# ZERO-CALIBRATED VARIANCE ESTIMATORS

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## 1. Introduction

We consider variance estimation in the case of a calibrated estimator of a finite population total of a variable of interest $y$ (the results can be generalized to more complicated finite population parameters, see Remark 3.3). An estimator is *calibrated* with respect to a particular variable $x$ when the estimator is such as to yield the actual population total of the $x$'s when $x$ is taken as the variable of interest (that is, when it is substituted for $y$ in the estimation formula), whatever the sample. In this situation, we say a variance estimator is *zero-calibrated* if, for $y = x$, it takes the value 0. Zero-calibration is a desirable property for calibrated estimators: actually if it is known that an estimate being applied to a particular variable has 0 variance then we would like the variance estimate to be equal to 0 too, since otherwise we may suspect overestimating the variance. Another, maybe more important, desirable property for any variance estimator is that it does not underestimate the variance, at least asymptotically. In this paper we show that for replication-type variance estimators under very general conditions these two desirable properties can be achieved by calibrating the estimate on each partial sample. As an illustration we consider variance estimation of a finite population total estimate under a particular 2-phase sampling scheme.

## 2. Definition and an example of Zero-calibrated Estimator

Consider a population $(y_i, x_i; i \in U)$ and a sample $(y_i, x_i; i \in S)$. Let our goal be to estimate a finite population total, $t_y \overset{def}{=} \sum_U y_i$.

Let $x_i$ be an auxiliary variable (the results of this paper can be easily generalized to vector valued $x$'s) and $w_i$ be a sample weight; the latter is not necessarily equal to the inverse selection probability; see, for example, Section 5.

**Definition**

**(i)** An estimate of $t_y$, $\hat{t}_y = g(y_i, x_i, w_i; i \in S)$, is *calibrated* with respect to $x$ if

$$\hat{t}_x = g(x_i, x_i, w_i; i \in S) = t_x \overset{def}{=} \sum_U x_i ;$$

**(ii)** a variance estimator (of calibrated with respect to $x$ estimator $\hat{t}_y$), $\hat{V}(\hat{t}_y) = h[g(y_i, x_i, w_i; i \in S)]$, is *zero-calibrated* if $\hat{V}(\hat{t}_x) = h[g(x_i, x_i, w_i; i \in S)] = 0$.

To illustrate this definition consider the following simple example. Let $\hat{t}_{\pi,y} = \sum_{i \in S} \frac{y_i}{\pi_i}$ be the Horvitz-Thompson's Estimator of a finite population total under probability sample of fixed size $n$ proportional to size variable $x_i$, $\pi_i = \frac{n x_i}{t_x}$. Obviously $\hat{t}_{\pi,x} = t_x$ and therefore $\hat{t}_{\pi,y}$ is calibrated with respect to $x$. Two classical variance estimators of $\hat{t}_{\pi,y}$ are the *Horvitz-Thompson's variance estimator,*

$$\hat{V}_{H-T}(y) = \sum_{k \in S} \sum_{l \in S} \frac{(\pi_{kl} - \pi_k \pi_l)}{\pi_{kl}} \frac{y_k}{\pi_k} \frac{y_l}{\pi_l},$$

and the *Yates-Grundy's variance estimator,*

$$\hat{V}_{Y-G}(y) = \sum_{k\in S}\sum_{l\in S}\frac{(\pi_{kl}-\pi_k\pi_l)}{\pi_{kl}}\left(\frac{y_l}{\pi_l}-\frac{y_k}{\pi_k}\right)^2.$$

One can easily see that the Horvitz-Thompson's variance estimator is not zero-calibrated since $\sum_{k\in S}\sum_{l\in S}\frac{(\pi_{kl}-\pi_k\pi_l)}{\pi_{kl}}\frac{x_k}{\pi_k}\frac{x_l}{\pi_l}\neq 0$; to check this, consider a sample of 2 units from population of 3 units with $x_i$ values 1,2 and 2. On the other hand the Yates-Grundy's variance estimator is zero-calibrated since $\frac{x_l}{\pi_l}-\frac{x_k}{\pi_k}=0$ for all $l$ and $k$.

