
9/17	$\{3, 113, 137, 211, 239, 241\}$	$\{2, 5, 101, 109, 239, 271, 373\}$
10/17	$\{3, 17, 127, 137, 239, 307, 337\}$	$\{2, 5, 17, 89, 101, 109, 373\}$
11/17	$\{3, 13, 29, 43, 239\}$	$\{2, 3, 29, 59, 109, 373, 509\}$
12/17	$\{3, 7, 43, 127, 239, 307\}$	$\{2, 3, 11, 53, 67, 271, 431\}$
13/17	$\{3, 7, 13, 239, 241, 337, 421, 1021\}$	$\{2, 3, 7, 29, 67, 151, 569\}$
14/17	$\{3, 5, 29, 43, 103, 239\}$	$\{2, 3, 5, 19, 101, 109, 373, 509\}$
15/17	$\{3, 5, 11, 41, 137\}$	$P_{2/17}^+ \cup P_{13/17}^+$
16/17	$\{3, 5, 7, 73, 137, 307\}$	$P_{4/17}^+ \cup P_{12/17}^+$

Table 5: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among 18 and 19

r	P_r^-	P_r^+
1/18	$\{19\}$	$\{17\}$
5/18	$\{5, 37\}$	$\{3, 53, 107\}$
7/18	$\{5, 13, 19\}$	$\{2, 17\}$
11/18	$\{3, 13, 37\}$	$\{2, 3, 43, 197\}$
13/18	$\{3, 7, 19\}$	$\{2, 3, 7, 71\}$
17/18	$\{3, 5, 7, 37\}$	$\{2, 3, 5, 7, 17, 71\}$
1/19	$\{37, 137, 191, 229, 331, 397, 761, 1021\}$	$\{37, 107, 227, 239, 311, 359, 701, 911\}$
2/19	$\{13, 101, 151, 191\}$	$\{13, 59, 223, 251, 269, 359, 863, 911\}$
3/19	$\{11, 29, 127, 229, 271, 379, 457, 761\}$	$\{7, 71, 151, 239, 311, 379, 683, 1039\}$
4/19	$\{7, 53, 131‘, 157, 211, 281, 457\}$	$\{5, 41, 139, 223, 311, 379, 607, 1039\}$
5/19	$\{7, 17, 61, 191, 229, 241, 457, 761\}$	$\{5, 13, 79, 239, 311, 389, 727, 797\}$
6/19	$\{5, 29, 71, 191, 211, 281, 457\}$	$\{3, 23, 83, 239, 311, 379, 797, 1039\}$
7/19	$\{5, 13, 61, 101, 241, 401, 571\}$	$\{2, 53, 227, 269, 307, 359, 659, 1063\}$
8/19	$\{5, 11, 17, 229, 241\}$	$\{2, 13, 227, 263, 307, 379, 769, 1063\}$
9/19	$\{5, 7, 31, 67, 229, 419, 571\}$	$\{2, 11, 23, 167, 251, 359, 683, 839\}$
10/19	$\{3, 101, 151, 191, 229\}$	$\{2, 5, 71, 239, 311, 379, 683, 1039\}$
11/19	$\{3, 29, 37, 127, 281, 457, 571\}$	$\{2, 5, 17, 71, 227, 379, 719, 911\}$
12/19	$\{3, 11, 61, 211, 229, 281, 457\}$	$\{2, 3, 29, 151, 307, 379, 659, 1063\}$
13/19	$\{3, 11, 23, 37, 181, 331, 419\}$	$\{2, 3, 11, 127, 227, 383, 607, 911\}$
14/19	$\{3, 7, 31, 41, 191, 229, 457\}$	$\{2, 3, 7, 59, 151, 359, 719, 911\}$
15/19	$\{3, 5, 67, 101, 151, 191, 419\}$	$\{2, 3, 5, 37, 127, 383, 607, 911\}$
16/19	$\{3, 5, 17, 61, 191, 241, 457, 761\}$	$\{2, 3, 5, 13, 71, 227, 683, 1063\}$
17/19	$\{3, 5, 11, 31, 211, 281, 571, 761\}$	$\{2, 3, 5, 7, 71, 379, 569, 683\}$
18/19	$\{3, 5, 7, 61, 151, 229, 601, 761\}$	$\{2, 3, 5, 7, 17, 71, 503, 1063\}$

