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LINE COLORED TREES WITH EXTENDABLE AUTOMORPHISMS*

Georg Gati[1] *Frank Harary*[2] *Robert W. Robinson*[3]

Abstract

An E-tree is a rooted line-colored tree such that every automorphism of a subtree containing the root can be extended to an automorphism of the entire tree. When only one color is used, E-trees correspond both to achiral planted trees and to partitions with successively divisible parts. The exact numbers of E-trees with n points and colors from a store of c are found, with and without the restriction that each color should be used at least one. Asymptotic formulas for these quantities are discussed for c fixed and $n \to \infty$.

§ 1 Introduction

An "E-tree" was first introduced by Gati[3], motivated by an application to the theory of complexity of algorithms based on Engeler[1]. By recursive definition, a rooted tree T with each line colored with one of c given colors and having root point w is an E-tree if

(*a*) T is the trivial rooted tree, or

(*b*) all principal branches with the same stem color are isomorphic, and for each principal branch B of T, the rooted tree B-w is also an E-tree.

When $c=1$, these correspond precisely to the achiral planted trees

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1) Institut fur Informatik, ETH Zurich, Switzerland. Present address: Institut fur Informatik, Universitat Wien, Austria. Supported by Swiss National Science Foundation Grants Nos. 2.861-0°77 and 82.813-0.80.

2) Ovrseas Fellow (1980-81), Churchill College, University of Cambridge, England. Present address, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.

3) University of Newcastle, New South Wales 2308, Australia. Temporary address, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A. until Jan. 1982.

counted in [6]. Figure 1 shows all six E-trees with $p=6$ points having one color.

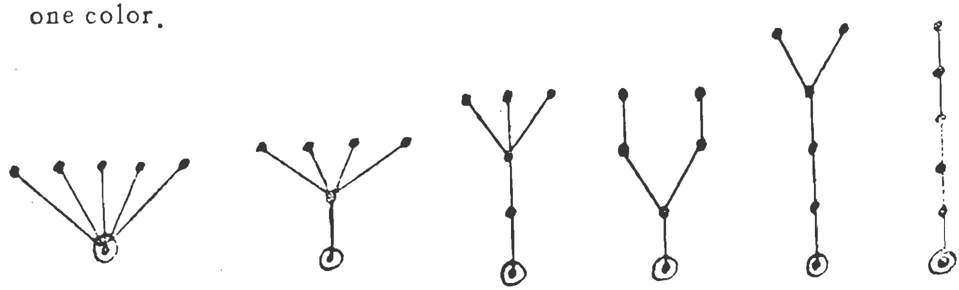

Figure 1. The E-trees with one color having six points

When $c \geqslant 2$, the structure of E-trees is more complex. The original concept[3] of an E-tree was a line-colored rooted tree T such that every automorphism of a subtree containing the root can be extended to an automorphism of T. Obviously any E-tree has this property. We now show the converse.

Suppose P is a planted tree with root u and stem uv, and T is the tree $p-u$ rooted at v. Then it is also clear that P is an E-tree if and only if T is. We say that T is obtained from P by *pruning the root*.

Now to prove the converse, (a) holds as the trivial tree trivially has the extendability property. To verify (b) we first note for a nontrivial tree T that the root and its incident lines form a (star) subtree in which any two lines of the same color are interchanged by some automorphism. Thus to have the extendability property, any two principal branches of T having the same stem color must be isomorphic. Secondly, any automorphism of a nontrivial subtree of a principal branch B is extendable to an automorphism α of T. The stem being fixed in the automorphism of the subtree, it is fixed by α in T, so the restriction of α to B is an automorphism of B. Thus each principal branch has the extendabilityp property. By pruning the root, it then follows by induction that each principal branch is an E-tree.

Our object is to count E-trees in several ways; first those with at most c colors, deriving both generating functions and recurrence relations; then those needing exactly c colors; and finally, the asymptotic numbers.

