Rational Approximation Using Farey Sequence : Review

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***Abstract*:** **Farey sequence is a part of Number theory, it has always made everyone attracted by its pattern. As interesting as its pattern it has interesting results and applications also. In this article, some results related to Farey Sequence, rational approximation of real number by Hurwitz theorem with the help of Farey sequence and use of approximation of rational numbers in clock making by Stern Brocot tree are to be discussed.**

***Keywords*:** **Farey Sequence, Mediant, Hurwitz theorem, Stern Brocot tree**

I. Introduction

The Farey sequence has crucial values in different and advanced branches of number theory. The Farey sequence of order n is the sequence of all reduced fractions between 0 and 1 with denominator less than or equal to n, arranged in order of increasing size [3]. The Farey Sequence, sometimes called the Farey series, is a series of sequences in which, each sequence consists of rational numbers ranging in value from 0 to 1 [7]. Origin of the name of Farey sequence from the list of simple or vulgar fraction is quite interesting. The sequence is named after John Farey, who thought that he was the first person to note the properties of this sequence. He note some results of sequences of rational numbers.

Farey published many articles between 1804 to1824 in different magazines like Rees's Encyclopedia, The Monthly Magazine, and Philosophical Magazine [7]. But article on the curious property of vulgar fractions was similar to previously discovered works. This article included of four criterias. The first notes the curious property. The second included definition and statement on the Farey sequence. In the third he gave an example of 5th order of Farey sequence. In the end, he wrote, "I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of some easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers ..."[7]

John Farey's note was then republished in the French magazine "Bulletin de la Soci´et´e Philomatique" [10]. One of the readers of Farey's article, Augustin-Louis Cauchy, provided the proof that mediant property holds (crediting John Farey) in one his writings the same year Farey's article was released, and since it was believed that Farey was the first to notice this property the sequence was named after him [7]. But in 1802, Charles Haros explained the construction of the 99 th Farey sequence using mediant property and also provided some important results related to sequence [3]. For these reasons Farey was not looked fondly upon in the mathematical community, in G H Hardy's, A mathematicians apology, he writes, "... Farey is immortal because he failed to understand a theorem which Haros had proved perfectly fourteen years before ..." [11].

Pattern of Farey Sequence can be visualized geometrically with the help of Ford circles which is named on L.R. Ford. Ford circle is defined as: For every proper fraction $\dfrac{a}{b}$ where $\gcd(a,b)=1$, there exists a Ford Circle $C_{\frac{a}{b}}$ which is tangent to the x axis at the point $\left(\dfrac{a}{b},0\right)$ with $centre\left(\dfrac{a}{b},\dfrac{1}{2b^2}\right)$ and

$radius = \dfrac{1}{2b^2}$ [8]. It helps to represent the concept of mediant and the patterns associated with Farey fractions [8].

Stern Brocot tree was invented independently but it has similar properties as of Farey sequence which is formed by the mediant property of the fractions. Farey sequence is between *0* to *1* but Stern Brocot tree is from 0 to infinity [3].

II. Farey Sequence and Results

Definition 2.1: [1] Farey sequence of order *n*

A Sequence of all proper fraction and reduced fraction between 0 to 1 arranged in increasing order of size with the denominator less than or equal to *n* is said to be a Farey sequence of order *n*.

Farey sequence of order *n* is denoted by F_n.

For example:

Farey sequence of order 1 : $\frac{0}{1}$ and $\frac{1}{1}$.

Farey sequence of order 2 : $\frac{0}{1}, \frac{1}{2}$ and $\frac{1}{1}$.

Farey sequence of order 3 : $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $\frac{1}{1}$.

Farey sequence of order 4 : $\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}$ and $\frac{1}{1}$.

Table 1: Tabular form of Farey sequence is as follows [1]:

$\frac{0}{1}$										$\frac{1}{1}$
$\frac{0}{1}$					$\frac{1}{2}$					$\frac{1}{1}$
$\frac{0}{1}$			$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$			$\frac{1}{1}$
$\frac{0}{1}$		$\frac{1}{4}$	$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$	$\frac{3}{4}$		$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{1}$

Definition 2.2: [1] Mediant property of Farey sequence

If $\frac{a}{b}$ and $\frac{c}{d}$ are any two consecutive fractions in n^{th} order of Farey sequence then $\frac{e}{f} = \frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}$ in between $\frac{a}{b}$ and $\frac{c}{d}$ if $b+d \le n+1$ in $(n+1)^{th}$ order of Farey sequence [1].