### 3. Zero-calibrated Jackknife Variance Estimator

As above write our estimator as, $\hat{t}_y = g(y_i, x_i, w_i; i\in S)$, where $g$ is a known function for a fixed population, for example if $\hat{t}_y$ is the ratio estimator then $\hat{t}_y = t_x\frac{\sum_{i\in s} y_i}{\sum_{i\in s} x_i} = g(y_i, x_i; i\in S)$ and $t_x$ is treated as a constant. Consider a partition of the complete sample into $B\le n$ disjoint subsamples, $S = S_1\cup\ldots\cup S_B$, and let $b$'th partial sample be equal to the initial sample without $b$'th subsample, $S_{[b]} = S - \{S_b\}$. Define an *estimate based on $b$'th partial sample* (ebop) by $\hat{t}_{y[b]} = g(y_i, x_i, w_i; i\in S_{[b]})$. Consider $B$ ebop's. Then the Jackknife estimator of the variance of $\hat{t}_y$ is defined by,

$$\hat{V}_{Jack}(\hat{t}_y) = h[g(Y_i, X_i, w_i; i\in S)] = \frac{B-1}{B}\sum_{b=1}^{B}(\hat{t}_{y[b]}-\hat{t}_y)^2.$$

(One can find alternative definitions of Jackknife variance estimators in Wolter 1985.)

The Jackknife estimator satisfies the following property. For the Horvitz-Thompson's Estimator, $\hat{t}_{\pi,y} = \sum_s y_i/\pi_i$, under a probability sample *without* replacement ,

$$E_D[\hat{V}_{Jack}(\hat{t}_{\pi,y})] = E_D[\hat{V}_{PPSWR}(\hat{t}_{\pi,y})], \quad (3.1)$$

where $E_D$ denotes the expectation with respect to the design distribution and

$$\hat{V}_{PPSWR}(\hat{t}_{\pi,y}) = \frac{1}{n(n-1)}\sum_{i\in S}\left(\frac{ny_i}{\pi_i}-\frac{1}{n}\sum_{i\in S}\frac{ny_i}{\pi_i}\right)^2$$

is a design unbiased variance estimate of $\hat{t}_y$ as if the sample would be *with* replacement, see Wolter (1985), p.170. One may show that

$$V_{PPSWR}(\hat{t}_{\pi,y}) - E_D[\hat{V}_{PPSWR}(\hat{t}_{\pi,y})] = \frac{1}{n-1}(V_D(\hat{t}_{\pi,y}) - V_{PPSWR}(\hat{t}_{\pi,y})), \quad (3.2)$$

and therefore $E_D[\hat{V}_{PPSWR}(\hat{t}_{\pi,y})]$ is close to $V_{PPSWR}(\hat{t}_{\pi,y})$ if the sample size is big. Eq. 3.1 and 3.2 imply that if the given design is more efficient than the probability sampling with replacement (with the same selection probabilities) then the expectation over the design distribution of the Jackknife Estimator *does not underestimate* the correct variance of the estimator, which is the second desirable property of the Variance Estimator that was mentioned in the introduction.

Return now to calibrated estimators and consider them as they were introduced in Deville and Särndal (1992).

**Definition** An estimate of a population total $t_y$, $\hat{t}_{C,y} = g_C(y_i, x_i, w_i; i\in S)$, is calibrated in the sense of Deville and Särndal (1992) with respect to $x$ and a given distance function $\rho$ if

$$\hat{t}_{C,y} = g_C(y_i, x_i, w_i; i\in S) = \sum_{i\in S} d_i y_i,$$

where

$$\mathbf{d} = (d_1,\ldots,d_n) = \underset{\mathbf{v}:\ \sum_{k\in S} v_k x_k = t_x}{\operatorname{argmin}} \rho(\mathbf{v},\mathbf{w})$$

and $\mathbf{w} = (w_1,\ldots,w_n)$.

In words, an estimator calibrated with respect to $x$ is a weighted estimator with the weights as similar to the original sample weights as possible, given that it estimates perfectly the known information, where "as similar as possible" is defined by some given distance function $\rho$.

