On the Reciprocal of the Binary Generating Function for the Sum of Divisors

Joshua N. Cooper *
Department of Mathematics
University of South Carolina
Columbia, SC 29201
USA
cooper@math.sc.edu

Alexander W. N. Riasanovsky
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104
USA
alexneal@sas.upenn.edu

Abstract

If A is a set of natural numbers containing 0, then there is a unique nonempty "reciprocal" set A^{-1} of natural numbers such that every positive integer can be written in the form $a+a'$, where $a \in A$ and $a' \in A^{-1}$, in an even number of ways. It is straightforward to see that the generating functions of (the characteristic functions of) A and A^{-1} over $\mathbb{F}_2$ are reciprocals in $\mathbb{F}_2[[q]]$. Let Σ denote the set containing 0 and all positive integers such that $\sigma(n)$ is odd, where $\sigma(n)$ is the sum of all the positive divisors of n. Euler showed that $\sigma(n)$ satisfies an almost identical recurrence as that given by his Pentagonal Number Theorem, a corollary of which is that the set P of natural numbers n so that the partition function $p(n)$ is odd is the reciprocal of the set of generalized pentagonal numbers (integers of the form $k(3k+1)/2$, where k is an integer). Therefore, motivated by the 1967 Parkin-Shanks Conjecture that the density of P is $1/2$, we analyze the density ρ of Σ^{-1}, conjecturing that $\rho = 1/32$ and proving that $0 \le \rho \le 1/16$. We also discuss a few surprising connections between Σ and certain so-called "Beatty sequences".

*This work is funded in part by NSF grant DMS-1001370.

1 Introduction

For any sets containing nonnegative integers A and B, the asymmetric additive representation function is defined by

$$R(n) = \#\left\{(a,b) : n = a+b, a \in A, b \in B\right\}$$

Alternatively, we can define $R(n)$ as

$$\left(\sum_{a\in A} q^a\right)\left(\sum_{b\in B} q^b\right) = \sum_{n=0}^{\infty} R(n)q^n$$

We are interested in the case where $R(n) \equiv 0 \pmod 2$ for $n \geq 1$ and $R(0) = 1$ which is illustrated in the power series ring $\mathbb{F}_2[[q]]$. Here, A and B are called reciprocals. For a set A, we write its reciprocal as $\overline{A}$. Given A, we focus on the relative density of $\overline{A}$,

$$\delta\left(\overline{A}, n\right) = \frac{\left|\overline{A} \cap [0,n]\right|}{n+1}$$

and natural density $\delta\left(\overline{A}\right) = \lim_{n\to\infty} \delta\left(\overline{A}, n\right)$. Consider the following statement of Euler's Pentagonal Number Theorem:

$$\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}\right)\left(\sum_{n=0}^{\infty} p(n)q^n\right) = 1, \tag{1}$$

where $p(n)$ is the partition function of n, i.e., the number of ways to write n as an unordered sum of positive integers. If we rewrite (1) mod 2, the result is

$$\left(\sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}\right)\left(\sum_{n=0}^{\infty} p(n)q^n\right) = 1,$$

where the power series are now elements of the ring $\mathbb{F}_2[[q]]$. In this sense, P, the set of integers with an odd number of partitions (including 0), is the reciprocal of the set of generalized pentagonal numbers G, i.e., $\overline{G} = P$. A well-known and difficult conjecture of Parkin and Shanks states that $\delta(P) = 1/2$. The current best lower bounds on the density of P still tend towards 0 [1, 6].

The paper which precedes this attempts to shed light on the question by studying reciprocals mod 2 in general. In particular, the authors found that a loosely-defined "typical" reciprocal set has density 1/2 [4]. With this in mind, we continue the line of work by studying the analogous reciprocal for the function $\sigma(n)$, the sum-of-divisors function defined as

$$\sigma(n) = \sum_{d|n} d.$$

The motivating connection between $p(n)$ and $\sigma(n)$ is the fact that they satisfy almost identical recurrences [2]

$$p(n) = \sum_{n=-\infty}^{\infty} (-1)^n p\left(n - \frac{k(3k-1)}{2}\right)$$

and

$$\sigma(n) = \sum_{n=-\infty}^{\infty} (-1)^n \sigma\left(n - \frac{k(3k-1)}{2}\right)$$

with the only difference being that $\sigma(n-n)$ is interpreted to mean n (since $\sigma(0)$ is undefined, whereas $p(0) = 1$). The reciprocal sets G and $P = \overline{G}$ have densities 0 and, assuming the Parkin-Shanks Conjecture, 1/2, respectively. Then letting Σ denote the set containing 0 and all positive integers n such that $\sigma(n) \equiv 1 \pmod 2$, we ask, what are $\delta(\Sigma)$ and $\delta(\overline{\Sigma})$?