Theorem 2.1: [1] If $\frac{a}{b}$ and $\frac{c}{d}$ are simultaneous fractions then $|bc - ad| = 1$.

Proof : This result can be proved by induction.

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For $k = 1$, $1(1)-0(1)=1$. So, it is true for $k = 1$.

Let $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fraction in Farey sequence of order $k = n - 1$ is true so, $|bc - ad| = 1$.

Now there are two possibilities for $k = n$, either $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fraction or $\frac{a}{b}, \frac{a+c}{b+d}$ and $\frac{c}{d}$ are consecutive fraction.

If $\frac{a}{b}, \frac{a+c}{b+d}$ and $\frac{c}{d}$ are consecutive fraction then,

$\therefore b(a+c)-a(b+d)=ba+bc-ab-ad=bc-ad=1$

or $(b+d)c-(a+c)d=bc+dc-ad-cd=bc-ad=1$

Hence proved.

Corollary 2.1: [1] Every fraction $\frac{a}{b}$ in table 1 is in reduced form. In other words $\gcd(a,b)=1$.

Proof: Let $n \in N$ and $\frac{a}{b}$ be a fraction in the n^{th} row.

To prove this result by using diophantine equation, $qx-py=1$ which has the solution if and only if $\gcd(p,q)=1$,

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the two consecutive fractions in n^{th} row.

By the result, $|bc-ad|=1$.

Comparing with the diophantine equation,

$\gcd(a,b)=1$ or $\gcd(c,d)=1$.

Theorem 2.2: [1] Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ be any three consecutive fractions in the Farey sequence of order n. Prove that $\frac{c}{d}=\frac{a+e}{b+f}$.

Proof: Using mathematical induction,

It is true for $n=1$ and $n=2$.

Assume the statement is true for n = k and let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ be any three consecutive fractions in the Farey sequence of order $k+1$.

If $\frac{c}{d}$ is in the Farey sequence of order $k+1$ but was not in the Farey sequence of order k, then by construction of the Farey sequence, have $\frac{c}{d}=\frac{a+e}{b+f}$. So now assume $\frac{c}{d}$ is also in the Farey sequence of order k.

Let $\frac{p}{q}, \frac{c}{d}, \frac{r}{s}$ be consecutive fractions in the Farey sequence of order k. There are four possibilities now:

(1) $\frac{a}{b} = \frac{p}{q}$ and $\frac{e}{f} = \frac{r}{s}$

(2) $\frac{a}{b} = \frac{c+p}{d+q}$ and $\frac{e}{f} = \frac{r}{s}$

(3) $\frac{a}{b} = \frac{c+p}{d+q}$ and $\frac{e}{f} = \frac{c+r}{d+s}$

(4) $\frac{a}{b} = \frac{p}{q}$ and $\frac{e}{f} = \frac{c+r}{d+s}$.

Let $\frac{c}{d} = \lambda$.

By the induction hypothesis, $\lambda = \frac{p+r}{q+s}$

For case (1), it is done by the induction hypothesis.

For case (2), $c+p+r = \lambda q + \lambda(q+s) = \lambda(d+q+s)$.

For case (3),

$c+p+c+r = 2c+(p+r) = 2\lambda d + \lambda(q+s) = \lambda(2d+(q+s)) = \lambda(d+q+d+s)$.

For case (4), there is a simmilar condition as in case (2), so it is done.

For proving the edge cases, when the middle Farey fraction is an integer, it succes to show that in the Farey sequence of order n, the term to the immediate right of $\frac{0}{1}$, say $\frac{g}{h}$ when added to the term to the immediate left of $\frac{1}{1}$ gives 1.

This can also be proved by induction on n. This is clear for $n=1$ and $n=2$. Assume that for $n=k$ result is true and let $\frac{g}{h}$ be the term to the immediate right of $\frac{0}{1}$ in the Farey sequence of order k.