Table 6: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among 20 and 21

r	P_r^-	P_r^+
1/20	$\{29, 71\}$	$\{19\}$
3/20	$\{11, 29, 71\}$	$\{7, 71, 89\}$
7/20	$\{5, 11\}$	$\{2, 59\}$
9/20	$\{5, 7, 31\}$	$\{2, 11, 29\}$
11/20	$\{3, 29, 71\}$	$\{2, 5, 19\}$
13/20	$\{3, 11, 29, 71\}$	$\{2, 3, 17, 89\}$
17/20	$\{3, 5, 11\}$	$\{2, 3, 5, 11, 59\}$
19/20	$\{3, 5, 7, 31\}$	$\{2, 3, 5, 7, 23, 29\}$
1/21	$\{31, 71\}$	$\{47, 107, 167, 179, 269, 431\}$
2/21	$\{17, 43, 113\}$	$\{17, 43, 167, 197, 251, 503\}$
4/21	$\{7, 43\}$	$\{7, 23, 83, 167, 251, 503\}$
5/21	$\{7, 17, 113\}$	$\{5, 19, 103, 179, 233, 503\}$
8/21	$\{5, 11, 61, 71\}$	$\{2, 31, 167, 223, 251, 503\}$
10/21	$\{5, 7, 29, 43\}$	$\{2, 7, 131, 197, 307, 503\}$
11/21	$\{3, 43\}$	$\{2, 5, 83, 167, 251, 503\}$
13/21	$\{3, 13, 29\}$	$\{2, 3, 41, 167, 251, 503\}$
16/21	$\{3, 7, 17, 43, 113\}$	$\{2, 3, 5, 167, 251, 503\}$
17/21	$\{3, 5, 29, 43\}$	$\{2, 3, 5, 19, 139, 419\}$
19/21	$\{3, 5, 13, 17, 113\}$	$\{2, 3, 5, 7, 41, 167\}$
20/21	$\{3, 5, 7, 29\}$	$\{2, 3, 5, 7, 13, 167\}$

Table 7: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among 22 and 24

r	P_r^-	P_r^+
1/22	$\{23\}$	$\{53, 107, 149, 199, 263, 449\}$
3/22	$\{11, 41, 89\}$	$\{11, 29, 109, 179, 197\}$
5/22	$\{7, 23, 67\}$	$\{5, 23, 71, 197\}$
7/22	$\{5, 37, 67, 73, 89\}$	$\{3, 23, 71, 131, 197\}$
9/22	$\{5, 13, 19, 67, 199\}$	$\{2, 17, 89, 109\}$
13/22	$\{3, 17, 61, 199, 241, 397\}$	$\{2, 5, 11, 131\}$
15/22	$\{3, 7, 67\}$	$\{2, 3, 13, 59, 139, 307\}$
17/22	$\{3, 7, 19, 23, 199\}$	$\{2, 3, 5, 43\}$
19/22	$\{3, 5, 11, 181, 199, 331\}$	$\{2, 3, 5, 11, 43, 131\}$
21/22	$\{3, 5, 7, 37, 127, 463\}$	$\{2, 3, 5, 7, 13, 167, 461\}$
1/24	$\{37, 73\}$	$\{23\}$
5/24	$\{7, 37, 73\}$	$\{5, 41, 139, 223, 239, 479\}$
7/24	$\{5, 37, 73\}$	$\{3, 41, 139, 223, 239, 479\}$
11/24	$\{5, 7, 37, 73\}$	$\{2, 11, 31, 223, 263, 461\}$
13/24	$\{3, 37, 73\}$	$\{2, 5, 31, 223, 263, 461\}$
17/24	$\{3, 7, 37, 73\}$	$\{2, 3, 11, 29, 167, 419\}$
19/24	$\{3, 5, 37, 73\}$	$\{2, 3, 5, 29, 167, 419\}$
23/24	$\{3, 5, 7, 37, 73\}$	$\{2, 3, 5, 7, 13, 83\}$

