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U.S. Department of Labor
U.S. Bureau of Labor Statistics
Office of Compensation and Working Conditions

Mixed Search Over Commutable Space

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Working Paper 481
August 2015

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Abstract

We present a model of search equilibrium in a labor market characterized by more than one locus of employment, in which workers may commute between loci. Unlike previous work with this feature, the model produces outcomes in which unemployed workers sometimes choose to conduct "mixed search" – simultaneously searching for jobs in more than one location. We examine the labor market properties of such equilibria and circumscribe the conditions under which they are preferred to other equilibria such as the "autarkic" equilibrium in which no inter-area search is conducted. We then explore the effects of changes in inter-area commuting costs.

JEL Classification Codes: R23, J64

Keywords: Regional Labor Markets, Cities, Job Search, Unemployment

In his groundbreaking book, *Urban Labor Economics*, Zenou (2009) establishes a model synthesizing the land equilibrium concepts of the archetypical, monocentric urban model with the wage and unemployment dynamics of search-matching models; this relatively simple model is able to characterize the interplay of space with the labor market. He then uses the model as a baseline to explore a wide range of topics by extending it in various ways. In one group of the extensions, Zenou considers multiple job centers, focusing first on the existence of an alternative, rural job locus and the consequent rural-urban migration equilibrium, and second on a model in which jobs are spread continuously across space.

Many metropolitan areas today are characterized by a somewhat different set-up: multiple job centers are spread discretely across space, but are included within one, larger job market. In some cases, job centers are further away and have less labor market integration with other job loci; in still others, residents living near a small job locus participate in a larger job center further away, but "reverse commuting" to the small center is rare. What are the economics underlying these patterns? Why do some job centers become integrated with others in a single labor market while others remain separate? How do the labor markets, and their outcomes, differ between these scenarios? Some answers to these questions are suggested by the results of Zenou's various models, but none of the models are directly on point.

In this paper, we present a model of search equilibrium in a labor market characterized by two employment loci, in which workers may commute between loci. Under some circumstances, agents will find it optimal to arrange activity in such a way that unemployed workers simultaneously search for jobs in both locations – a behavior that we refer to as "mixed search." Although this feature seems realistic and straightforward in the context of many modern metropolitan areas in the US, we believe it is a novel feature in the search literature. Models of search in an urban setting such as Wasmer and Zenou (2002) usually assume, as in the traditional urban model, that employment has only one locus. Models of search over multiple loci, such as Ortega (2000) tend to assume that the loci are far enough apart as to be completely separate. An intermediate case is that of Coulson, Laing and Wang (2001), in which workers can feasibly commute to one locus or another; assumptions in their model, however, dictate that workers never engage in mixed search – each worker chooses one locus toward which to dedicate search. Zenou (2009) presents a model in which jobs and workers are both spread evenly over continuous space[1], which can be thought of as an extension of a special case of the model we present below.

The key assumptions underlying the model's dynamics are a) that searching in more than one location allows unemployed workers to find jobs more quickly than if they only search in one location; and b) commuting from one location to another is costly. The resulting tradeoff between increased search efficiency and additional costs associated with multi-location search has some resemblance to the inherent trade-offs in models of costly search effort, with one difference being that the additional costs are borne after a job match is made, not during (unemployed) search.

The description of the model is as follows. We first describe the basic set-up of the model with two districts and describe its equilibrium behavior when no commuting is observed (the two districts are autarkic). In order to make progress on the spatial aspects we pursue later, we make as many simplifying assumptions as possible in this set-up. We then describe an alternative equilibrium in which workers from one district search for jobs in both districts, but workers living in the other district search and work only in their home

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[1] This model is an adaptation of Salop's (1979) model in which heterogeneity in goods is represented by a circle. See section 3 of Chapter 3, on pp. 132-142.

district. Finally, we describe another possible equilibrium: the "two-way-commuting" equilibrium, in which workers commute between districts in both directions and residents of each district conduct mixed search. We note a special case of the two-way commuting equilibrium: the "symmetrical" equilibrium occurring when underlying conditions in the two districts are identical. Focusing on this symmetrical case, we then describe the conditions under which the autarkic equilibrium or the two-way-commuting equilibrium is preferred.

Model Set-up

Consider an economy in which workers live in one of two districts (indexed 0 and 1), between which regular travel is possible but entails a cost to the traveler. The population in district i is P_i. U_i workers in district i are unemployed, creating an unemployment rate of $u_i=U_i/P_i$, and unemployed workers may search for new jobs in their own district, the neighboring district, or both. Let $s_{ij} \in [0,1]$ be an indicator of whether unemployed residents of district i search for a job in district j, and let $S_j = \sum_i s_{ij} * U_i$ be the number of workers searching for jobs in district j. Workers employed in district j receive a wage of w_j, which, net of any commuting costs accrued, is their flow utility from employment. Let the cost of commuting between districts to a job be φ; to simplify notation we denote $\varphi_{ij}= 0$ if i=j and $\varphi_{ij}= \varphi$ if i≠j, so that the flow utility of employment to district j workers living in district i is $w_j - \varphi_{ij}$. Unemployed workers receive a utility of b, which includes any monetary benefit they receive as well as the utility of leisure and home production.

The number of employers in district j is F_j. Each employer employs 1 worker when operational, and the worker produces an output worth y_j; an operational employer earns a profit of y_j-w_j. But the employer may instead have a vacancy, both because newly-formed employers are assumed to start in vacancy and because all employer-employee matches are subject to an exogenous separation probability of δ. In vacancy, no output is produced and a cost of c is incurred.[2] At any point in time, the number of vacancies in district j is V_j, so that the vacancy rate is $V_j/ F_j = v_j$. In addition, a key outcome parameter to keep track of will be the labor market "tightness" characterizing any equilibrium, denoted as $\tau_j= V_j/S_j$. Equilibrium will be determined when employers choose a wage rate, and unemployed workers choose an optimal search strategy; i.e., they determine whether to search in both districts or only in their home district.[3] Note that employers are not allowed to offer different wages to different workers upon the formation of a match. There is no heterogeneity in employers, workers, or job matches, apart from the workers' residential locations (and the implied commuting costs, φ, borne by working in one district or another) and the location-specific output values, y_j, of employers.

Matching of unemployed workers to job vacancies is assumed to be governed by an aggregate matching function in each district. We begin with a generic version of the function, with very limited restrictions on its form, although later we will impose a Cobb-Douglas form. Let the flow of matches in district j at any

[2] Note: the vacancy cost might include (or consist of) a fixed cost of capital or rent for a physical facility, in which case it is also to be incurred while the job is filled. To this extent, our output variable y_j should considered as being net of all non-wage costs.

[3] Note: we assume that the productivity of employers in each district is high enough that workers always search (at least) in their home district.

given point in time be defined by $M_j = m(s_j,v_j;k)$, where k is a scaling parameter such that $m_k>0$ and $m_{sk}= m_{vk}= 0$. We denote the flow rate of job arrival to unemployed workers searching in j as $\mu_j=M_j/S_j$ and the flow rate of job arrival to employers with vacancies in j as $\eta_j=M_j/V_j$. We assume that m is increasing and convex in both s_j and v_j, and we impose the boundary (M_j is 0 when either s_j or v_j is 0) and Inada (slope of m is infinite around $s_j=0$ and $v_j=0$ and zero around $s_j=\infty$ and $v_j=\infty$) conditions customary to the search literature. In addition, we assume that $\lim_{sj\to\infty}M_j = \infty$ and $\lim_{vj\to\infty}M_j = \infty$, but that this convergence in M_j is slower than that of either argument. This last assumption implies that $\lim_{sj\to\infty}\mu_j = 0$ and $\lim_{vj\to\infty}\eta_j = 0$.