Then the term to the immediate left of $\frac{1}{1}$ is $\frac{h-g}{h}$ by the induction hypothesis. If $1+f > k+1$, then the terms to the immediate right of $\frac{0}{1}$ and to the immediate left of $\frac{1}{1}$ remain unchanged from order n to order $n+1$.

If $1+f \le k+1$, then in the Farey sequence of order $k+1$, the term to the immediate right of $\frac{0}{1}$ becomes $\frac{g}{h+1}$ and the term to the immediate left of $\frac{1}{1}$ becomes $\frac{h-g+1}{h+1} = \frac{(h+1)-g}{h+1} = 1 - \frac{g}{h+1}$ and the proof is done by induction.

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Results 2.1

1. Let $\frac{a}{b}, \frac{c}{d}$ be any consecutive fractions in the Farey sequence of order n then $b+d \geq n+1$ [1].
2. If $0 \leq a \leq b, \gcd(a,b)=1$, then the fraction $\frac{a}{b}$ appears in the b^{th} and all later rows.[1]
3. Let $\frac{a}{b}, \frac{c}{d}$ be any consecutive fractions of $\frac{1}{2}$ in the Farey sequence of order n then $b=d=1+2\left(\frac{n-1}{2}\right)$, b is the greatest odd integer $\leq n$ and $a+c=b$ [1].
4. Farey length: $|F_n|=|F_{n-1}|+\phi(n)$ [2].
5. Asymptotic behaviour of Farey sequence is : $|F_n|=\frac{3n^2}{\pi^2}$ [2].

By using above results of Farey sequence another properties can be proved which will eventually lead us to a basic theorems about rational approximation.

Theorem 2.3: [1] If $\frac{a}{b}, \frac{c}{d}$ are Farey fractions of order n such that no other Farey fraction of order n lies between them, then $\left|\frac{a}{b}-\frac{a+c}{b+d}\right|=\frac{1}{b(b+d)} \leq \frac{1}{b(n+1)}$ and $\left|\frac{c}{d}-\frac{a+c}{b+d}\right|=\frac{1}{d(b+d)} \leq \frac{1}{d(n+1)}$.

Proof : By using the result 2.1 (1), If $\frac{a}{b}, \frac{c}{d}$ are consecutive then $|cb-ad|=1$ and $b+d \geq n+1$,

$$\left|\frac{a}{b}-\frac{a+c}{b+d}\right|=\frac{|ad-bc|}{b(b+d)}$$

$$\therefore \left|\frac{a}{b}-\frac{a+c}{b+d}\right| \leq \frac{1}{b(n+1)}$$

$$\therefore \left|\frac{a}{b}-\frac{a+c}{b+d}\right| \leq \frac{1}{b(b+d)}$$

Similarly, $\left|\frac{c}{d}-\frac{a+c}{b+d}\right|=\frac{1}{d(b+d)} \leq \frac{1}{d(n+1)}$.

Theorem 2.4: [1] If n is a positive integer and x is a real, there is a rational number $\frac{a}{b}$ such that $0<b \leq n$ and $\left|x-\frac{a}{b}\right| \leq \frac{1}{b(n+1)}$.

Proof : For some Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$, the number x will lie between or on, and so by interchanging $\frac{a}{b}$ and $\frac{c}{d}$ if necessary, x lies in the closed interval between $\frac{a}{b}$ and $\frac{a+c}{b+d}$.

$$\therefore \left|x-\frac{a}{b}\right| \le \left|\frac{c}{d}-\frac{a+c}{b+d}\right|$$

By Theorem 2.3 , If $\frac{a}{b}$ and $\frac{c}{d}$ are Farey fractions of order n such that no other Farey fraction of order n lies between them, then $\left|\frac{a}{b}-\frac{a+c}{b+d}\right| = \frac{1}{b(b+d)} \le \frac{1}{b(n+1)}$ and $\left|\frac{c}{d}-\frac{a+c}{b+d}\right| = \frac{1}{d(b+d)} \le \frac{1}{d(n+1)}$.

$$\therefore \left|x-\frac{a}{b}\right| \le \left|\frac{c}{d}-\frac{a+c}{b+d}\right| \le \frac{1}{d(n+1)}.$$

Theorem 2.5: [1] If ξ is a real and irrational, there are infinitely many distinct rational numbers $\frac{a}{b}$ such that $\left|x-\frac{a}{b}\right| < \frac{1}{b^2}$.