Throughout this sequel, we use the following notation.

Definition 1.1. *For any set containing nonnegative integers F, we write $F(q)$ for the ordinary generating function over $\mathbb{F}_2$ of (the indicator function of) F. In other words,*

$$F(q) = \sum_{f \in F} q^f.$$

Furthermore, for a set of nonnegative integers F, we write F^k for the set of indices of nonvanishing monomials in $F(q)^k$.

Definition 1.2. *For any set $F \subseteq \mathbb{N}$ with $0 \in F$, let $\overline{F}$ be the unique set obtained from $F(q)$ by defining $F(q)\overline{F}(q) = 1$.*

Definition 1.3. *For any set $F \subseteq \mathbb{N}$, let the set of even elements of F be denoted F_e and the odd elements F_o, so that $F(q) = F_e(q) + F_o(q)$.*

We repeatedly employ the following result, sometimes known as the "Children's Binomial Theorem".

Theorem 1. *For any $f,\ g \in \mathbf{F}_2[[q]]$, $(f+g)^2 \equiv f^2 + g^2$.*

2 The Sum of Divisors Function

Definition 2.1. *Let $\Sigma(q)$ be the binary generating function for $\sigma(n)$ for nonnegative integers n. By definition, Σ is the set containing nonnegative integers n with $\sigma(n)$ odd.*

$$\Sigma(q) = \sum_{n=0}^{\infty} \sigma(n) q^n$$

In order to find the density of Σ, we need a description of those integers that have an odd divisor sum. Let n be a positive natural number. Then $\sigma(n)$ is defined to be the sum of all (positive) divisors of n, including n itself, i.e.,

$$\sigma(n) = \sum_{d|n} d.$$

We can write the prime factorization of n as

$$n = \prod_{i=1}^{k} p_i^{e_i}$$

where the p_i are distinct primes and e_i is a positive integer. Clearly, $d \in \mathbb{N}$ divides n if and only if it can be written in the form

$$d = \prod_{i=1}^{k} p_i^{f_i}$$

where $0 \le f_i \le e_i$ for each i. Therefore,

$$\begin{aligned}\sigma(n) &= \sum_{d|n} d \\ &= \sum_{f_1=0}^{e_1} \cdots \sum_{f_k=0}^{e_k} \prod_{i=1}^{k} p_i^{f_i} \\ &= \prod_{i=1}^{k} \sum_{f_i=0}^{e_i} p_i^{f_i}. \end{aligned} \tag{2}$$

This product is odd precisely when all of its factors $\sum_{f_i} p_i^{f_i}$ are odd.

Lemma 2. *For an odd prime p, the quantity $\sum_{f=0}^{e} p^f$ is odd precisely when e is even.*

Proof. If e is even, we may write

$$\sum_{f=0}^{e} p^f = (1+p)(1+p^2+p^4+\cdots+p^{e-2}) + p^e.$$

Since p is odd, $1+p$ is even, so the first summand is even. The second summand, being a nonnegative power of an odd integer, is odd. Therefore, the sum is odd. If e is odd, we may write

$$\sum_{f=0}^{e} p^f = (1+p)(1+p^2+p^4+\cdots+p^{e-1}).$$

Again, $1+p$ is even, so the sum is even. □

Lemma 3. *The quantity $\sum_{k=0}^{r} 2^r$ is odd for any natural number r.*

Proof. This follows immediately from the observation that

$$\sum_{k=0}^{r} 2^r = 2^{r+1} - 1.$$

□

Theorem 4. *The quantity $\sigma(n)$ is odd if and only if the odd part of n is a square, i.e., if $n = 2^r \cdot p_1^{e_1} \cdots p_k^{e_k}$ is the prime factorization of n, then e_i is even for each i, $1 \leq i \leq k$.*

Proof. By (2), we may write

$$\sigma(n) = \left(\sum_{j=0}^{r} 2^j\right) \cdot \left(\prod_{i=1}^{k} \sum_{j=0}^{e_i} p_i^j\right).$$

The left-hand factor is always odd, by Lemma 3. The right-hand factor is odd if and only if *all* of the sums $\sum_j p_i^j$ are odd. By Lemma 2, this occurs if and only if each of the e_i is even. Therefore, $\sigma(n)$ is even if and only if its odd part is a square. □