With this basic framework established, we can begin analysis of the model by writing down the Bellman's equations describing the values of filled and vacant jobs to employers in district j (π_j^F and π_j^V) and the values of employment in j and unemployment of workers living in district i (π_{ij}^E and π_j^U). We have the following 10 equations.

$$r\pi_j^F = (y_j - w_j) + \delta[\pi_j^V - \pi_j^F], j \in (1,2) \tag{1}$$

$$r\pi_j^V = -c + \eta_j[\pi_j^F - \pi_j^V], j \in (1,2) \tag{2}$$

$$r\pi_{ij}^E = w_j + \varphi_{ij} + \delta[\pi_i^U - \pi_{ij}^E], i \in (1,2), j \in (1,2) \tag{3}$$

$$r\pi_i^U = b + \mu_1 s_{i1}[\pi_{i1}^E - \pi_i^U] + \mu_2 s_{i2}[\pi_{i2}^E - \pi_i^U], i \in (1,2) \tag{4}$$

To solve the model, these equations can be matched up with equations describing profit maximization by employers and equations that set these profits equal to zero (reflecting free entry by new employers). As we will see, the equations setting profits equal to zero will be of the same form no matter which equilibrium the model is in, but specification of the profit maximization equation depends on whether workers commute between districts. Wages are posted by firms. Since all workers in a given location are paid the same wage, setting wages to maximize profits in district j implies finding the lowest wage at which all relevant unemployed workers search for a job in j and then accept a job in j when a match is made. This reduces to a straightforward expression: the relevant values of employment and unemployment are equated.

Equilibrium 1: Autarky

We first establish the equilibrium in which intra-district commuting is not observed. Intuitively, this is the equilibrium that would occur if the districts were very far apart, or separated by an impenetrable barrier. Borrowing from the international trade literature, we call it autarky. In autarky, this profit-maximizing wage is simply equal to the exogenous flow utility of unemployment, b. Mathematically, this result can be attained by setting $\pi_i^U = \pi_{ii}^E$ and restricting s_{ij} to zero for $i\neq j$. Intuitively, the result attains because if workers are restricted to two outcomes (working in their home districts and unemployment) and are made indifferent between the two outcomes, there is no uncertainty about future utility flows, so the current flows are equated.

The model in autarky can then be solved by $\tau_j=v_j/s_j$, by imposing the zero-profit condition. To be more specific, since new entrants into the pool of employers are assumed to start in vacancy, free entry drives the value of a vacancy, π_j^V, to zero. Setting equation (3) equal to zero and merging it with equation (4), the zero profit condition can be expressed as:

$$\eta_j(y_j - w_j) = c(r + \delta). \quad (5)$$

This expression has the intuitive interpretation of setting the probability-weighted benefits of receiving profit flows from an operational plant equal to the discounted and probability-weighted costs of vacancy flows. Equilibrium, which depends on the exact form of the matching function governing η_j, is then found by setting w_j=b. Figure 1 shows the determination of the equilibrium level of τ_j, which we call τ_j^A, by plotting the profit maximizing curve w_j=b and the zero-profit curve described by equation (5) in τ_j - w_j space. It is straightforward to show that τ_j^A is greater (the labor market is tighter) the higher is the productivity of employed workers, y_j. Similarly, τ_j^A depends positively on the matching efficiency parameter, k, and negatively on the flow utility of unemployment b, the flow cost of vacancy c, the discount rate r, and the match separation rate δ.

The final piece to the characterization of the autarkic equilibrium is to note that the equilibrium number of employers in district j, F_j^A, can be determined by applying the identities $E_j^A + V_j^A = F_j^A$ and $E_i^A + U_i^A = P_i$ (=1) to the following steady-state employment equation:

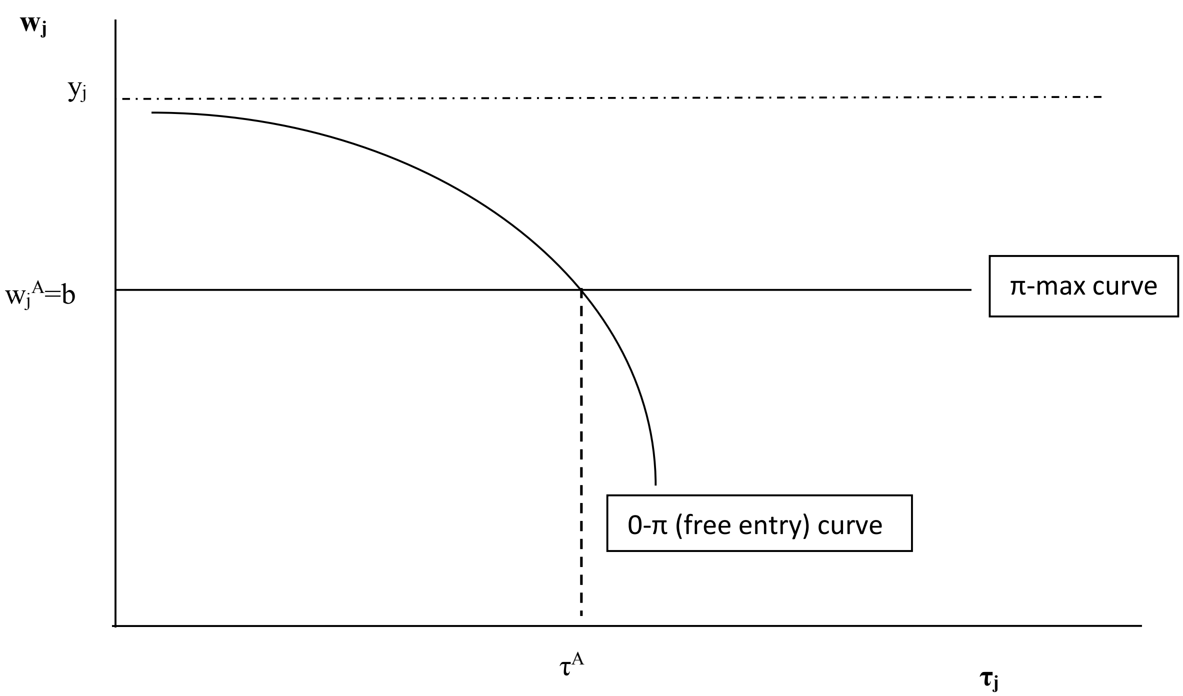
$$\delta E^A = \eta^A V^A = \mu^A U^A, \quad (6)$$

which leads to the following equilibrium levels of employment, unemployment, and vacancy:

$$E^A = \frac{\mu^A}{\mu^A+\delta}\,;\, U^A = \frac{\delta}{\mu^A+\delta};\, V^A = \frac{\tau^A \cdot \delta}{\mu^A+\delta} \quad (7)$$

Under the assumptions we have made about the matching function, this equation requires that higher levels of τ_j^A are associated with higher levels of F_j^A. We will return to this relation when we place more structure on the matching function.

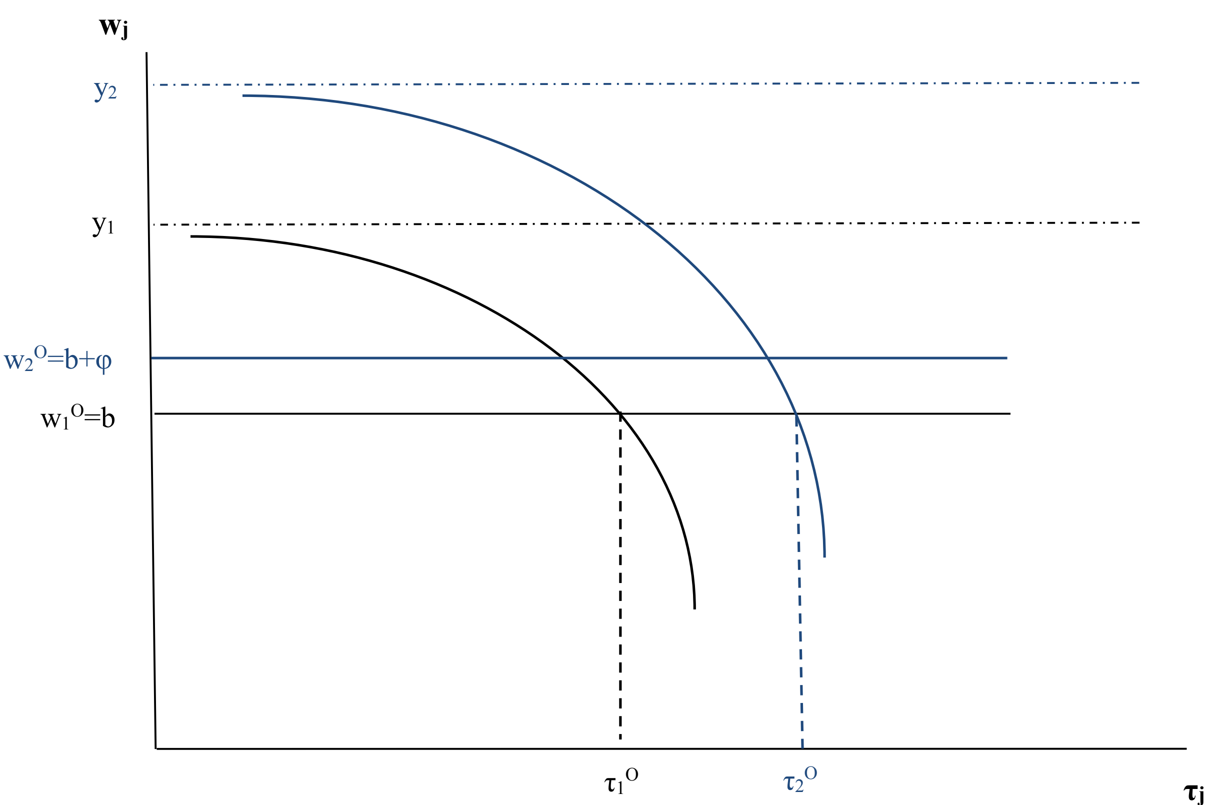
Figure 1: Wage and Labor Market Tightness in Autarky