Proof: For each $= 1,2,3...$ the value of a_n and b_n can be obtained. If n is a positive integer and ξ is a real, there is a rational number $\frac{a}{b}$ such that $0 < b \le n$ and $\left|x-\frac{a}{b}\right| \le \frac{1}{b(n+1)}$, such that $0 < b_n \le n$ and

$$\left|\xi-\frac{a}{b}\right| \le \frac{1}{b_n(n+1)} < \frac{1}{{b_n}^2}.$$

Many of the $\frac{a_n}{b_n}$ may be equal to each other, but there will be infinitely many distinct ones. For, if there were not infinitely many distinct ones, there would be only a finite number of distinct values taken by $\left|\xi-\frac{a_n}{b_n}\right|$ for some n, say $n = k$.

Here $|\xi - \frac{a_n}{b_n}| \ge |\xi - \frac{a_k}{b_k}|$ for all $n = 1,2,3...$ but $\left|\xi-\frac{a_n}{b_n}\right| > 0$ since ξ is irrational, and so sufficiently large n can be obtained such that, $\left|\frac{1}{n+1}\right| < \left|\xi-\frac{a_k}{b_k}\right|$.

This leads to contradiction so ,

$$\left|\xi-\frac{a_k}{b_k}\right| \le \left|\xi-\frac{a_n}{b_n}\right| \le \frac{1}{b_n(n+1)} \le \left|\frac{1}{n+1}\right| < \left|\xi-\frac{a_k}{b_k}\right|.$$

The condition that ξ be irrational is necessary in the theorem .

For if x is any rational number, say, $x = \frac{r}{s}, s > 0$.

If $\frac{a}{b}$ is any fraction such that $\frac{a}{b} \ne \frac{r}{s}, b > s$, then

$$\left|\frac{r}{s}-\frac{a}{b}\right|=\frac{|rb-as|}{sb}\geq\frac{1}{sb}>\frac{1}{b^2}.$$

Hence, all the fraction $\frac{a}{b}$, $b>0$, satisfying $\left|x-\frac{a}{b}\right|<\frac{1}{b^2}$ have denominators $b\leq s$, and there can only be a finite number of such fractions.

Corollary 2.2: [1] If x and y are positive integers then not both of the inequalities $\frac{1}{xy}\geq\frac{1}{5}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$ and $\frac{1}{x(x+y)}\geq\frac{1}{5}\left(\frac{1}{x^2}+\frac{1}{(x+y)^2}\right)$ can hold.

Proof : The two inequalities can be written as,

$\sqrt{5}\geq y^2+x^2, \sqrt{5}x(x+y)\geq(x+y)^2+x^2$.

Adding these inequalities, $\sqrt{5}(2xy+x^2)\geq 3x^2+2xy+y^2$.

So, $2y^2-2(\sqrt{5}-1)xy+(3-\sqrt{5})x^2\leq 0$.

Now multiplying this by 2 ,

$4y^2-4(\sqrt{5}-1)xy+(5-2\sqrt{5}+1)x^2\leq 0, (2y-(\sqrt{5}-1)x)^2\leq 0$.

This is impossible for positive integers x and y because $\sqrt{5}$ is irrational.

Hurwitz Theorem 2.6: [1] [2] Given any irrational number ξ, there are infinitely many different rational numbers $\frac{h}{k}$ such that $\left|\xi-\frac{h}{k}\right|<\frac{1}{\sqrt{5}k^2}$.

Proof: Let n be a positive integer. There exist any two consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ in the Farey sequence of order n, such that $\frac{a}{b}<\xi<\frac{c}{d}$.

To show at least one of the three fractions $\frac{a}{b},\frac{c}{d},\frac{a+c}{b+d}$ can serve as $\frac{h}{k}$.

Suppose this is not so either $\xi<\frac{a+c}{b+d}$ or $\xi>\frac{a+c}{b+d}$.