§ 2 Counting

Let ${}_cE_n$ be the number of different E-trees on n points using colors from a fixed set of cardinality c, and let

$$ {}_cE(x)=\sum_{n=1}^{\infty}{}_cE_nx^n $$

be the ordinary generating function. We first consider the case $c=1$, for which the lower left subscript c is omitted. Then $E(x)$ satisfies the functional equation,

$$ E(x)=x\left(1+\sum_{i=1}^{\infty}E(x^i)\right). \quad (1) $$

In fact, x stands for the trivial tree (on one point) and $x\ E(x^i)$ enumerates E-trees of a single color in which the root is incident with precisely i lines. In that case the i principal branches must all be isomorphic, and i identical copies of a pruned branch are enumerated by $E(x^i)$.

A recurrence suitable for computation is readily deduced from (1) on equating coefficients of x^{n+1}. One finds $E_1=1$, and for $n\geqslant 1$,

$$ E_{n+1}=\sum_{d\mid n}E_d. \quad (2) $$

A slight improvement in computatonal facility is obtained by considering E_{n+1} as a function of n, so that standard moebius inversion gives

$$ E_{n+1}=E_n-\sum_{1<d\mid n}\mu(d)E_{1+n/d}. \quad (3) $$

The first few values of E_n are shown in Table 1. It may be noted that $E_n=P_{n+1}$, where P_{n+1} is the number of achiral planted trees discussed in [6]. A 1–1 correspondence is obtained simply by pruning the root from the planted trees.

The same reasoning applies to ${}_cE(x)$, $c>1$, as for $E(x)$ except that now there are c different stem colors possible at the root. Since the choices for the different colors are independent, one can simply take the c'th power of the sum on the right with $E(x)$' replaced by ${}_cE(x)$. Then

$$ {}_cE(x)=x\left(1+\sum_{i=1}^{\infty}{}_cE(x^i)\right)^c. \quad (4) $$

In expressing a recurrence for ${}_cE_n$ it is convenient to give the sum on the right of (4) a name, say ${}_cS(x)$. If we denote the coefficient of x^n by ${}_cS_n$ then ${}_cS_0=1$ and for $n>0$,

$$ {}_cS_n=\sum_{d\mid n}{}_cE_d. \quad (5) $$

Now on differentiating (4) we have

$$ {}_cS(x)\cdot({}_cE(x)/x)' = c\cdot({}_cE(x)/x)\cdot{}_cS'(x). $$

Equating coefficients of x^{n-2} gives for $n>1$,

$$ {}_cE_n = \frac{1}{n-1}\sum_{i=1}^{n-1}((c+1)i-n+1)\,{}_cS_i\,{}_cE_{n-i}. \qquad (6) $$

With ${}_cE_1=1$, (5) and (6) together provide an efficient recurrence for computing numerical values. The first few of these are shown in Table 1.

Of course (5) and (6) also apply to the earlier case $c=1$, although somewhat different from (2) or (3). By (2) it can be seen that $S_n = E_{n+1}$, and when that is substituted into (6) we have the curious identity,

$$ \sum_{i=0}^{n-1}(2i-n+1)E_{i+1}E_{n-i}=0 \qquad (7) $$

for $n>1$. As (7) is true of any sequence, this gives a circuitous demonstration that (5) and (6) for $c=1$ are equivalent to (2).

The standard method of inclusion-exclusion can be used to compute the numbers of E-trees according to how many clors are actually used. Let ${}_cF_n$ be the number of E-trees of order n using all of the colors from a fixed list of c. It is natural in this context to take ${}_0F_0=1$ and ${}_0F_i=0$ for all $i>0$. The number of ways in which a subset of j colors can be chosen in $\binom{c}{j}$. The number of E-trees of order n on the remaining $c-j$ colors is ${}_{(c-j)}E_n$. Thus one finds that

$$ {}_cF_n = \sum_{j=0}^{c} {}_{(c-j)}E_n\binom{c}{j}(-1)^j \qquad (8) $$

for all $c\geqslant 0$ and $n\geqslant 1$. In this way the first few values of ${}_cF_n$ were computed, as displayed in Table 2.