Equilibrium 2: One-way commuting

When commuting between the districts is feasible, unemployed workers have another option to consider: they may conduct "mixed search" by searching for a job in both districts simultaneously. When wages are at their autarky levels as shown in Figure 1, workers have no incentive to conduct mixed search; their flow utility from work in the non-home district, b-φ, is lower than their flow utility in unemployment, b. But note that employers will be better off, all else equal, if unemployed workers engage in mixed search, because they stand to gain from the associated increase in match efficiency. In terms of the model, mixed search results in a higher level of s_j and thus a lower level of market tightness (τ_j) and a higher probability of exiting vacancy, η_j. As such, employers have an incentive to offer a higher wage in order to draw workers into mixed search. One possible equilibrium that incorporates mixed search is one in which one district's employers draw in searchers from both districts, while the other district only attracts its own residents. Intuitively, this one-way commuting equilibrium could occur if employers in one district are more productive than those in the other. Suppose the output flowing from a match in district 2 is greater than that in district 1 ($y_2>y_1$), and district 2 employers set wages high enough to entice workers to commute between districts while district 1 employers continue to target only district 1 workers. Then the employers' profit maximizing conditions are: $\pi_1^U = \pi_{12}^E, \pi_1^U = \pi_{11}^E$. In the resulting equilibrium, employers in district 2 offer w=b+φ, while employers in district 1 continue to offer w=b. If the difference in productivity between employers in the two districts is large enough, the labor market may be tighter in district 2 than in district 1, even though τ_2 now has a much bigger denominator ($s_2 = u_1 + u_2$); Figure 2 depicts such a scenario.

Figure 2: Wage and Labor Market Tightness in One-Way Commuting Equilibrium



The steady-state levels of employment, unemployment, and vacancy in this equilibrium are:

$$E_{11}^O = \frac{\mu_1^O}{\mu_1^O+\mu_2^O+\delta}\,;\, E_{12}^O = \frac{\mu_2^O}{\mu_1^O+\mu_2^O+\delta};\, U_1^O = \frac{\delta}{\mu_1^O+\mu_2^O+\delta};\, V_1^O = \frac{\tau_1^O\cdot\delta}{\mu_1^O+\mu_2^O+\delta}$$

$$E_{22}^O = \frac{\mu_2^O}{\mu_2^O+\delta};\, U_2^O = \frac{\delta}{\mu_2^O+\delta};\, V_2^O = \frac{\tau_2^O\cdot\delta(2\delta+\mu_1^O+2\mu_2^O)}{(\mu_1^O+\mu_2^O+\delta)\cdot(\mu_2^O+\delta)} \tag{8}$$

Equilibrium 3: Two-way commuting

A third type of equilibrium that could be attained when commuting between the districts is possible is one in which workers from both districts engage in mixed search and commuting is observed in both directions. We now characterize this two-way commuting equilibrium. To solve for the equilibrium values of w and τ, we start again with the profit maximizing conditions of employers. Profits are again maximized by offering the lowest wage at which all relevant unemployed workers search for a job in j and then accept a job in j when a match is made. The corresponding profit maximizing conditions are thus: $\pi_i^U = \pi_{ij}^E, i \neq j$.

Applying these conditions to the value of employment and unemployment equations, we obtain the following expression for the equilibrium wage under two-way commuting, which we call w_j^T.

$$w_j^T = b + \varphi + \left[\frac{\mu_k\cdot(r+\delta+2\mu_j)}{(r+\delta)(r+\delta+\mu_j+\mu_k)}\right]\varphi; \quad j\neq k \tag{9}$$

We see from this equation that equilibrium wages are no longer fixed at a given point: in addition to the flow utility of unemployment b, they depend positively on the commuting cost, φ, and the worker probabilities of finding a job match, μ_j, and μ_k. These latter parameters, in turn, reflect the labor market tightness measures, τ_j and τ_k. Note that, if φ=0, we again have the autarky wage, b. This reflects our simplistic set-up: since workers are paid just enough to lure them out of unemployment and not, for example, a share of the match surplus, their wages do not benefit directly from the increases in match efficiency gained through mixed search, even when commuting costs are zero.

It's worth considering, then, the nature of the increased wages we see in the two-way commuting equilibrium, relative to autarky. The key to understanding the expression in (9) is to note that, although there is only one wage rate offered within each district, the fact that unemployed workers may randomly match with an employer in their home or non-home district introduces variability into the utility they enjoy when employed. Wages are set so that workers matching with employers in their non-home districts receive the same utility by accepting the job that they would by remaining unemployed. But workers who match with an employer in their home districts receive a windfall: their wages are higher than the value of unemployment because they incur no commuting costs.

The potential for a windfall opportunity causes unemployed workers to command a higher wage from employers in the non-home district. When employers set their wages to match the value of unemployment, they must pay more than the flow utility of unemployment (b), because unemployed workers have the additional option value inherent in the possibility of finding a job in their home district. And since employers in both districts face the same issue, a feedback loop is created that drives equilibrium wages upward. The resulting wage premium is seen in the last term of equation (9). The size of this wage component is positively related to the tightness of the labor market in both the home and the non-home districts, with positive interaction between the two. The partial derivatives of the two districts' wages with respect to τ_j are:

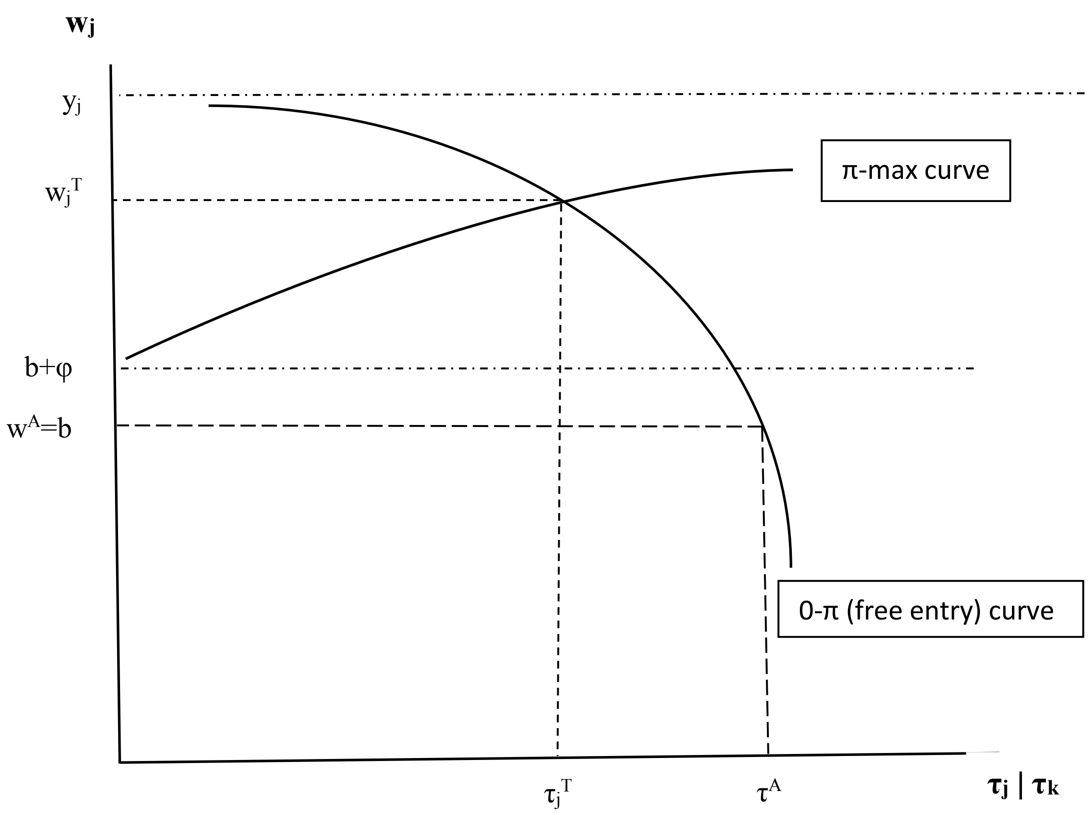
$$\partial w_j^T / \partial \tau_j|_{\mu_k} = \frac{\varphi(r+\delta+2\mu_k)\mu_k}{(r+\delta)(r+\delta+\mu_j+\mu_k)^2} \cdot \frac{\partial \mu_j^T}{\partial \tau_j} \tag{10}$$