Given this definition, the Jackknife variance estimator for calibrated in the sense of Deville and Särndal (1992) estimate has to be based on ebop's each of which is a weighted estimator with the weights as similar to the partial sample weights as possible, given that it estimates perfectly the known information, i.e.

$$\hat{t}_{C,y[b]} = g_C(y_i, x_i, w_i; i \in S_{[b]}) = \sum_{i \in S_{[b]}} d_{[b]i} y_i, \quad (3.3)$$

where

$\mathbf{d}_{[b]} = \arg\min_{\mathbf{v}: \sum_{k \in S_{[b]}} v_k x_k = t_x} \rho(\mathbf{v}, \mathbf{w}_{[b]})$ and

$\mathbf{w}_{[b]} = (w_i, i \in S_{[b]})$.

In particular, under this definition the Jackknife Variance Estimator,

$$\hat{V}_{0-Cal-Jack}(\hat{t}_{C,y}) = \frac{B-1}{B} \sum_{b=1}^{B} (\hat{t}_{C,y[b]} - \hat{t}_{C,y})^2,$$

is zero-calibrated automatically.

**Remark 3.1** Eq. 3.1 implies that if the given design is more efficient than the sampling with replacement (with the same selection probabilities) and if ebop's $g_C(y_i, x_i, w_i; i \in S_{[b]})$ can be approximated as $C_U + \sum_{k \in S_{[b]}} \frac{z_k}{\pi_k}$ with some $z_k = f(y_k, x_k, w_k)$ and some constant $C_U$ which depends only on the finite population then $E_D[\hat{V}_{0-Cal-Jack}(\hat{t}_{C,y})] \geq V(\hat{t}_{C,y})$ asymptotically. Illustrate the last statement by the following .

**Example 3.1** Consider the Generalized Regression Estimator (GREG), i.e. calibrated in the sense of Deville and Särndal (1992) estimator with respect to $x$ and $\rho(\mathbf{v}, \mathbf{w}_{[b]}) = \sum_{i \in S_{[b]}} (v_i - w_i)^2 / w_i$. This estimator can be written as,

$$\hat{t}_{C,y} = \sum_{i \in S} w_i y_i + (t_x - \sum_{i \in S} w_i x_i) B_{WLS},$$

where $B_{WLS} = [\sum_{i \in S} w_i x_i^2]^{-1} \sum_{i \in S} w_i x_i y_i$ is the weighted estimate of the census linear regression coefficient,

$$B_U = [\sum_{i \in U} x_i^2]^{-1} \sum_{i \in U} x_i y_i .$$

According to Eq. 3.3,

$$\hat{t}_{C,y[b]} = \sum_{i \in S_{[b]}} w_i y_i + (t_x - \sum_{i \in S_{[b]}} w_i x_i) B_{WLS_{[b]}} .$$

If the weighted estimates of the census linear regression coefficients on partial and complete samples respectively are approximately equal to $B_U$, $B_{WLS_{[b]}} \cong B_{WLS} \cong B_U$ and if $w_i = \pi_i^{-1}$, then

$$\hat{t}_{C,y[b]} \cong C_U + \sum_{k \in S_{[b]}} \frac{z_k}{\pi_k}$$

with $C_U = t_x B_U$ and $z_i = y_i + x_i B_U$.

Therefore in the case of GREG, under mild assumptions the suggested zero-calibrated Jackknife variance estimator does not underestimate the correct variance.

**Remark 3.2** Note that well-known Deville and Särndal (1992) variance estimator based on the residuals is also zero-calibrated.

**Remark 3.3** Following Woodruff (1971) one can easily generalize (at least asymptotically) the results of this and next sections to estimates of any finite population parameter that can be written as a function of finite population totals.