The maximum possible number of colors on the lines of a tree with n points is $n-1$, since that is the number of lines. In that case a root is specified and each line is labeled by a separate color, so that the tree is effectively labeled. A 1—1 correspondence can be established at once by assigning the colors numbers $1,\cdots,n-1$. Then an E-tree of order n using these $n-1$ colors is point-labeled by assigning 1 to the root, and to each other point the number $1+i$ where i is the label of the first line on the unique path to the root. From this and the well known formula of Cayley[5, p.20] for counting labeled trees, we have ${}_{n-1}E_n = n^{n-2}$ for $n\geqslant 1$. This can be seen in Table 2 for $1\leqslant n\leqslant 6$.

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n	$1^E n$	$2^E n$	$3^E n$	$4^E n$	$5^E n$
1	1	1	1	1	1
2	1	2	3	4	5
3	2	7	15	26	40
4	3	22	73	172	335
5	5	75	387	1243	3070
6	6	250	2106	9364	29526
7	10	886			
8	11	3150			
9	16				
10	19				

Table 1 E-trees with at most c colors.

n	$1^F n$	$2^F n$	$3^F n$	$4^F n$	$5^F n$
1	0	0	0	0	0
2	1	0	0	0	0
3	2	3	0	0	0
4	3	16	16	0	0
5	5	65	177	125	0
6	6	238	1874	2432	1296
7	10	866			
8	11	3138			
9	16				
10	19				

Table 2 E-trees with exactly c colors.

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§ 3 Asymptotics

We have already mentioned that the E-trees counted by $E(x)$ are precisely the achiral rooted trees enumerated in [6]. One of us (RWR) noticed that these particular trees are faithfully represented by the partitions of n in which each part divides the next one, and asked Paul Erdős in 1975 how many there are. This led to the paper by Erdős and Loxton[2] in which their asymptotic behavior was studied.

They used the notations

$$q(n) = E_{n+1}, \qquad Q(x) = \sum_{0 \le n \le x} q(n)$$

and established that

$$\log Q(x)=\frac{1}{2\log 2}\left(\log\frac{x}{\log x}\right)^{2}+\left(\frac{1}{2}+\frac{1}{\log 2}+\frac{\log\log 2}{\log 2}\right)\log x$$

$$-\left(1+\frac{\log\log 2}{\log 2}\right)\log\log x+V\left(\frac{\log x-\log\log x}{\log 2}\right)+0(1)$$

where $V(t)$ is periodic with period l but V is not available explicitly

For $c>1$ the asymptotic behavior of ${}_cE_n$ is much simpler than that of E_n. Consider, for example, $c=2$. The path of length $n-1$ rooted at one end is an E-tree no matter how its lines are colored, so ${}_2E_n\geqslant 2^{n-1}$ for all $n\geqslant 1$. Conversely, if T_n is the number of rooted trees of order n, then ${}_2E_n\leqslant 2^{n-1}T_n$ since 2^{n-1} is an upper bound for the number of distinct ways of line-coloring any one of those trees. It follows at once that if $\eta=0.3383219\cdots$ denotes the radius of convergence of $T(x)$ and ζ_2 is that of ${}_2E(x)$, then

$$\frac{r}{2}\leqslant\zeta_2\leqslant\frac{1}{2}.$$

Hence the methods of [7] can be applied to show that

$${}_2E_n=a_2n^{-3/2}\zeta_2^{-n}(1+0(1/n)).$$

Here the constants a_2 and ζ_2 can be determined as closely as desired. again by the standard methods. More generally but in the same way, for any $c>1$ one has

$$\frac{n}{c}\leqslant\zeta_c\leqslant\frac{1}{c}$$

and

$${}_cE_n=a_cn^{-3/2}\zeta_c^{-n}(1+0(1/n)).$$

Here c is fixed while $n\to\infty$, ζ_c is the radius of convergence of ${}_cE(x)$, and a_c depends on c but not n, as does the bound implicit in the $0(1/n)$ remainder term.

It seems clear that $\zeta_c>\zeta_{c+1}$ for all $c\geqslant 1$. (That $\zeta_1=1$ was shown in [2].) Assuming this, it follows from (8) that ${}_cF_n\sim{}_cE_n$ for fixed $c\geqslant 1$ as $n\to\infty$.

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