and

$$\partial w_k^T / \partial \tau_j|_{\mu_k} = \frac{\varphi(r+\delta+2\mu_k)(r+\delta+\mu_k)}{(r+\delta)(r+\delta+\mu_j+\mu_k)^2} \cdot \frac{\partial \mu_j^T}{\partial \tau_j}. \tag{11}$$

These partial derivatives illustrate a consequence of the fact that we developed our wage rates based on the need to incentivize commuting from the non-home district: all else equal, the profit-maximizing wage rate is more sensitive to the tightness of the labor market in the non-home district than it is to tightness in the home district. Figure 3 illustrates the two-way commuting equilibrium for a given district in *conditional* w_j-τ_j space – i.e., holding labor market tightness in the other district (τ_k) constant. It also re-illustrates the autarky equilibrium values w^A and q^A as a point of comparison.

Figure 3: Wage and Labor Market Tightness in a Two-Way Commuting Equilibrium



Note that the zero-profit (free entry) curve of equation (5) has remained unchanged from Figure 1. The profit-maximization curve (9), however, is higher than that of the autarkic market (its lowest point is at b+φ, when τ_j=0) and has taken on a positive slope; the resulting equilibrium wage, w_j^T, is higher than w_j^A.

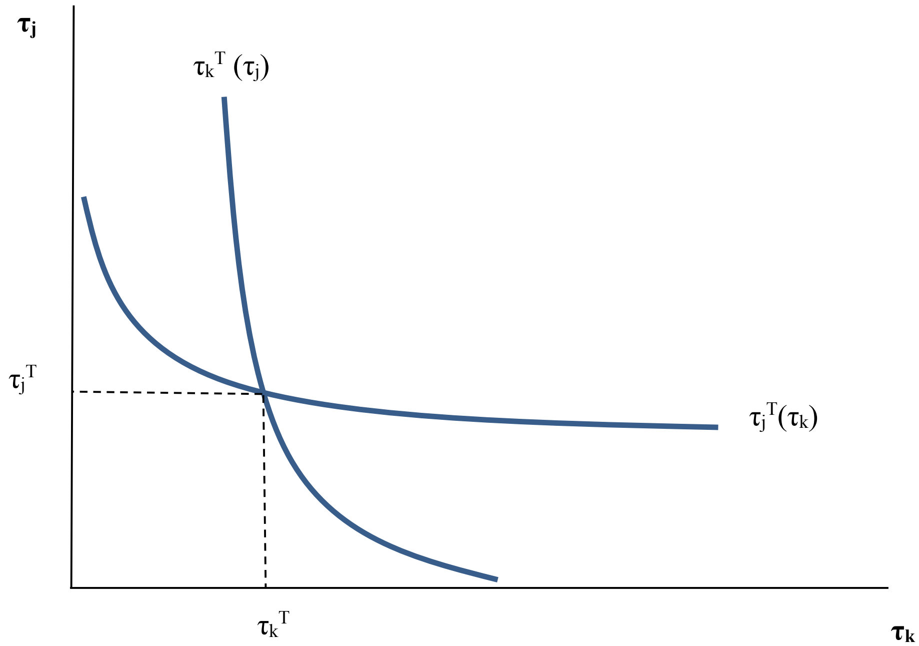
Figure 3 also shows that $\tau_j^{\,T}$ is lower than $\tau_j^{\,A}$, depicting "looser" labor market conditions in the two-way commuting equilibrium. Note that this does not imply higher unemployment rates among workers; rather, it reflects the fact that unemployed workers from both districts, and not just from the home district, are searching in each district (recall that the numerator of τ_j is $s_j = \sum_i s_{ij} * u_i$). In fact, under the conditions (developed below) establishing employers "prefer" the two-way-commuting equilibrium, the unemployment rates in each district are lower in the two-way commuting equilibrium than they would be under autarky.[4]

A unique two-way commuting equilibrium ($w_j^T, \tau_j^T, w_k^T, \tau_k^T$) exists for any values of y_j and y_k greater than $b+\varphi$. To see this, note that, for any given value of τ_k, a unique set of parameters (w_j^T, τ_j^T) must be determined since the zero-profit curve and the conditional (on the value of τ_k) profit-maximization curve for district j have opposite slopes. Conversely, the determination of τ_j^T implies that a unique value w_k^T must accompany these values. Proving existence of a unique equilibrium thus amounts to proving existence of a (τ_j^T, τ_k^T) pair.

Given a set of parameters y,b,r, c and δ, equations (5) and (9) imply a mapping of any value of τ_k into a value of τ_j^T. Let us sketch out the shape of this mapping $\tau_j^T(\tau_k)$. When the district k labor market is maximally loose (τ_k approaches 0), the conditional (on τ_k) profit maximization curve in district j approaches $w_j^T = b+\varphi$, and τ_j^T is uniquely determined by district j's zero profit equation at $w_j^T = b+\varphi$. As τ_k rises, the conditional (on τ_k) profit maximization curve in district j becomes steadily steeper, tracing out district j's zero profit equation so that τ_j^T declines; $\tau_j^T(\tau_k)$ is monotonically decreasing over the range of τ_k from 0 to infinity. Similarly, we can trace out a mapping $\tau_k^T(\tau_j)$ that is likewise monotonically decreasing over its whole support. It's straightforward to show that these two mappings must cross uniquely, as in Figure 4. By definition, at the crossing point, equations (5) and (9) are both satisfied in both districts.

[4] The rationale for this statement is as follows. Holding y, b, and the matching function constant, only two things affect the relative levels of u_j in the different equilibria: efficiency differences in matching (always favors T) and the number of employers (when T is preferred, $F^T > F^A$). Both of these favor lower unemployment in T.

Figure 4: Joint Determination of Market Tightness in a Two-Way Commuting Equilibrium



Equilibrium 3a: Symmetric, Two-way commuting

A special case of the two-way commuting equilibrium is one in which two districts are identical; e.g., y_j=y_k=y. We refer to this case as the symmetric equilibrium. In it, the profit-maximizing wage described by equation (9) simplifies to:

$$w^S = b + \varphi + \left[\frac{\mu^S}{(r+\delta)}\right]\varphi, \tag{12}$$

where subscripts have been removed as unnecessary (w and μ are the same in both districts), and we denote the symmetric case with a superscript of S instead of T. We can also consider the derivative expressing the sensitivity of the common wage rate in the symmetric equilibrium to the common labor market tightness, which is simply:

$$\partial w^S/\partial \tau = \frac{\varphi}{(r+\delta)} \cdot \frac{\partial \mu^S}{\partial \tau} \quad . \tag{13}$$

Note that the expression in equation (13) is greater in magnitude than those of equations (10) and (11) because it considers the simultaneous movement of both τ_j and τ_k, rather than holding one of them constant.