Definition 2.2. *Let $S(q)$ be the power series with positive squares as the exponents of q:*

$$S(q) = \sum_{n=1}^{\infty} q^{n^2}$$

Lemma 5. $\Sigma(q) = 1 + S(q) + S(q)^2$

Proof. The power series $S(q)$ has a nonzero coefficient for q^n if and only if n is a positive integer of the form k^2. By the Children's Binomial Theorem, $S(q)^2$ has exactly the positive integers of the form $2k^2$ as exponents of q. By adding 1, $S(q)$, and $S(q)^2$ together, we obtain a power series whose nonzero monomials are all the nonnegative integer powers of q of the form k^2 or $2k^2$. By Theorem 4, $\Sigma(q)$ has a nonzero coefficient of q^n if and only if the odd part of n is a square, i.e., n is any nonnegative integer of the form k^2 or $2k^2$. The claimed equality follows immediately. □

Corollary 6. $\delta(\Sigma) = 0$.

Proof. The relative density of Σ, $\delta(\Sigma, n)$, is just the number of nonnegative integers less than or equal to n that are either a square or twice a square, divided by $n + 1$, so

$$\delta(\Sigma, n) = \frac{1 + \lfloor\sqrt{n}\rfloor + \left\lfloor\sqrt{n/2}\right\rfloor}{n+1}$$

Taking the limit as n tends to infinity yields

$$\delta(\Sigma) = 0.$$

□

3 The Reciprocal

Definition 3.1. *Let D denote the odd (positive) squares, i.e.,*

$$D(q) = \sum_{n=0}^{\infty} q^{(2n+1)^2}$$

Lemma 7. $S(q) = \sum_{n=0}^{\infty} D(q)^{4^n}$.

Proof. Consider the set of all odd squares, D. If we quadruple the members of D, we obtain those even squares divisible by 4, but not divisible by 16. If we quadruple again, we obtain the even squares divisible by 16, but not 64. Applying the Children's Binomial Theorem,

$$S(q) = \sum_{n=0}^{\infty} D(q)^{4^n}.$$

□

Corollary 8. $\Sigma(q) = 1 + \sum_{n=0}^{\infty} D(q)^{2^n}$

Proof. Substituting the equation in the statement of Lemma 7 into the expression in Lemma 5 and applying the Children's Binomial Theorem once again, we obtain the desired statement. □

The next results will allow us to decompose $\overline{\Sigma}$ into congruence classes, so that we are able to analyze $\delta\left(\overline{\Sigma}\right)$ "piecewise".

Lemma 9.

$$\overline{\Sigma}(q) = \sum_{n=0}^{\infty} D(q)^{2^n-1}$$

Proof. We begin by squaring $\Sigma(q)$ and rewriting:

$$\Sigma(q)^2 = \left(1 + \sum_{n=0}^{\infty} D(q)^{2^n}\right)^2 = 1 + \sum_{n=1}^{\infty} D(q)^{2^n}.$$

If we add $D(q)$, we have

$$\Sigma(q)^2 + D(q) = \Sigma(q),$$

or

$$D(q) = \Sigma(q) + \Sigma(q)^2,$$

which we can divide by $D(q)\Sigma(q)$ to obtain

$$\overline{\Sigma}(q) = \frac{1+\Sigma(q)}{D(q)} = \sum_{n=0}^{\infty} D(q)^{2^n-1} \tag{3}$$

by Corollary 8. □

Definition 3.2. *Let $\overline{\Sigma}_k$ denote the subset of $\overline{\Sigma}$ of integers congruent to $k \pmod 8$, i.e., $\overline{\Sigma}_k = \overline{\Sigma} \cap (8\mathbb{Z} + k)$.*

Lemma 10. *Using the above definition, the following hold:*

i. $\overline{\Sigma}_0(q) = 1$, *i.e.,* $\overline{\Sigma}_0 = \{0\}$.

ii. $\overline{\Sigma}_1(q) = D(q)$, *i.e.,* $\overline{\Sigma}_1 = \left\{(2k+1)^2 \mid k \in \mathbb{N}, k \geq 0\right\}$.

iii. $\overline{\Sigma}_3(q) = D(q)^3$.

iv. $\overline{\Sigma}_7(q) = D(q)^7 + D(q)^{15} + D(q)^{31} + \cdots$.

Proof. The proof proceeds as follows. For any $F(q) \in \mathbb{F}_2[[q]]$, $F(q)^k$ is the power series whose exponents are those integers that can be represented as a sum of k of $F(q)$'s monomial exponents in an odd number of ways. Note that the exponents of q accompanying nonzero coefficients in $D(q)$ are congruent to $1 \pmod 8$. Therefore, when $D(q)$ is raised to a power congruent to $k \pmod 8$, the exponents of the resulting series are all congruent to $k \pmod 8$. Proceeding from (3), then, we may write

$$\overline{\Sigma} = \overline{\Sigma}_0 \cup \overline{\Sigma}_1 \cup \overline{\Sigma}_3 \cup \overline{\Sigma}_7,$$

because the powers of $D(q)$ on the right-hand side of (3) are congruent only to 0, 1, 3, and 7 $\pmod 8$. Indeed, by examining those exponents which appear in the terms of (3), it is straightforward to see the above lemma holds. □

Our next steps concern further classifications of $\overline{\Sigma}_3$ and $\overline{\Sigma}_7$. For now, we put $\overline{\Sigma}_3$ aside until the next section. Presently, we provide the following definition and lemma.

