$k = 2$, X is a 2-cell and $f \epsilon C$, then the set of all those y for which $S(f, y) < M(f, y)$ is countable. However, if $k > 2$, this inequality may hold on a set of positive measure; for instance f may wrap a solid cube several times around an arc of positive volume. Hence the results of the preceding section indicate that M is more suitable than S for the theory of area.

[1] Here Bdry A refers to the relative topology of X.

[2] Cesari, L., "Una ugualianza fondamentale per l'area delle superficie," *Atti della Reale Accademia d'Italia, Memorie*, **14**, 891–951 (1944).

[3] Rado, T., and Reichelderfer, P. V., "A Theory of Absolutely Continuous Transformations in the Plane," *Trans. Am. Math. Soc.*, **49**, 258–307 (1941).

[4] Federer, H., "Coincidence Functions and Their Integrals," *Ibid.*, **59**, 441–466 (1946).

[5] Cesari, L., "Sui punti di diramazione delle trasformazioni continue e sull' area delle superficie in forma parametrica," *Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl.*, ser. 5, **3**, 37–62 (1942).

SOME NEW RESULTS ON PARTITIONS

By Nathan J. Fine

Department of Mathematics, University of Pennsylvania

Communicated by J. L. Walsh, October 20, 1948

This paper contains the statements of five theorems on partitions, the proofs of which will appear in a series of papers in another American journal, devoted entirely to mathematics.

Euler's identity

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = (1+x)(1+x^2)(1+x^3)\cdots \quad (1)$$

may be paraphrased as follows:

The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Definition 1: The *rank* of a partition is the excess of the maximum part over the number of parts.[1]

Definition 2: $D_r(n)$ = the number of partitions of n into distinct parts, the rank of each partition being r.

Definition 3: $U_{2r+1}(n)$ = the number of partitions of n into odd parts, the maximum part being $2r + 1$.

It is clear that the rank of a partition into distinct parts cannot be negative. Hence Euler's theorem may be written

$$U_1(n) + U_3(n) + U_5(n) + \cdots = D_0(n) + D_1(n) + D_2(n) + \cdots \quad (2)$$

Our first theorem sets up a more refined correspondence between the terms in the right and left members of (2).

THEOREM 1. *For all $n \geqslant 1$, $r \geqslant 0$, we have*

$$U_{2r+1}(n) = D_{2r+1}(n) + D_{2r}(n). \tag{3}$$

Our second theorem is a paraphrase of the double identity

$$\begin{aligned} 1 - \frac{x}{1+x} &+ \frac{x^3}{(1+x)(1+x^3)} - \frac{x^5}{(1+x)(1+x^3)(1+x^5)} + \cdots \\ &= 1 - x + (1+x)x^2 - (1+x)(1+x^2)x^3 + (1+x)(1+x^2)(1+x^3)x^4 - \cdots \\ &= 1 - x + x^2 - x^5 + x^7 - x^{12} + x^{15} - x^{22} + \cdots, \end{aligned} \tag{4}$$

the exponents in the last series being the pentagonal numbers $^1/_2(3k^2 \pm k)$.

Definition 4: $Q_a(n)$ = the number of partitions of n into distinct parts, the maximum part being $\equiv a \pmod 2$, $a = 0, 1$.

Definition 5: $Q_b^*(n)$ = the number of partitions of n into odd parts, the maximum part being $\equiv b \pmod 4$, $b = 1, 3$.

THEOREM 2.

(i) $Q_1^*(2n) = Q_0(2n)$; $\quad Q_3^*(2n) = Q_1(2n)$.

(ii) $Q_1^*(2n+1) = Q_1(2n+1)$; $\quad Q_3^*(2n+1) = Q_0(2n+1)$.

(iii) $$\begin{aligned} Q_0(n) - Q_1(n) &= +1 \text{ if } n = {}^1/_2(3k^2 + k),\ k \geqslant 0, \\ &= -1 \text{ if } n = {}^1/_2(3k^2 - k),\ k > 0, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Part (iii) of Theorem (2) bears some resemblance to the famous pentagonal number theorem of Euler, but we have not been able to establish any real connection between the two theorems.

Definition 6: $p(n)$ = the number of unrestricted partitions of n; $p(0) = 1$; $p(n) = 0$ for $n < 0$.

Definition 7: $P_r(n)$ = the number of partitions of n with rank r; $P_0(0) = 1$; $P_r(n) = 0$ for $r \neq 0$, $n \leqslant 0$.

THEOREM 3. *For $k > 1$, $r > \max(0, k - 5)$,*

$$P_r(r + k) = p(k-1) - p(k-2). \tag{5}$$

THEOREM 4.

(i) $P_0(n+1) + P_0(n) + 2P_3(n-1) = p(n+1) - p(n) \quad (n > 0).$

(ii) $P_0(n-1) - P_1(n) + P_3(n-2) - P_4(n-3) = 0 \quad (n > 1).$

(iii) $P_0(n) - P_1(n-1) - P_2(n-1) + P_3(n-2) = 0 \quad (n > 1).$

(iv) $P_{r+1}(n) - P_r(n-1) - P_{r+3}(n-r-2) + P_{r+4}(n-r-3) = 0 \quad (r > 0, n > 0).$

We observe that (i) enables us to determine the unrestricted partition function from the two functions $P_0(n)$ and $P_3(n)$. In fact, if we sum (i) for $1 \leqslant n \leqslant N$, we obtain

$$p(N + 1) = P_0(N + 1) + 2P_0(N) + 2 \sum_{n=1}^{N-1}(P_0(n) + P_3(n)). \quad (6)$$

Equations (ii), (iii), (iv), together with the obvious remark that $P_r(n) = P_{-r}(n)$, yield the result that the functions $P_r(n)$ can all be determined from the functions $P_0(n)$, $P_1(n)$, $P_2(n)$.

Definition 8: $L(n)$ = the number of partitions of n into distinct parts, the minimum part being odd.

THEOREM 5. *For $n \geqslant 1$, $L(n)$ is odd if and only if n is a square.* It is possible to obtain more precise information about the arithmetic properties of $L(n)$ for special forms of n. We shall restrict ourselves to the remark that

$$L(p^{2m}) \equiv 1 \pmod 4$$

if p is a prime.

[1] To the best of our knowledge, this concept was first introduced by F. J. Dyson in the *American Mathematical Monthly*, August-September, 1947, p. 418.