Steady-state equilibrium then requires the following identities:

$$\delta E^S = \eta^S V^S = 2\mu^S U^S, \tag{14}$$

which lead to the following steady state expressions for employment, unemployment, and vacancy:

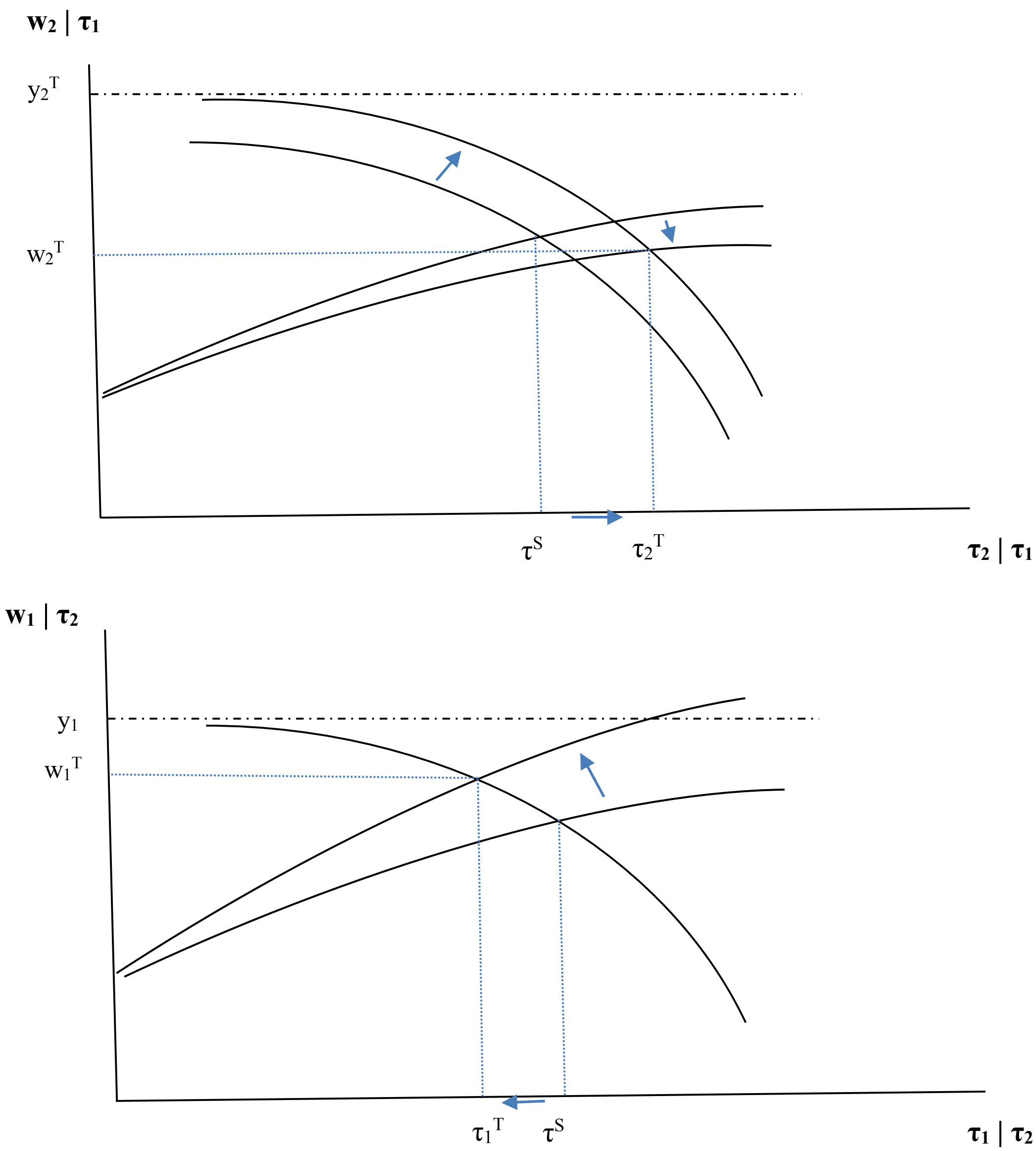
$$E^S = \frac{2\mu^S}{2\mu^S+\delta}\, ;\, U^S = \frac{\delta}{2\mu^S+\delta};\, V^S = \frac{2\tau^S \cdot \delta}{2\mu^S+\delta}, \tag{15}$$

As in the autarkic equilibrium, F^S (= $E^S + V^S$) and τ^S are positively related, albeit with relatively smaller movements in F^S associated with a given change in τ^S.

Equilibrium 3b: Asymmetric, Two-way commuting

The symmetric equilibrium provides a starting point for a full characterization of the two-way commuting equilibrium when $y_j \neq y_k$: the asymmetrical case. In particular, consider the case where the system starts in symmetric equilibrium and y_2 is increased. To see how the asymmetric equilibrium evolves, consider Figure 5. The increase in y_2 causes the zero-profit curve in district 2 to shift upward, with an immediate (partial) effect of increasing w_2 and τ_2. But the increase in τ_2 is accompanied by a shifting in of the profit-maximization curve in district 1, which reduces τ_2, causing the profit-maximization curve in district 2 to move outward. This causes τ_2 to increase further, but it also has a partial effect of lowering w_2. The net effect is to raise τ_2, lower τ_1, and raise w_1. The sign of the effect on w_2 is ambiguous.

Figure 5: Asymmetric Two-Way Commuting Equilibrium



Comparison of equilibiria: Beveridge Curves

We have already seen that the symmetric, two-way commuting equilibrium results in a higher wage level and a lower level of labor market tightness than is observed in autarky, *ceteris paribus*. But how do other outcomes of these equilibria compare? For the rest of the analysis, we focus on these two equilibria to study how other outcomes of the two equilibria differ, and to describe the conditions under which one equilibrium or the other may be preferred by market participants.

To enable this analysis, we impose a Cobb-Douglas form on the matching function:[5]

$$m_j = M(s_j, v_j; k) = k \cdot s_j^{\alpha} \cdot v_j^{(1-\alpha)}, \tag{16}$$

where we require that $0<\alpha<1$. It follows that $\mu_j = k \cdot \tau_j^{(1-\alpha)}$ and $\eta_j = k \cdot \tau_j^{-\alpha}$. This structure is entirely consistent with the foregoing discussion.

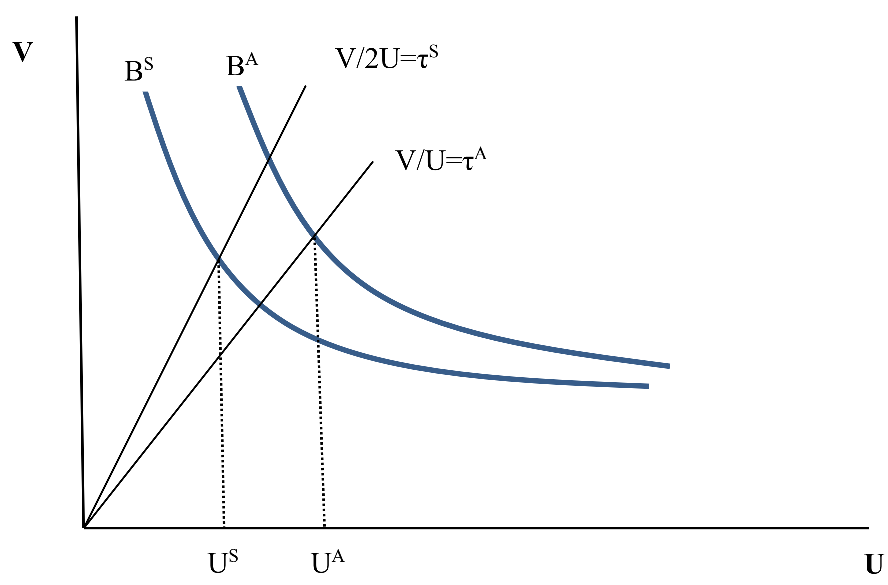
Combining the structure on the matching function from equation (16) with the steady state conditions in equations (6) and (14), we can obtain the following relationships between total vacancies (V) and total unemployment (U) in a district:

$$V^A = \left(\frac{\delta}{k}\right)^{\frac{1}{1-\alpha}} \cdot (1-U^A)^{\frac{1}{1-\alpha}} \cdot U^{A\frac{1}{1-\alpha}} \tag{17}$$

$$V^S = \left(\frac{\delta}{k \cdot 2^{\alpha}}\right)^{\frac{1}{1-\alpha}} \cdot (1-U^S)^{\frac{1}{1-\alpha}} \cdot U^{S\frac{1}{1-\alpha}} \tag{18}$$

These equations trace out a form of Beveridge curve; we depict equation (17) as B^A and equation (18) as B^S in Figure 6. The fact that B^S lies everywhere below B^A illustrates the fact that matching is more efficiently carried out in the two-way commuting equilibrium. By adding lines portraying τ^A and τ^S, we can get a sense of the relative values of U and V that result from the 2 equilibria. Figure 6 depicts unemployment in autarky as being lower than unemployment in the two-way commuting equilibrium; as we will see below, this is a feature of the model whenever the two-way commuting equilibrium is preferred to autarky.