Table 8 (Qing-Hu Hou): P_r^- and P_r^+ for $r \in (0,1)$ with denominator 23

r	P_r^-	P_r^+
1/23	$\{29, 139, 1933\}$	$\{23, 643, 3863\}$
2/23	$\{13, 277\}$	$\{11, 367, 1103\}$
3/23	$\{11, 47, 139, 691\}$	$\{7, 229, 919\}$
4/23	$\{7, 139\}$	$\{5, 137\}$
5/23	$\{11, 13, 31, 1381\}$	$\{5, 19, 2069, 4139\}$
6/23	$\{5, 139, 277\}$	$\{3, 137, 367, 1103\}$
7/23	$\{5, 31, 61, 277, 1381\}$	$\{3, 19, 229\}$
8/23	$\{5, 13, 79, 599\}$	$\{2, 139, 229, 410, 1609\}$
9/23	$\{5, 11, 47, 61, 461, 1381\}$	$\{2, 17, 643, 1609, 4139\}$
10/23	$\{5, 11, 13, 691\}$	$\{2, 19, 29, 59, 827, 4139\}$
11/23	$\{5, 11, 13, 31, 139, 277, 1381\}$	$\{2, 11, 19, 137, 229\}$
12/23	$\{3, 47\}$	$\{2, 5, 47, 1103\}$
13/23	$\{3, 29, 47, 139, 1933\}$	$\{2, 5, 17, 181, 467, 1091, 1103, 4783\}$
14/23	$\{3, 11, 139, 691\}$	$\{2, 5, 17, 29, 89, 139, 643\}$
15/23	$\{3, 11, 31, 61, 461\}$	$\{2, 5, 19, 29, 41, 59, 83, 137, 139, 643, 1931\}$
16/23	$\{3, 7, 47, 139\}$	$\{2, 3, 11, 47, 137, 1103\}$
17/23	$\{3, 7, 23, 67, 139, 277, 1013\}$	$\{2, 3, 11, 19, 59, 229, 827, 4139\}$
18/23	$\{3, 5, 47, 139, 277\}$	$\{2, 3, 5, 31, 1103, 2207\}$
19/23	$\{3, 7, 11, 31, 47, 277, 1381\}$	$\{2, 3, 5, 13, 229, 5519, 7727\}$
20/23	$\{3, 5, 11, 61, 461, 1381\}$	$\{2, 3, 5, 11, 29, 367, 5519\}$
21/23	$\{3, 5, 11, 31, 61, 139, 277, 461\}$	$\{2, 3, 5, 11, 13, 137, 1103, 7727\}$
22/23	$\{3, 5, 11, 13, 47, 691\}$	$\{2, 3, 5, 7, 17, 71, 137, 229, 2069\}$