## 4. Zero-calibrated Balance Repeated Replication (BRR) Variance Estimators

In this section we assume that our sampling scheme is a 2-step sample where the first step is a Stratified Sample of two Primary Sampling Units (PSU). Let $S = (S_{11} \cup S_{12}) \cup ... \cup (S_{H1} \cup S_{H1})$ where the first sub-index denotes stratum and

$S_{h1}$ and $S_{h2}$ denote two PSU's selected from stratum $h$. Let $A$ be a multiple of 4 greater than $H+1$, $\varsigma_h^{(\alpha)}$, $h=1,...,H;\ \alpha=1,...,A$ be an orthogonal array, i.e. $\varsigma_h^{(\alpha)}$ takes values -1 and 1, $\sum_{\alpha=1}^{A}\varsigma_h^{(\alpha)}=0$ for any $h$ and $\sum_{\alpha=1}^{A}\varsigma_h^{(\alpha)}\varsigma_{h'}^{(\alpha)}=0$ for any $h\neq h'$. For $\alpha=1,...,A$ define a set of $A$ half samples as, $S^{(\alpha)}=$

$$\left(\bigcup_{h:\varsigma_h^{(\alpha)}=1} S_{h1}\right)\bigcup\left(\bigcup_{h:\varsigma_h^{(\alpha)}=-1} S_{h2}\right).$$

and define the estimate based on $\alpha$'s half sample by $\hat{t}_y^{(\alpha)}=g(y_i,x_i,w_i;i\in S^{(\alpha)})$. Then the BRR estimator of the variance of $\hat{t}_y$ can be defined as,

$$\hat{V}_{BRR}(\hat{t}_y)=\frac{1}{A}\sum_{j=1}^{A}(\hat{t}_y^{(\alpha)}-\hat{t}_y)^2,$$

(see also Wolter 1985 for alternative definitions of BRR variance estimators)**.**

By analogy with the Jackknife, the BRR variance estimator of an estimator calibrated in the sense of Deville and Särndal (1992) is,

$$\hat{V}_{0-cal-BRR}(\hat{t}_y)=\frac{1}{A}\sum_{\alpha=1}^{A}(\hat{t}_{Cal-BRR,y}^{(\alpha)}-\hat{t}_y)^2,$$

where $\hat{t}_{Cal-BRR,y}^{(\alpha)}=\sum_{i\in S^{(\alpha)}}d_i^{(\alpha)}y_i$,

$\mathbf{d}^{(\alpha)}=\underset{\mathbf{v}:\ \sum_{k\in S^{(\alpha)}}v_k x_k=t_x}{\arg\min}\ \rho(\mathbf{v},\mathbf{w}^{(\alpha)})$ and

$\mathbf{w}^{(\alpha)}=(w_i,i\in S^{(\alpha)})$.

Obviously the last estimator is zero-calibrated. On the other hand, since the BRR variance estimator tends to estimate the variance as if the sample were selected with replacement (see Wolter (1985) pp. 123-124) it, like the Jackknife, does not underestimate the correct variance.

## 5. Calibrated Estimators with respect to Sampling Designs

Some estimators are calibrated under particular designs. For example, under simple random sampling without replacement, $\hat{t}_{\pi,y}$ is calibrated with respect to a constant, $x=C$,

$$\hat{t}_{\pi,C}=\sum_{i\in S}C/\pi_i=\sum_{i\in S}\frac{C}{(n/N)}=N\times C.$$

The same is correct for the stratified sampling with proportional allocation, $\pi_i=nN_h/N$. Under many 2-phase designs, see Särndal, Swensson and Wretman (1992), section 9.3, weighted estimates of finite population totals are also self calibrated. One such design is the National Compensation Survey (NCS). The specifics of this survey is described in details in Guciardo et *al*. (2004). Here we consider a part of NCS, the "Remaining 99 primary sampling strata". For this part, after some simplifications, the sampling scheme can be defined as follows.

Data is collected by a 2-step procedure.

*Step 1*. The population (the "Remaining 99 primary sampling strata") is divided into PSU's grouped into *area strata*; one PSU is selected from each stratum with a probability proportional to the employment in the PSU.