Figure 6: Beveridge Curves in Autarky and Two-Way Commuting Equilibrium



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[5] This is a common assumption in the literature on search. It does not have a convincing rationale from a theoretical perspective, but empirical evidence doesn't seem to contradict it. From empirical studies, the value of α seems to be about 0.3, implying that the vacancy rate is about twice as important a determinant of the match rate as is the unemployment rate.

Equilibrium Preference: Autarky vs. Symmetric Two-Way Commuting

Having documented the contours of each of the several possible equilibria in this model, we are left with the question of which equilibrium we will observe under different circumstances. Equilibrium choice is, in this case, difficult to derive, because it involves coordinated action among the employers. Suppose we are in autarky – no worker searches for a job outside of their home district. Working together, employers can elicit mixed search by offering a high enough wage. But a single employer acting alone cannot elicit such a response; all the firms in the district must act together. And yet, in the model, all employers must make zero profits in the long run. To make progress on this topic, we consider the notion of equilibrium "preference." That is, starting from an autarkic equilibrium, would extant employers prefer to be in another equilibrium? The answer is, yes, if those firms would make positive profits in the short run (before entry of additional firms drove profits back to zero). Equivalently, we can say that employers prefer the other equilibrium if, in the long run, there are more firms observed in the other equilibrium.

We focus on a particular choice: the choice between autarky and the symmetric, two-way commuting equilibrium when $y_1=y_2$ (and the districts are in other ways identical). Following from the above discussion, we judge that autarky is preferred if and only if $F^A>F^S$. Tracking F^A and F^S, though, also turns out to be difficult to determine algebraically. Instead, we study an alternative measure that provides an equivalent test.

Consider a new equilibrium, which we call the "restricted" two-way commuting equilibrium, which we will refer to with the superscript R (w^R, τ^R, U^R, V^R, etc.). In this equilibrium, the number of employers is restricted to remain at the autarky level of F^A, but two-way commuting is observed, and employers set wages so as to maximize profits as in equation (9). The district attains levels of employment, unemployment and vacancy so as to maintain a steady state. Intuitively, in this equilibrium, wages and labor market tightness are determined as the point along the profit-maximization curve consistent with a steady state firm count of F^A. Needless to say, comparing F^R to F^S to determine whether autarky or two-way commuting is preferred is equivalent to comparing F^A to F^S. We make two additional transformations of this test. First, since F^S is everywhere positively related to τ^S, and the same can be said for F^R and τ^R, we can also say that autarky is preferred to symmetric, two-way commuting if and only if $\tau^R > \tau^S$. Second, we define "relative" labor market tightness as $t^R = \tau^R/\tau^A$ and $t^S = \tau^S/\tau^A$. Then we can say that autarky is preferred to symmetric, two-way commuting if and only if $t^R > t^S$.

We begin with the relative tightness of the restricted, two-way commuting equilibrium (t^R). Combining the steady-state relations in equations (6) and (14) with the Cobb-Douglas matching function, we obtain:

$$t^R = \frac{1}{2} + \frac{\eta^A}{2(\delta+\mu^A)}(1-\tau^A)\left(1-2{t^R}^{1-\alpha}\right). \tag{19}$$

Depending on the tightness of the autarkic labor market, t^R can be above or below 1/2 – the value it would have if moving to the restricted entry equilibrium mechanically doubled the number of unemployed workers available to employers in each district without any further adjustments. An intuition about this relation can be developed by considering the move from autarky to two-way-commuting with F_j fixed at F^A. Such a move would increase the efficiency of the matching process by increasing the rate of contact between unemployed workers and vacant employers, as is implicit in the matching function itself. As a result, both the unemployment rate and the vacancy rate can be expected to fall when inter-district commuting is introduced. Which falls by a greater percentage? That depends on the initial, autarkic conditions. If $\tau^A > 1$ (i.e., if the autarkic market is tight), then the percentage reduction in unemployment will exceed the percentage reduction in vacancies, and τ^S will be more than half the autarkic level. But if $\tau^A < 1$ (i.e., if the autarkic market is loose), then the opposite is true. And if $\tau^A = 1$, then $t^R = 1/2$.

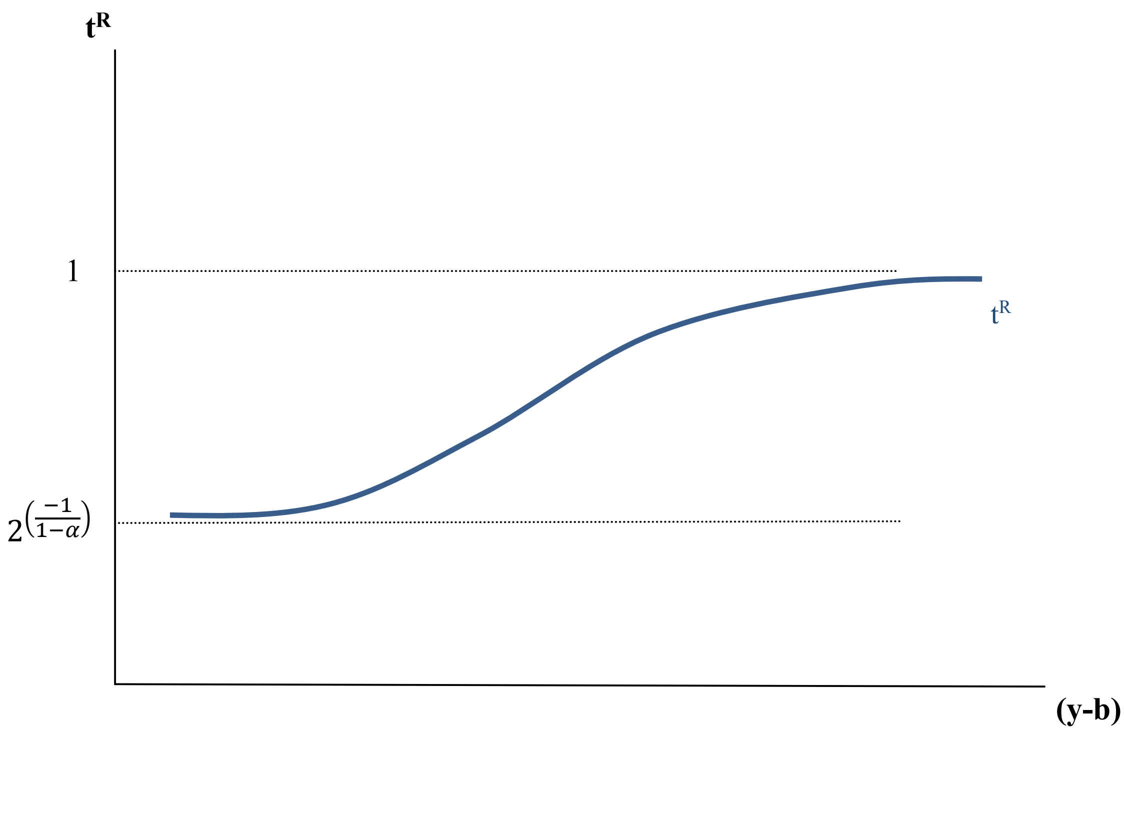
Since τ^A, μ^A, and η^A are functions of (y-b), we can translate equation (19) into a relationship between t^R and (y-b). Proposition 1 gives some characterization of this relationship:

<u>Proposition 1:</u>

a) $t^R \rightarrow 1$ as $(y-b) \rightarrow \infty$
b) $t^R \rightarrow \left(1/2\right)^{\frac{1}{1-\alpha}} = 2^{\left(\frac{-1}{1-\alpha}\right)}$ as $(y-b) \rightarrow 0$
c) $\partial t^R / \partial (y-b) > 0$
d) $\partial t^R / \partial \varphi = 0$

Figure 7 provides an illustration of the relationship. As (y-b) approaches infinity, the autarkic labor market becomes increasingly tight (τ^A goes to infinity); the convergence of t^R toward 1 reflects the shrinking scope for additional matches. As (y-b) approaches zero, autarkic labor market becomes increasingly loose (τ^A goes to zero), and τ^R converges to a fixed ratio of τ^A. Since the usual avenue for φ to affect equilibrium conditions (employer entry) is restricted, τ^R is insensitive to the level of φ.