If $\xi<\frac{a+c}{b+d}$,

$$\xi-\frac{a}{b}\geq\frac{1}{b^2\sqrt{5}};<\frac{a+c}{b+d}-\xi\geq\frac{1}{(b+d)^2\sqrt{5}};\frac{c}{d}-\xi\geq\frac{1}{d^2\sqrt{5}}.$$

Adding inequalities, $\frac{c}{d}-\frac{a}{b}\geq\frac{1}{b^2\sqrt{5}}+\frac{1}{d^2\sqrt{5}};\frac{a+c}{b+d}-\frac{a}{b}\geq\frac{1}{(b+d)^2\sqrt{5}}+\frac{1}{b^2\sqrt{5}}$.

Hence, $\frac{1}{bd}=\frac{cb-ad}{bd}=\frac{c}{d}-\frac{a}{b}\geq\frac{1}{\sqrt{5}}\left(\frac{1}{b^2}+\frac{1}{d^2}\right)$ and

$$\frac{1}{b(b+d)}=\frac{(a+c)b-a(b+d)}{b(b+d)}\geq\frac{1}{\sqrt{5}}\left(\frac{1}{b^2}+\frac{1}{(b+d)^2}\right).$$

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These two inequalities contradict Corollary 2.2 . So one of $\frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d}$ can serve as $\frac{h}{k}$.

If $\xi > \frac{a+c}{b+d}$,

$$\xi - \frac{a}{b} \geq \frac{1}{b^2\sqrt{5}}; \xi - \frac{a+c}{b+d} \geq \frac{1}{(b+d)^2\sqrt{5}}; \frac{c}{d} - \xi \geq \frac{1}{d^2\sqrt{5}}.$$

Adding above inequalities, $\frac{c}{d} - \frac{a}{b} \geq \frac{1}{b^2\sqrt{5}} + \frac{1}{d^2\sqrt{5}}; \frac{c}{d} - \frac{a+c}{b+d} \geq \frac{1}{(b+d)^2\sqrt{5}} + \frac{1}{d^2\sqrt{5}}$.

Hence, $\frac{1}{bd} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2} + \frac{1}{d^2}\right)$ and $\frac{1}{d(b+d)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2} + \frac{1}{(b+d)^2}\right)$.

Again with same contradiction to Corollary 2.2.

Let $\frac{a}{b} < \xi < \frac{c}{d}$, by Theorem 2.4 ,

$$\therefore \left|\xi - \frac{h}{k}\right| < \left|\frac{c}{d} - \frac{a}{b}\right| = \left|\frac{c}{d} - \frac{a+c}{b+d}\right| + \left|\frac{a+c}{b+d} - \frac{a}{b}\right| \leq \frac{1}{d(n+1)} + \frac{1}{b(n+1)} \leq \frac{2}{n+1} \leq \frac{2}{n}.$$

To show there are infinitely many $\frac{h}{k}$, suppose $\frac{x}{y}$ which satisfies this condition then, $\left|\xi - \frac{x}{y}\right|$ is positive and $n > \frac{2}{\left|\xi - \frac{x}{y}\right|}$.

$$\therefore \left|\xi - \frac{h}{k}\right| \leq \frac{2}{n+1} < \left|\xi - \frac{x}{y}\right|.$$

Hence, there are infinitely many $\frac{h}{k}$ such that for any rational numbers there exist another which is closer to ξ .

Corollary 2.3: [1] [2] The constant $\sqrt{5}$ in the Hurwitz theorem is the best possible or The theorem does not hold if $\sqrt{5}$ is replaced by any larger value.

Proof : To show that result does not hold if $\sqrt{5}$ is replaced by any larger value find any ξ.

Let us take $\xi = \frac{1+\sqrt{5}}{2}$. (1)

$$(x-\xi)\left(x - \frac{1-\sqrt{5}}{2}\right) = x^2 - x - 1 \quad (2)$$

For integers p, q with $q > 0$,

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then, $\left|\frac{p}{q}-\xi\right|\left|\frac{p}{q}-\xi+\sqrt{5}\right|=\left|\left(\frac{p}{q}-\xi\right)\left(\frac{p}{q}-\frac{1-\sqrt{5}}{2}\right)\right|$.