*Step 2*. The selected sample, $S_{PSU}$, is regrouped into *industrial strata*, each industrial stratum is divided into Secondary Sampling Units (SSU), defined by the establishments in the stratum; for each industrial stratum SSU's are sampled by systematic probability sampling proportionally to the *employment level in SSU* $\times$ the *inverse of probability of selection of PSU containing the establishment*.

Denote by $w_i=\dfrac{1}{\pi_i^{PSU}\times\Pr(i\in S\mid S_{PSU})}$ the sample weight, where $\pi_i^{PSU}$ denotes the selection probability of the PSU containing the $i$'th SSU and $\Pr(i\in S\mid S_{PSU})$ denotes the selection probability of $i$'th SSU from $S_{PSU}$. Note that $w_i$ depends on the sample selected on the first step and therefore it is not equal to the inverse selection probability,

$$\pi_i^{-1} = \frac{1}{\pi_i^{PSU} \times E_D[\Pr(i \in S \mid S_{PSU})]}.$$

On the other hand, one can easily check that $\hat{t}_{w,y} = \sum_{i \in S} w_i y_i$ is a design unbiased estimate of $t_y$. Moreover, in Wang et *al.* (2004) it is shown that for this sampling scheme $\sum_{i \in S} w_i x_i = t_x$, where $x_i$ denotes the employment level in $i$'s SSU.

For this sampling scheme $\hat{t}_{w,y}$ can be easily written in a form of a calibrated in the sense of Deville and Särndal (1992) estimator; indeed if $\hat{t}_y = \sum_{i \in S} d_i y_i$, such that $\mathbf{d} = \arg\min_{\mathbf{v}: \sum_{k \in S} v_k x_k = t_x} \rho(\mathbf{v}, \mathbf{w})$ and $\mathbf{w} = (w_1, ..., w_n)$ then, for any sample (under this design) and any distance function $\rho$ (i.e. $\rho(\mathbf{a}, \mathbf{b}) \geq 0$ and $\rho(\mathbf{a}, \mathbf{b}) = 0$ if and only if $\mathbf{a} = \mathbf{b}$), $d_i = w_i$ automatically for all $i$.

Therefore for calibrated estimators with respect to a given design we suggest choosing a simple distance $\rho$, for example the uniform or the euclidean distance, and estimating the variance of the estimator as if it is calibrated estimator in the sense of Deville and Särndal (1992). Such an estimator does not underestimate the variance if the conditions of Remark 3.1 hold.

**Example 5.1** For simplicity let a distance function be, $\rho(\mathbf{v}, \mathbf{w}_{[j]}) = \sum_{i \in S_{[j]}} (v_i - w_i)^2 / w_i$, and let us check when $\hat{t}_{w,y}$ satisfies the conditions of Remark 3.1.

$$\hat{t}_{w,y} = \sum_{i \in S} w_i y_i + (t_x - \sum_{i \in S} w_i x_i) B_{WLS} =$$

$$t_x B_{WLS} + \sum_{i \in S} \pi_i^{-1} (y_i - x_i B_{WLS}) +$$

$$\sum_{i \in S} (w_i - \pi_i^{-1})(y_i - x_i B_{WLS}) \cong$$

$$C_U + \sum_{i \in S} \pi_i^{-1} z_i +$$

$$\sum_{i \in S} (w_i - \pi_i^{-1})(y_i - x_i B_{WLS}),$$

where $B_{WLS}$, $B_U$, $C_U$ and $z_i$ are defined in Example 3.1. Thus $\hat{t}_{w,y}$ satisfies the conditions of Remark 3.1 if: (a) the variance over given design distribution is not greater than the variance under with replacement sampling design; (b) the weighted estimates on partial and complete samples of the census linear regression coefficient are approximately equal to $B_U$; (c) $\sum_{i \in S} (w_i - \pi_i^{-1})(y_i - x_i B_S) \cong 0$. The latter is satisfied, for example, when the sample residuals, $y_i - x_i B_S$, do not correlate with the deviations of sample weights $w_i$ over the inverse selection probabilities.

**Remark 5.1** Sometimes, for calibrated estimators with respect to a given design, it can be easier and more effective, instead of using the distance function, to calibrate the estimate for each partial sample, for example, on the basis of sampling design information. Note that in the latter case it could be difficult to check the conditions of Remark 3.1 and therefore one can not guarantee that the estimator will not underestimate the correct variance.