Figure 7: Relative Labor Market Tightness in the Restricted-Entry Equilibrium



Next, let us consider the relative tightness of the (unrestricted) symmetric, two-way commuting equilibrium (t^S).

Combining equations (5) and (9) and again imposing the Cobb-Douglas structure on the matching function, we can characterize t^S by the following equation:

$$(y-b-\varphi)=\left[\frac{\varphi c}{y-b}\right]\tau^A {t^S}^{(1-\alpha)}+(y-b){t^S}^{\alpha}, \tag{20}$$

Or, equivalently,

$$1=\psi\cdot\left(\frac{1}{y-b-\varphi}\right)(y-b)^{\frac{1-\alpha}{\alpha}}{t^S}^{1-\alpha}+\left(\frac{y-b}{y-b-\varphi}\right){t^S}^{\alpha} \tag{21}$$

where $\psi=\left[\frac{k}{(r+\delta)c^{1-\alpha}}\right]^{\frac{1}{\alpha}}\varphi$. As long as φ>0, it follows that, $t^S < 1$ for all values of (y-b)>φ. It also follows that $t^S = 1$ for all values of (y-b)>0 if φ=0; i.e., when there are no commuting costs, a variant of the autarkic equilibrium is attained.[6]

Some key properties of the relationship between t^S and (y-b) are described in Propositions 2 and 3 and illustrated in Figure 8.

<u>Proposition 2:</u> When φ>0, (y-b) > φ, and α<1/2:

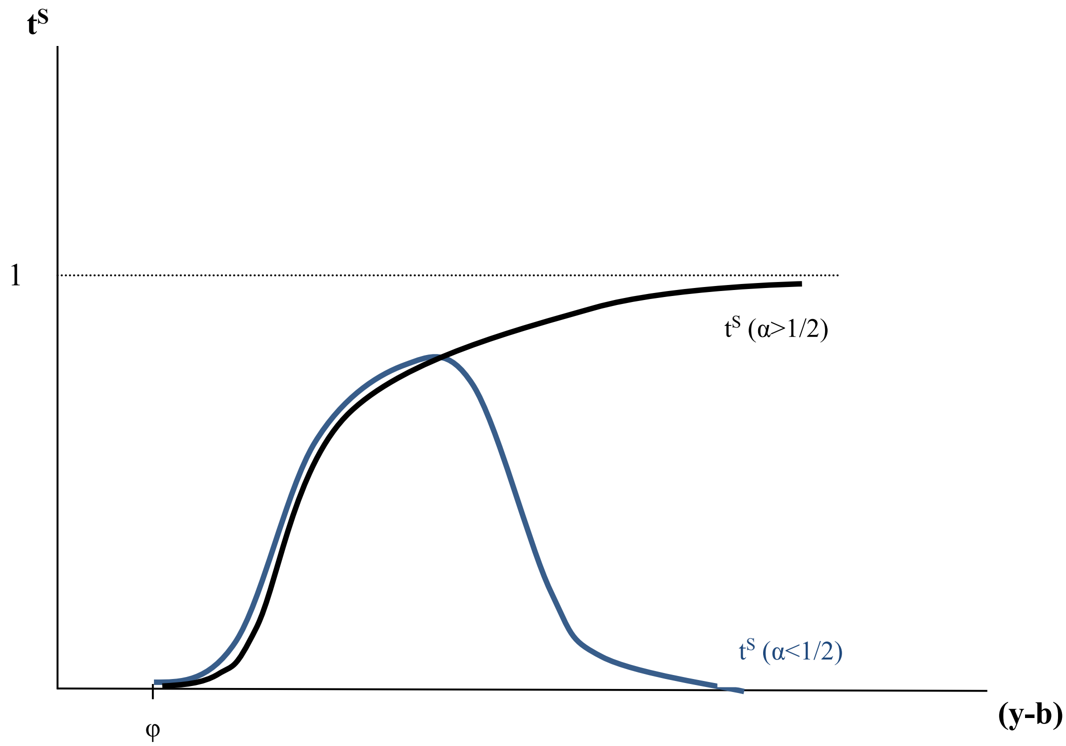
a) $t^S \to 0$ as (y-b)→ φ
b) $t^S \to 0$ as (y-b)→ ∞
c) $\partial t^S/\partial(y-b)>0$ and $\partial t^S/\partial(y-b)\to 0$ as (y-b)→ φ
d) $\partial t^S/\partial(y-b)<0$ and $\partial t^S/\partial(y-b)\to 0$ as (y-b)→ ∞
e) There exists a unique value of (y-b) at which t^S reaches a maximum and $\partial t^S/\partial(y-b)=0$.

<u>Proposition 3:</u> When φ>0, (y-b) > φ, and α>1/2:

a) $t^S \to 0$ as (y-b)→ φ
b) $t^S \to 1$ as (y-b)→ ∞
c) $\partial t^S/\partial(y-b)>0$ and $\partial t^S/\partial(y-b)\to 0$ as (y-b)→ φ
d) $\partial t^S/\partial(y-b)<0$ and $\partial t^S/\partial(y-b)\to 0$ as (y-b)→ ∞

[6] Note: this version is one with a larger number of employers and lower unemployment rate (albeit a higher number of job searchers in each district) than the 1-district autarkic equilibrium has, as the efficiency gains to search are absorbed by workers (and new employers).

Figure 8: Relative Labor Market Tightness in Symmetric, 2-Way Commuting Equilibrium With Positive Commuting Costs



Having described the shapes of both t^R and t^S in t – (y-b) space, we can now characterize some of the conditions under which the symmetric, two-way commuting equilibrium is preferred to autarky by employers. Recall that the two-way commuting equilibrium is preferred when $t^S > t^R$. Focusing on the case where α<1/2, Proposition 4 follows from Propositions 1 and 2[7].

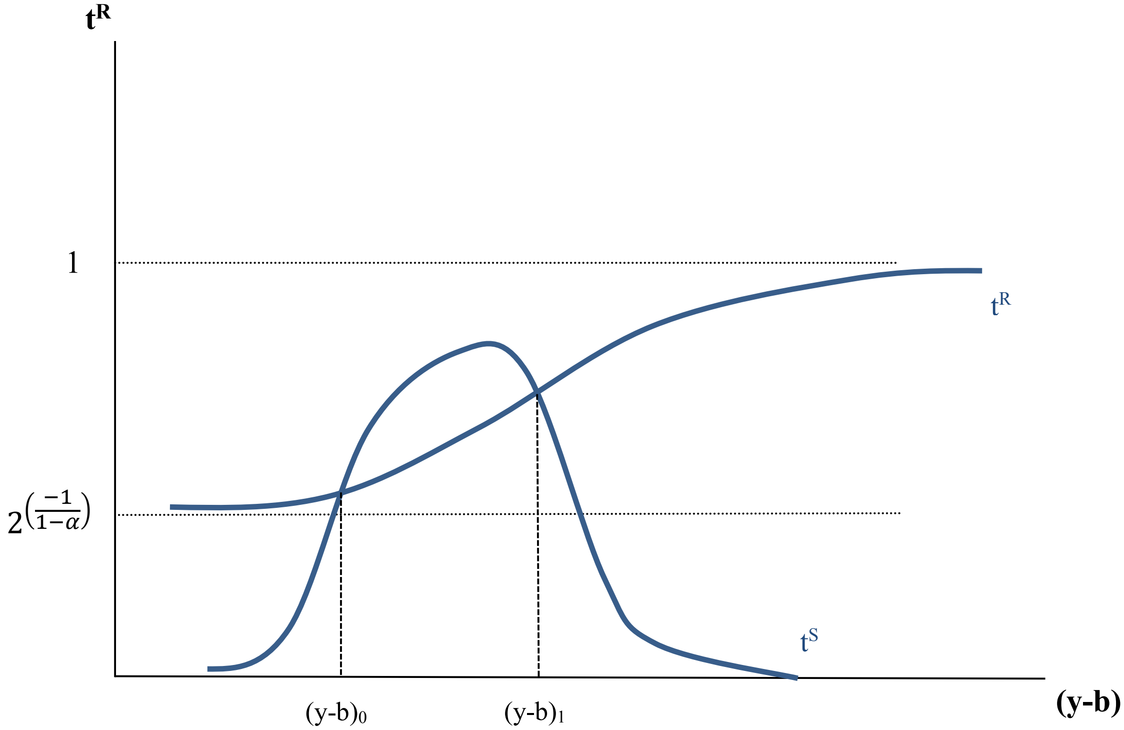
Proposition 4: When φ>0 and α<1/2,

a) At levels of (y-b) lower than φ, the two-way commuting equilibrium is infeasible.
b) As (y-b)→ φ from above, autarky is preferred to the two-way commuting equilibrium.
c) As (y-b)→ ∞ from below, autarky is preferred to the two-way commuting equilibrium.
d) For some intermediate range, $(y-b)_0 < (y-b) < (y-b)_1$, the two-way commuting equilibrium may be preferred to autarky.