$\therefore \left|\frac{p}{q}-\xi\right|\left|\frac{p}{q}-\xi+\sqrt{5}\right|=\left|\frac{p^2}{q^2}-\frac{p}{q}-1\right|$.

$$\therefore \left|\frac{p}{q}-\xi\right|\left|\frac{p}{q}-\xi+\sqrt{5}\right|=\frac{1}{q^2}|p^2-pq-q^2|. \quad (3)$$

Since both ξ and $\sqrt{5}-\xi$ are irrational the LHS of above is not zero.

$$\text{As } |p^2-pq-q^2|\geq 1,\ \left|\frac{p}{q}-\xi\right|\left|\frac{p}{q}-\xi+\sqrt{5}\right|\geq\frac{1}{q^2}. \quad (4)$$

Now, suppose an infinite sequence of rational numbers $\frac{p_j}{q_j}, q_j>0$, and the positive real number m

$$\text{such that } \left|\frac{p}{q}-\xi\right|<\frac{1}{mq_j^{\,2}}. \quad (5)$$

Then $q_j\xi-\frac{1}{mq_j}<p_j<q_j\xi+\frac{1}{mq_j}$ and this shows that there are only a finite number of p_j corresponding to each value of q_j .

Therefore, $q_j\to\infty$ as $j\to\infty$.

Also, by (4), (5) and the triangle inequality

$\frac{1}{q_j^{\,2}}\leq\left|\frac{p}{q}-\xi\right|\left|\frac{p}{q}-\xi+\sqrt{5}\right|<\frac{1}{mq_j^{\,2}}\left(\frac{1}{mq_j^{\,2}}+\sqrt{5}\right)$.

So, $m<\frac{1}{mq_j^{\,2}+\sqrt{5}}$.

$\therefore m<\lim_{j\to\infty}\frac{1}{mq_j^{\,2}+\sqrt{5}}=\sqrt{5}$

Hence proved.

III. Stern Brocot Tree

Stern Brocot tree is an extension of a Farey sequence on real line to infinity. The construction of the Stern-Brocot tree begins with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$. It is useful to think of here $\frac{1}{0}$ as representing infinity. Now, adding the terms by the mediant property $\frac{1}{1}$ is obtained. For the next step of

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tree add $\frac{1}{0}$ and $\frac{1}{1}$ to get $\frac{1}{2}$; And add $\frac{1}{1}$ and $\frac{1}{0}$ to get $\frac{2}{1}$. In similar way proceed by adding the terms to get tree which is shown in Fig. 3.1 .[1]

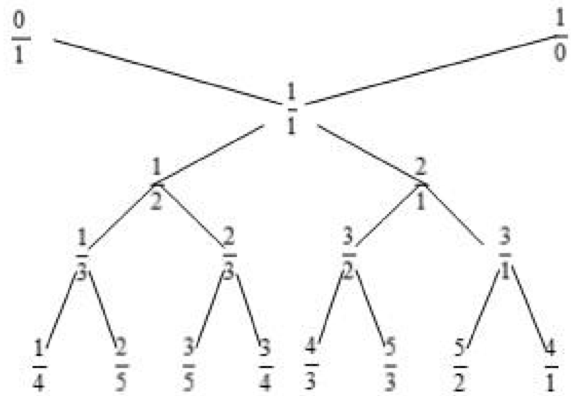

Fig. 3.1 Stern Brocot tree.

Properties3.1

1. Every positive rational number appears in the Stern-Brocot tree [5].
2. A rational number appears in the Stern-Brocot tree only one time.
   This is because the rationals added to the tree are always between consecutive numbers that are already in the Stern-Brocot tree [5].
3. The rational numbers always appear in simplest form [5].
4. By using Stern Brocot tree one can represent any real number x by a string, either finite or infinite, of *L's* and *R's*.

Begin with $\frac{1}{1}$, check wether the number is on the left or right of $\frac{1}{1}$. Mark L for left and R for right . Repeat the process and mark L or R. A finite string for rational number and infinite string for a irrational number is obtained [5].

Example 3.1: $e = 2.7182818...$

Table 2: *L's* and *R's* string for *e*.

R	*2/1*	*2.0000000...*
RR	*3/1*	*3.0000000...*
RRL	*5/2*	*2.5000000...*
RRLR	*8/3*	*2.6666666...*
RRLRR	*11/4*	*2.7500000...*
RRLRRL	*19/7*	*2.7142857...*
...	...	...
RRLRRLRLLLL	*106/39*	*2.7179487...*
...	...	...