## 6. Monte-Carlo simulation study

In order to illustrate the performance of zero-calibrated Jackknife variance estimators discussed in Section 3 we designed a simulation study using the sampling scheme described in Section 5. More complicated studies based on real NCS data are presented in Guciardo et *al.* (2004). The study was carried out as follows.

All pseudo random variables below are generated independently of each other.

Steps to generate the finite population:

(1) let $I = 30$ be the number of areas and $J = 6$ be the number of industries;

(2) for each area $i$ generate the number of PSU's in the area, $L_i$, from U[2,6], where U[a,b] denotes the Uniform distribution on {a, a+1,…,b};

(3) generate the number of establishments in each PSU $l_i$ and industry $j$, $K_{l_i j}$, from U[5,30];

(4) generate the "employment level" for each establishment, $x_{ijlk}$, $i=1,...,I$, $J=1,...,J$, $l=1,...,L_i$, $k=1,...,K_{l_i j}$, from the Gamma distribution with shape parameter $a=10$ and scale parameter $b=0.2$, so that the mean is $a/b=50$ and the variance is $a/b^2=250$;

(4) generate an auxiliary variable, $z_{ijlk}$, $i=1,...,I$, $J=1,...,J$, $l=1,...,L_i$, $k=1,...,K_{l_i j}$, from the Gamma distribution with shape parameter $a=10$ and scale parameter $b=1$, and define 4 target variables:

$y_{ijlk}(1)=x_{ijlk}$,

$y_{ijlk}(2)=z_{ijlk}$,

$y_{ijlk}(3)=z_{ijlk}+x_{ijlk}$,

$y_{ijlk}(4)=z_{ijlk}^{2/3}+x_{ijlk}^{1/2}$.

After the population was obtained, 10,000 samples, $S(1),...,S(10000)$, were drawn from the population independently. In each run, a sample of size $n=120$ was generated using the sampling scheme described in Section 5: on the *first step* one PSU was selected from each area with a probability proportional to the employment in the PSU; on the *second step* the selected sample, $S_{PSU}$, was regrouped into 6 industrial strata (by $j$), from each industrial stratum $n_j=20$ establishments were selected using systematic probability sampling proportionally to the *employment level in the establishment,* $x_{ijlk}$, multiplied by the *inverse probability of selection of PSU containing the establishment,* $\pi_i^{PSU}$.

For each sample $S(a)$, $a=1,...,10000$, and each target variable $y_{ijlk}(q)$, $q=1,...,4$, we calculate

$$\hat{t}_{w,y}(a,q)=\sum_{(i,j,l,k)\in S(a)} w_{ijlk}(a)y_{ijlk}(q),$$

where

$$w_{ijlk}(a)=\left(\frac{n_j\alpha_{ijlk}}{\sum_{(r,t):l_r\in S_{PSU}(a)}\alpha_{rjll_rt}}\right)^{-1},$$

$S_{PSU}(a)$ is a set of establishments selected on the first step, $\alpha_{ijl_ik}=\pi_{il_i}^{-1}x_{ijl_ik}$ and $\pi_{il}=\frac{x_{i.l.}}{\sum_u x_{i.u.}}$ is the ratio of employment level in the $l$'th PSU and the area stratum $i$ to employment level in the stratum $i$; see Wang et *al.* (2004) for the details. In addition, for the first 1,000 samples we calculate two Jackknife variance estimates for each $\hat{t}_{w,y}(a,q)$. Both estimates are based on partial samples defined by the areas, i.e. $b$'th partial sample, $S_{[b]}(a)$, $b=1,...,B=I$, is equal to the initial sample, $S(a)$, without units that belong to area $b$. The first (uncalibrated) Jackknife variance estimator is,

$$\hat{V}_{Jack}(\hat{t}_{w,y}(a,q))=\frac{B-1}{B}\sum_{b=1}^{B}(\hat{t}_{w,y[b]}(a,q)-\hat{t}_{w,y}(a,q))^2,$$

with $\hat{t}_{w,y[b]}(a,q)=\sum_{(i,j,l,k)\in S_{[b]}(a)} w_{ijlk}(a)y_{ijlk}(q)$.