Table 9: P_r^- and P_r^+ for $r \in (0,1)$ with denominator 25

r	P_r^-	P_r^+
1/25	$\{31, 151\}$	$\{59, 149, 167, 223, 239, 479\}$
2/25	$\{29, 61, 71, 151, 241, 401\}$	$\{17, 139, 167, 199, 251, 419\}$
3/25	$\{13, 71, 151, 211, 241, 337, 421, 701\}$	$\{11, 53, 149, 179, 269, 449\}$
4/25	$\{11, 29, 101, 181, 271, 379, 421\}$	$\{7, 53, 167, 251, 269, 349\}$
6/25	$\{7, 29, 43, 211, 281, 337, 401\}$	$\{7, 17, 149, 197, 263, 439\}$
7/25	$\{5, 109, 151, 211, 241, 379, 401\}$	$\{3, 71, 197, 199, 263, 439\}$
8/25	$\{5, 23, 113, 241, 337, 401, 421, 463, 701\}$	$\{3, 19, 139, 149, 251, 449\}$
9/25	$\{5, 13, 101, 127, 271, 379, 421\}$	$\{2, 103, 167, 233, 251, 349\}$
11/25	$\{5, 7, 109, 211, 241, 379, 401\}$	$\{2, 11, 103, 149, 233, 359\}$
12/25	$\{5, 7, 29, 71, 151, 241, 401\}$	$\{2, 7, 103, 179, 233, 449\}$
13/25	$\{5, 7, 13, 109, 379, 401, 433, 541, 701\}$	$\{2, 7, 103, 179, 233, 359\}$
14/25	$\{3, 29, 71, 101\}$	$\{2, 5, 23, 89, 199, 449\}$
16/25	$\{3, 11, 31, 151\}$	$\{2, 3, 23, 139, 199, 349\}$
17/25	$\{3, 11, 17, 151, 157, 401, 521\}$	$\{2, 3, 11, 139, 251, 449\}$
18/25	$\{3, 7, 29, 131, 401, 433, 541, 547, 701\}$	$\{2, 3, 11, 23, 149, 199\}$
19/25	$\{3, 5, 101\}$	$\{2, 3, 7, 23, 139, 349\}$
21/25	$\{3, 5, 13, 151\}$	$\{2, 3, 5, 13, 83, 149\}$
22/25	$\{3, 5, 11, 61, 151, 241, 401\}$	$\{2, 3, 5, 11, 23, 199\}$
23/25	$\{3, 5, 11, 19, 181, 379, 401, 433, 701\}$	$\{2, 3, 5, 7, 29, 149, 199\}$
24/25	$\{3, 5, 7, 41, 101, 241, 433, 541\}$	$\{2, 3, 5, 7, 17, 41, 349, 359\}$

Table 10: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among 26 and 28

r	P_r^-	P_r^+
1/26	$\{71, 127, 211, 271, 313, 379, 521\}$	$\{67, 139, 181, 263, 311, 461, 509\}$
3/26	$\{13, 71, 157, 241, 337, 421, 547\}$	$\{11, 67, 197, 233, 263, 439, 509\}$
5/26	$\{7, 109, 181, 271, 337, 433, 547\}$	$\{5, 109, 181, 263, 307, 439, 571\}$
7/26	$\{7, 13, 131, 229, 313, 457, 571\}$	$\{5, 11, 139, 251, 311, 359, 467\}$
9/26	$\{5, 17, 53, 137, 337, 443, 547\}$	$\{3, 13, 83, 233, 311, 359, 389\}$
11/26	$\{5, 11, 19, 157, 181, 271, 541\}$	$\{2, 13, 167, 233, 263, 461, 467\}$
15/26	$\{3, 19, 103, 239, 307, 443, 547\}$	$\{2, 5, 17, 79, 239, 389, 467\}$
17/26	$\{3, 11, 29, 113, 241, 313, 547\}$	$\{2, 3, 17, 181, 233, 311, 503\}$
19/26	$\{3, 7, 23, 131, 157, 421, 463\}$	$\{2, 3, 7, 67, 311, 389, 509\}$
21/26	$\{3, 5, 29, 79, 281, 313, 421\}$	$\{2, 3, 5, 23, 89, 359, 467\}$
23/26	$\{3, 5, 11, 61, 79, 313, 521\}$	$\{2, 3, 5, 7, 233, 311, 467\}$
25/26	$\{3, 5, 7, 37, 73, 313\}$	$\{2, 3, 5, 7, 11, 311\}$
1/28	$\{29\}$	$\{41, 83\}$
3/28	$\{13, 43\}$	$\{11, 41\}$
5/28	$\{13, 17, 43, 113\}$	$\{5, 83\}$
9/28	$\{5, 17, 113\}$	$\{3, 13\}$
11/28	$\{5, 13, 29, 43\}$	$\{2, 19, 179, 251\}$
13/28	$\{5, 7, 31, 71\}$	$\{2, 7, 167\}$
15/28	$\{3, 29\}$	$\{2, 5, 131, 223\}$
17/28	$\{3, 13, 43\}$	$\{2, 3, 41\}$
19/28	$\{3, 13, 17, 43, 113\}$	$\{2, 3, 11, 83\}$
23/28	$\{3, 5, 17, 113\}$	$\{2, 3, 5, 13\}$
25/28	$\{3, 5, 13, 29, 43\}$	$\{2, 3, 5, 7, 71, 251\}$
27/28	$\{3, 5, 7, 31, 71\}$	$\{2, 3, 5, 7, 11, 167\}$