So, $e = RRLRRLRLLLL...$. In this manner, any ratioal or irrational number can represented in form of L and R stirng. This will help in approximation of rational number in clock making.

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IV. CLOCK MAKING USING STERN BROCOT TREE

There are many parts in any clock but here the main focus is on the formation of gears and shafts which are use to speed up or slow down the motion of the wheels.

Suppose a small wheel with *10* teeth and bigger wheel with *50* teeth then for one rotation of bigger wheel the smaller one have to rotate *5* times. So the ratio between the two wheels is $\frac{1}{5}$. In similar manner a gear train is obtaied by factorizing the number. For example, if the ratio of $\frac{6}{25}$ then we a gear train can be made which forms the same ratio by $\frac{2\times 3}{5\times 5}$.

So, to make clock for a tropical year which has *365* days, *5* hours and *49* minutes and other hand which completes one rotation per day, by converting into minutes it has a ratio of $\frac{720}{525949}$, but here the problem arises that it has no factors, the denominator number is prime [5]. Which means that without any factor the number of teeth of wheel cannot be reduced by gear train. To understand the solution to this problem let us take an example which shows to approximate the rational numbers by using Stern Brocot tree [5].

Approximate $\frac{13}{47}$ with $\frac{p}{q}$, which says that constructing a small wheel with p teeth on the shaft and letting it turn a wheel with q teeth. The small wheel makes a revolution every *13* minutes. This means that the wheel makes a revolution every $\frac{13q}{p}$ minutes.

So the error obtained is : $\frac{13q}{p} - 47 = \frac{13q - 47p}{p}$.

Let the number be $\frac{13}{47}$ and it lies between $\frac{1}{4}$ and $\frac{1}{3}$. Put values from top if value is positive and from bottom if value is negative as shown in table. Adding the latest value from the last columns of p and q to the top columns find the value of r and repeate the process till value of r equals to *0* that is same as finding *L's* and *R's* string step by step in Stern Brocot tree.

Table 3: As the value is negative insert at the bottom of the table.

p	**q**	**r =13q-47p**
1	*3*	*-8*

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Table 4: As the value is positive insert at the top of the table .

p	q	r =13q-47p
1	4	+5
1	3	-8

Table 5: Insert the values in table at top and bottom accordingly for the positive and negative values.

p	q	r = 13q-47p
1	4	+5
3	11	+2
2	7	-3
1	3	-8

Table 6: Final calculation for approximation.

p	q	r =13q-47p
1	4	+5
3	11	+2
8	29	+1
13	47	0
5	18	-1
2	7	-3
1	3	-8

Here table shows that $\frac{8}{29}$ and $\frac{5}{18}$ are the closest approximation. Hence $\frac{13}{47}$ can be approximated by $\frac{8}{29}$ with error of $\frac{1}{8}$ more or $\frac{5}{18}$ where it is $\frac{1}{5}$ minute less.

Now calculating approximation for a tropical year.

As from above example, approximating $\frac{720}{525949}$ with $\frac{p}{q}$,

the obtaied error is : $\frac{q}{p}720 - 525949 = \frac{720q - 525949p}{p}$.

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Table 7: Calculation for approximation of tropical year clock.

p	**q**	**720q-525949p**
0	1	720
1	0	- 525949
1	1	-525229
1	2	-524509
...	...	...
...	...	...
163	119069	-7
196	143175	-4

So, $\frac{196}{143175}$ can be approximated for $\frac{720}{525949}$. Hence a clock with 4 stage gear train can be made by this approximated value $\frac{196}{143175} = \frac{3\times 25\times 23\times 83}{2\times 2\times 7\times 7}$ [4].

And the error in seconds can be calcuated by :

$$60\left(\frac{196}{143175} - \frac{720}{525949}\right) = 1.2244...$$

The last wheel completes a rotation in $\frac{4}{196}$ minutes less than a tropical mean year and is almost 1.22 second less over one tropical year which shows it is a quite good approximation. So with the help of Stern-Brocot tree and the mediant property of the Farey sequence a better rational approximation of gear ratios can be obtained in the clock making [4].

ACKNOWLEDGEMENT

The authors thank DST-FIST file no. MSI-097 for techical support to department of Mathematics, Gujarat University.

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