One example in which the intermediate range described in part d) of Proposition 4 exists is seen in Figure 10, which overlays curves from Figures 8 and 9 (when α<1/2).

[7] Note: we focus on the case where α<1/2 because that is the usual case in the literature. In the alternative case (where α>1/2), the shape of the correspondence between t^S and (y-b) produces less of an obvious cross-over with the correspondence between t^R and (y-b). But, due to the two functions' speeds of convergence, we know that Proposition 4 holds whenever α<2/3.

Figure 10: Equilibrium Preference: Autarky vs. Symmetric Two-Way Commuting



An additional implication of equation (21) is that, for any value of (y-b)>φ, t^S will be larger, the smaller are commuting costs (φ). This means that, as φ decreases (increases), the range of (y-b) values over which the two-way commuting equilibrium is preferred expands (shrinks), and there exists a value φ*for which, if φ > φ*, the symmetrical, two-way commuting equilibrium is *never* preferred to autarky.

What we see in Proposition 4 and Figure 10 is that, in a world where different business districts are identical in their underlying characteristics, an equilibrium in which workers commute (both ways) between districts is only preferred to an equilibrium with no cross-commuting when productivity levels of employers are in some intermediate range, holding all other parameters of the model constant. The key to understanding the origin of this result is in realizing that inter-region commuting in this model acts as a way to increase the efficiency of the matching process, but that it does so at the cost of the commutes themselves. Thus, commuting is only preferred in situations where there are shortfalls in the autarkic matching process valuable enough to compensate for the increased costs associated with commuting. Higher levels of productivity increase the value of any such shortfalls, and so encourage commuting. But they also tend to make the shortfalls themselves smaller, because they encourage entry by new would-be employers.

Effects of Commuting Cost Changes

Since transportation policy is often concerned with increasing or decreasing the cost of commuting, it's interesting to consider the effects of changes in φ here. There are two: an intra-equilibrium effect and an inter-equilibrium effect.

Let us first consider the intra-equilibrium effect. If the two-way-commuting equilibrium is preferred (and, presumably, is operable) both before and after the change in θ, then the effect of changing commuting costs is to shift the profit-maximization curve shown in Figure 2. For example, a decrease in φ would shift the profit-maximization curve downward, causing wages to fall. Recall that, although workers in the model have little power to bargain over wages, employers must pay higher wages to incentivize search in the non-home district in order to maintain the two-way-commuting equilibrium, resulting in wind-fall gains for workers with jobs in their home district. For these workers, the reduction in φ is a negative, as they see their wages diminished. On the other hand, a fall in φ will also increase the amount of employer entry, causing labor market tightness (τ^S) to rise, and unemployment to fall.

The inter-equilibrium effect has different repercussions. If autarky is preferred (and, presumably, is operable) at the outset, then a decrease in φ may cause the two-way-commuting equilibrium to come into effect. If, in fact, the chosen equilibrium switches from autarky to two-way-commuting, we can expect an increase in wages and a decrease in labor market tightness, as seen by a comparison of w^S with w^A and τ^S with τ^A in Figure 3. So while the effect of diminished commuting costs on the unemployment rate is unambiguously downward, the direction of the effects on other variables, such as wages, depends on whether initial conditions imply that we are in an autarkic or an inter-area commuting equilibrium before the change in costs.

Conclusions

There seem to be two especially notable implications of the model. The first is the mechanism it portrays by which wages in the two-way commuting equilibrium are determined. This is a wage posting model in which workers are not imbued with any innate bargaining power: all of the surplus of the marginal worker-employer match is claimed by the employer. We see the consequence of this simplifying assumption in the fact that the equilibrium wage in autarky is no greater than the flow benefit available to unemployed workers. Similarly, in the one-way equilibrium, the equilibrium wage, even in the commuter-attracting district, is not greater than the unemployment benefit plus the cost of commuting. But in the two-way commuting equilibrium, employers in both districts have to raise wages above that level because they find themselves indirectly competing for each other's nearby workers. Due to the difference between the benefit of marginally-matched (i.e., commute-induced) workers and workers who work in their home district, coupled with the fact that employers post only one wage, workers are able to claim some share of the match surplus. In effect, workers have been given a measure of bargaining power where none originally existed. This description provides an alternative explanation for the high wages observed in cities: rather than reflecting improvements in the quality of matches (see, e.g., Wheeler 2001), they may reflect this increased worker bargaining power, enabled by a generalized efficiency in the rate of matching.

The second notable implication of the model lies in its depiction of the conditions under which the two-way commuting equilibrium is preferred by employers to autarky. The model indicates that very high levels of match productivity are not conducive to commuting; the two-way commuting equilibrium is preferred only for intermediate values of match productivity. This may seem counterintuitive: one might expect that more productive employers would try to attract workers from a larger market so as to fill vacancies as quickly as possible. But the model's free entry condition implies that higher match productivity causes more employers to enter the market, resulting a tighter labor market with less unemployment. As unemployment falls, the potential for improvement in the match rate declines; eventually the return to seeking workers from the other district becomes too small to justify the cost in higher wages. This dynamic illustrates the way the model confronts a fundamental trade-off: on one hand, integration of the job loci in one labor market increases job-matching efficiency of the labor market (resulting in lower vacancy rates and/or lower unemployment rates); on the other hand, integration of labor markets across space entail increased costs to workers who have to commute to jobs that are not close to their residences, which must be compensated for by employers.

We have formalized this tradeoff here in terms of the preferences of employers among different equilibria, with the implication that employers could act in concert to bring about one equilibrium or another. But we haven't specified a mechanism for this joint action. In future work, it might be worth considering some alternative ways of evaluating the equilibria, or specifying a path by which the system would gravitate to one equilibrium or another. Intuitively, it seems that the coordination involved in equilibrium choice would lead to a strong role for historical patterns, as is typical of many urban agglomeration stories.

Another extension of this work that would be promising to pursue is to generalize it to more than two employment districts. If the lessons of our model carry over, we would expect that, in a n-district model, employers in a given district would choose to include other districts within a certain distance in their labor market, resulting in higher wages and lower unemployment overall as more commuting is observed; we would also expect intermediate ranges of match productivity to result in the widest labor markets. One model that comes close to approximating such a model is the one proposed by Zenou (2009)[8] for a circular city, in which employers are spread out continuously over space; this is like a model with an infinite number of districts are present, with no distance between adjacent districts. The circular construct allows one to extrapolate from external limits of the city; all districts are equivalently situated with respect to the others. Zenou's model does result in the definition of local labor markets that extend an endogenously determined distance in either direction from a given employer. But the distance, and the resulting size of the labor markets, is monotonically related to the productivity of employer-employee matches. This difference appears to be due to the fact that, in continuous space, employers determine their preferred labor market radius individually without consideration of the choices of other employers: the external effects of their choice are not incorporated.[9]

Of course, as Zenou (2009) has documented throughout his book, the scope of potential extensions to this model is extensive: incorporating population movements and related effects on property values, allowing for heterogeneity among workers in skills, search efficiency, and transportation costs, and exploring restrictions on employer behavior are all fertile avenues for increased sophistication. Continuing progress in developing our understandings of urban labor markets through these models will have significant, lasting benefits for the optimal design of public policies in the fields of labor, geography and transportation.

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[8] See section 3 of Chapter 3, on pp. 132-142.

[9] Some other differences between Zenou's (2009) model and the one presented here are: it posits Nash bargaining over wages; it models workers choice of a search intensity (the "internal margin" of search determination) as well as a radius from which to accept offers; and the actual offers (or search) come in from all over the city, not just those within the acceptable segment (not clear whether or how this matters). It also builds in two different choices of transportation mode.

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