Table 11: P_r^- and P_r^+ for $r \in (0,1)$ with denominators among 27 and 30

r	P_r^-	P_r^+
1/27	$\{37, 109\}$	$\{43, 107, 197\}$
2/27	$\{29, 67, 127, 211, 271, 337, 433, 661\}$	$\{17, 53\}$
4/27	$\{11, 37, 157, 211, 271, 281, 521\}$	$\{7, 71, 107\}$
5/27	$\{11, 17, 109, 211, 241, 379, 541\}$	$\{5, 53\}$
7/27	$\{7, 17, 61, 211, 241, 379, 541\}$	$\{3, 107\}$
8/27	$\{5, 37, 181, 211, 271, 379, 541\}$	$\{3, 41, 53, 251\}$
10/27	$\{5, 11, 157, 211, 271, 281, 521\}$	$\{2, 41, 107, 251\}$
11/27	$\{5, 11, 29, 101, 151, 379, 421\}$	$\{2, 17, 53\}$
13/27	$\{5, 7, 23, 127, 181, 271, 463\}$	$\{2, 7, 71, 107\}$
14/27	$\{5, 7, 13, 109, 211, 379, 541\}$	$\{2, 5, 53\}$
16/27	$\{3, 17, 61, 211, 241, 379, 541\}$	$\{2, 3, 107\}$
17/27	$\{3, 11, 61, 211, 271, 379, 541\}$	$\{2, 3, 29, 107, 269\}$
19/27	$\{3, 7, 43, 241, 271, 337, 421\}$	$\{2, 3, 11, 29, 269\}$
20/27	$\{3, 7, 19, 109, 211, 379, 541\}$	$\{2, 3, 7, 53, 71\}$
22/27	$\{3, 5, 23, 127, 181, 271, 463\}$	$\{2, 3, 5, 17, 107\}$
23/27	$\{3, 5, 13, 109, 211, 379, 541\}$	$\{2, 3, 5, 11, 53\}$
25/27	$\{3, 5, 11, 17, 113, 379, 541\}$	$\{2, 3, 5, 7, 23, 107\}$
26/27	$\{3, 5, 7, 29, 181, 379, 421\}$	$\{2, 3, 5, 7, 17, 53, 71\}$
1/30	$\{31\}$	$\{29\}$
7/30	$\{7, 29, 61, 71\}$	$\{5, 17, 89\}$
11/30	$\{5, 11, 61\}$	$\{2, 29\}$
13/30	$\{5, 7, 61\}$	$\{2, 11, 59\}$
17/30	$\{3, 29, 61, 71\}$	$\{2, 5, 17, 89\}$
19/30	$\{3, 11, 31\}$	$\{2, 3, 19\}$
23/30	$\{3, 5, 61\}$	$\{2, 3, 5, 59\}$
29/30	$\{3, 5, 7, 29, 71\}$	$\{2, 3, 5, 7, 19, 23\}$

