# Pairwise Powers of 2 Problem

Alex Bradley

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## 1 Abstract

A set $S \subseteq \mathbb{Z}$ is a solution to the Pairwise Powers of 2 problem if and only if every pair of distinct elements of $S$ sum to a power of 2. That is,

$$\forall x, y \in S.\, \exists n \in \mathbb{Z}_+.\, (x \neq y \implies x + y = 2^n) \tag{1}$$

It is known that solutions of size 2 and 3 exist, for example $\{3, 5\}$ and $\{-1, 3, 5\}$. In this paper I will prove that no solutions exists of size 4 (or greater).

## 2 No Solutions of Size 4

**Observation 1.** *Any subset of a solution is also a solution.*

For example, the solution of size 3 listed about is $\{-1, 3, 5\}$. Notice that $\{-1, 3\}$, $\{-1, 5\}$, and $\{3, 5\}$ are all solutions of size 2.

**Lemma 1.** *Any solution of size 3 must contain a negative integer.*

*Proof.* Suppose by way of contradiction that $\{a, b, c\}$ is a solution of positive integers. Assume without loss of generality that $a < b < c$. Note that $a+b < a+c < c+b$ and $b+c < 2c < 2(a+c)$.

By definition, $a + c$ and $c + b$ are powers of 2, but note that $a + c$ and $2(a + c)$ consecutive powers of 2, thus $c + b$ cannot exist between them. We have reached a contradiction, so our assumption that an all-positive solution of size 3 exists must be false. □

**Theorem 1.** *There are no solutions of size 4.*

*Proof.* Suppose by way of contradiction that a solution $S = \{a, b, c, d\}$ exists. Then by Observation 1, $\{a, b, c\}$ is a solution of size 3. By Lemma 1, one of $a$, $b$, or $c$ must be negative. Let $a < 0$ without loss of generality.

Now note that again by Observation 1, $\{b, c, d\}$ is also a solution of size 3, and contains a negative value by Lemma 1. Let this negative value be $x \in \{b, c, d\}$.

Now we have $a, x \in S$ with $a, x < 0$ and $a \neq x$. Clearly $a + x < 0$ is not a power of 2, which means $S$ is not a solution of size 4. □

**Corollary 1.** *There are not solutions of size greater than 4.*

*Proof.* By Observation 1, any solution of size greater than 4 would contain a solution of size 4, which has already been shown to not exist. □