Cilleruelo's LCM Constants

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Let a, b be coprime integers such that $a \geq 1$, $a+b \geq 1$. The Prime Number Theorem for Arithmetic Progressions implies that

$$\ln\left(\mathrm{lcm}_{1\leq k\leq n}\left\{a\,k+b\right\}\right) \sim A\,n$$

as $n \to \infty$, where the constant A is

$$A = \frac{a}{\varphi(a)} \sum_{\substack{1\leq j\leq a,\\ \gcd(j,a)=1}} \frac{1}{j}$$

(independent of b) and φ is the Euler totient function [1, 2]. What happens if we replace the linear polynomial $ax+b$ by a quadratic polynomial ax^2+bx+c? On the one hand, if the quadratic is reducible over the integers, then there is not much change (the growth rate is still $A\,n$ for some new rational number A). On the other hand, if the quadratic is irreducible over the integers, then there is a more interesting outcome [3]:

$$\ln\left(\mathrm{lcm}_{1\leq k\leq n}\left\{a\,k^2+b\,k+c\right\}\right) = n\ln(n) + B\,n + o(n)$$

as $n \to \infty$, where the constant B will occupy our attention for the remainder of this essay.

Henceforth we set $a=1$, $b=0$, $c \in \{1,2,-2\}$. It follows that the fundamental discriminant $d \in \{-4,-8,8\}$. The constant B for our three special cases is

$$\begin{aligned} B &= \gamma - 1 - \frac{1}{2}\ln(2) - \sum_{k=1}^{\infty}\left(\frac{\zeta'(2^k)}{\zeta(2^k)} - \frac{L_d'(2^k)}{L_d(2^k)} + \frac{\ln(2)}{2^{2^k}-1}\right) + \frac{L_d'(1)}{L_d(1)} \\ &= \begin{cases} -0.0662756342... & \text{if } c=1, \\ -0.4895081630... & \text{if } c=2, \\ 0.3970903472... & \text{if } c=-2. \end{cases} \end{aligned}$$

As an example, if $c=1$, we have [4]

$$\frac{L_{-4}'(1)}{L_{-4}(1)} = \ln\left(2\pi e^{\gamma}\frac{\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2}\right) = \ln\left(\frac{\pi^2 e^{\gamma}}{2\Lambda^2}\right)$$

where Λ is Gauss' lemniscate constant [5]; it can be shown here that

$$B = -3 - \frac{3}{2}\ln(2) + 2\gamma + 4\tilde{C}$$

where $\tilde{C} = 0.7047534517...$ is the second-order constant corresponding to non-hypotenuse numbers [6, 7]. Similar relationships with second-order constants listed in [8] can be found.

Cilleruelo [3] further noted that, in the general case,

$$B = C_0 + C_d + C(f)$$

where

$$C_0 = \gamma - 1 - 2\ln(2) - \sum_{k=1}^{\infty} \frac{\zeta'(2^k)}{\zeta(2^k)} = -1.1725471674...$$

is universal,

$$C_d = \sum_{k=0}^{\infty} \frac{L_d'(2^k)}{L_d(2^k)} - \sum_{p|d} \sum_{k=1}^{\infty} \frac{\ln(p)}{p^{2^k} - 1}$$

depends only on d, and $C(f)$ is too complicated to reproduce (but is equal to $(3/2)\ln(2)$ for our three special cases). Although other irreducible quadratics are examined in [3], we note the absence of $x^2 \pm 3$ and wonder what can be deduced here. See also [9, 10, 11, 12].

References

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