Table 12 (Qing-Hu Hou): P_r^- and P_r^+ for $r \in (0,1)$ with denominator 29

r	P_r^-	P_r^+
1/29	$\{59, 61, 1741\}$	$\{31, 347, 4639, 6959\}$
2/29	$\{17, 241, 661, 1277\}$	$\{19, 59, 463, 6959\}$
3/29	$\{13, 59, 349\}$	$\{11, 59, 347, 1913, 19139\}$
4/29	$\{11, 31, 233, 4931, 11833\}$	$\{11, 19, 347, 811, 2029\}$
5/29	$\{7, 211, 1741, 2437\}$	$\{5, 173\}$
6/29	$\{7, 29, 281, 2437, 2521, 7309\}$	$\{5, 29, 271, 509, 1217, 4079, 7307, 17747\}$
7/29	$\{7, 19, 59, 523\}$	$\{5, 13, 521, 811, 7307\}$
8/29	$\{5, 41, 1451, 5801\}$	$\{5, 11, 47, 347, 463\}, \{3, 43, 347, 4871, 17863\}$
9/29	$\{5, 19, 233, 2089\}$	$\{3, 17, 347, 521\}$
10/29	$\{5, 13, 97, 929\}$	$\{2, 139, 419, 521, 18269\}, \{3, 19, 29, 89, 2609\}$
11/29	$\{5, 13, 37, 59, 1277, 5743\}$	$\{2, 23, 347, 811, 4871\}, \{3, 7, 347, 811, 4871\}$
12/29	$\{5, 13, 19, 79, 131, 349, 1171, 1741, 11311\}$	$\{2, 13, 167, 347, 463\}$
13/29	$\{5, 13, 19, 37, 59, 73, 2089\}$	$\{2, 11, 47, 173, 347, 463\}$
14/29	$\{5, 11, 13, 41, 61, 233, 349, 1741\}$	$\{2, 11, 17, 173, 347, 521\}$
15/29	$\{3, 59\}$	$\{2, 11, 19, 29, 89, 173, 2609\}, \{3, 5, 19, 29, 89, 173, 2609\}$
16/29	$\{3, 31, 59, 929, 13921\}$	$\{2, 5, 19, 811, 2029\}$
17/29	$\{3, 13, 349\}$	$\{2, 3, 347\}$
18/29	$\{3, 11, 59, 349, 1741\}$	$\{2, 3, 43, 131, 173, 811, 13397\}$
19/29	$\{3, 11, 23, 199, 349, 991, 1277\}$	$\{2, 3, 19, 59, 347, 463, 6959\}$
20/29	$\{3, 7, 67, 233, 419, 1103, 4409\}$	$\{2, 3, 11, 47, 463\}$
21/29	$\{3, 7, 19, 523\}$	$\{2, 3, 11, 17, 521\}$
22/29	$\{3, 5, 181, 349, 8353, 13921\}$	$\{2, 3, 5, 173, 347\}$
23/29	$\{3, 5, 41, 59, 1451, 5801\}$	$\{2, 3, 5, 23, 811, 4871\}$
24/29	$\{3, 5, 19, 59, 233, 2089\}$	$\{2, 3, 5, 13, 251, 811, 1217, 7307\}$
25/29	$\{3, 5, 13, 43, 349, 547, 7541, 11311\}$	$\{2, 3, 5, 11, 47, 173, 463\}$
26/29	$\{3, 5, 13, 19, 233, 349, 2089\}$	$\{2, 3, 5, 11, 17, 173, 521\}$
27/29	$\{3, 5, 13, 19, 37, 73, 2089\}$	$\{2, 3, 5, 11, 13, 41, 463, 4871, 9743\}$
28/29	$\{3, 5, 11, 13, 41, 241, 349, 6961\}$	$\{2, 3, 5, 7, 11, 173, 811, 4871\}$

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