The second (zero-calibrated) Jackknife estimator is,

$$\hat{V}_{0-Cal-Jack}(\hat{t}_{w,y}(a,q))=\frac{B-1}{B}\sum_{b=1}^{B}(\hat{t}_{C,y[b]}(a,q)-\hat{t}_{w,y}(a,q))^2,$$

with $\hat{t}_{C,y[b]}(a,q)=\sum_{(i,j,l,k)\in S_{[b]}(a)} v_{ijlk[b]}(a)y_{ijlk}(q)$

and $v_{ijlk[b]}(a)=$

$$= \left( \frac{\sum_{i} x_{i...}}{\sum_{i \neq b} x_{i...}} \times \frac{n_j \alpha_{ijlk}}{\sum_{(r,t) \in S_{PSU[b]}(a)} \alpha_{rjl_r t}} \right)^{-1},$$

where $x_{i...}$ is the employment level in the $i$'th area stratum.

Finally, for each $q = 1,...,4$, we calculate the following empirical statistics:

(a) the square root of the empirical variance of $\hat{t}_{w,y}(a,q)$ over 10,000 simulated samples,

$$\sqrt{V_{emp}(q)} = \sqrt{\frac{1}{10000} \sum_{a=1}^{10000} [\hat{t}_{w,y}(a,q) - \bar{t}(q)]^2},$$

where $\bar{t}(q) = \frac{1}{10000} \sum_{a=1}^{10000} \hat{t}_{w,y}(a,q)$;

(b) the square root of the empirical average of the uncalibrated Jackknife variance estimator over the first 1,000 samples,

$$\sqrt{E_{emp}\{\hat{V}_{Jack}(q)\}} = \sqrt{\frac{1}{1000} \sum_{a=1}^{1000} \hat{V}_{Jack}(\hat{t}_{w,y}(a,q))};$$

(c) the square root of the empirical average of the zero-calibrated Jackknife variance estimator over the first 1,000 simulated samples,

$$\sqrt{E_{emp}\{\hat{V}_{0-Cal-Jack}(q)\}} = \sqrt{\frac{1}{1000} \sum_{a=1}^{1000} \hat{V}_{0-Cal-Jack}(\hat{t}_{w,y}(a,q))}.$$

The values of these statistics (with the standard errors for the last two in the brackets) are shown in Table 1.

The conclusions from this limited simulation study are as follows:

- obviously for the estimate of population total of the calibration variable, the empirical variance and the empirical mean of the zero-calibrated variance estimator are equal to 0, whereas the estimator that ignores the calibration produces huge values for the variance estimate;

- the second column illustrate the other extreme situation, when the target variable is independent of the calibration variable. We expected insignificant or slightly significant overestimation for the zero-calibrated estimator. On the contrary, in our study the uncalibrated estimator produces small but statistically significant overestimation. One possible reason for this is that the uncalibrated estimator tends to estimate the variance as if the design is with replacement, when zero-calibrated one accounts partly for the sampling structure by calibrating its ebop's;

- the last two columns, where the target variables are functions of the calibration variable and the "noise", show that the zero-calibrated estimator gives correct estimate of the variance when the estimator that ignores the calibration hugely overestimates the correct variance.

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**Table 1.**

| | $q=1$ | $q=2$ | $q=3$ | $q=4$ |
|---|---|---|---|---|
| $\sqrt{V_{emp}(q)}$ | 0 | 5416 | 5416 | 3231 |
| $\sqrt{E_{emp}\{\hat{V}_{Jack}(q)\}}$ (Standard Error) | 7154 (73) | 5687 (22) | 10245 (75) | 3711 (13) |
| $\sqrt{E_{emp}\{\hat{V}_{0-Cal-Jack}(q)\}}$ (Standard Error) | 0 (0) | 5440 (33) | 5440 (33) | 3264 (19) |