Definition 3.3. *Let Δ denote the set of triangular numbers, i.e.,*

$$\Delta(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Lemma 11. *The following identities hold:*

i. $\overline{\Sigma}(q) - 1 - D(q) - D(q)^3 = D(q)^7 \overline{\Sigma}(q)^8$

ii. $q\Delta(q)^8 = D(q)$

Proof. By Corollary 8,

$$\overline{\Sigma}(q) = \sum_{n=0}^{\infty} D(q)^{2^n - 1}.$$

Therefore,

$$\overline{\Sigma}(q) - 1 - D(q) - D(q)^3 = \sum_{n=0}^{\infty} D(q)^{2^{n+3}-1}$$

$$= D(q)^7 \sum_{n=0}^{\infty} D(q)^{2^{n+3}-8}$$
$$= D(q)^7 \overline{\Sigma}(q)^8,$$

which is claim (i). Now, observe that multiplying a triangular number $n(n+1)/2$, $n \geq 0$, by 8 and adding 1 yields an odd positive square, and, in fact, every odd positive square can be uniquely obtained in this manner. Therefore,

$$q\Delta(q)^8 = q \left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \right)^8 = D(q),$$

yielding claim (ii). $\square$

The following result is [8, Theorem 357] reduced modulo 2.

Theorem 12.
$$\Delta(q) = \prod_{n \geq 1} (1 + q^n)^3$$

We believe the following theorem, though strictly speaking is not needed in its full generality for our main result, holds some independent interest.

Theorem 13. *Let $k \in \mathbb{Z}$ and $G(q) = \prod_{n \geq 1}(1 + q^n)$. Then, if k is odd,*

$$\Sigma(q) G^k(q) = G_e^k(q).$$

In particular, since $G^3(q) = \Delta(q)$ and $G^{-1}(q) = P(q)$, we have $\Delta(q)\Sigma(q) = \Delta_e(q)$, $P(q)\Sigma(q) = P_e(q)$, and $\overline{\Delta}(q)\Sigma(q) = \overline{\Delta}_e(q)$.

Proof. Over $\mathbb{F}_2$, the derivative with respect to q of q^n is 0 if n is even and q^{n-1} if n is odd. Taking the derivative of the expression $G^k(q) = \prod_{n \geq 1}(1 + q^n)^k$ where k is odd, we obtain

$$\frac{d}{dq}(G^k(q)) = G^k(q) \sum_{n \geq 1} \frac{nq^{n-1}}{1 + q^n},$$

which simplifies to

$$\frac{G_o^k(q)}{q} = G^k(q) \sum_{n \geq 0} \frac{q^{2n}}{1 + q^{2n+1}}.$$

This may be rewritten as

$$\frac{G_o^k(q)}{G^k(q)} = \sum_{n \geq 0} \frac{q^{2n+1}}{1 + q^{2n+1}}.$$

If we add $G^k(q)/G^k(q)$ to the left and 1 to the right, we obtain

$$\frac{G_e^k(q)}{G^k(q)} = 1 + \sum_{n \geq 0} \frac{q^{2n+1}}{1 + q^{2n+1}}. \tag{4}$$

The right-hand side of (4) has monomial terms 1 and q^n for all positive integers n which are divisible by exactly an odd number of odd numbers. Note that, if r is the largest integer so that $2^r | n$, then

$$\begin{aligned}\sigma(n) &= (2^{r+1}-1)\sum_{2\nmid d|n} d \\ &= \sum_{2\nmid d|n} 1 \\ &= \begin{cases} 1 & \text{if } n \text{ has an odd number of odd divisors;} \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The desired conclusion is then the reciprocal of (4). □

Corollary 14. $\overline{\Sigma}(q) = \Delta(q)/\Delta_e(q)$.

Definition 3.4. *Let $V(q)$ be the power series such that $\overline{\Sigma}_7(q) = q^7 V(q)^8$.*

This definition is meaningful because $\overline{\Sigma}_7(q)$ has only monomial terms of the form q^k where $k \equiv 7 \pmod 8$.

Definition 3.5. *Let $T(q) = \frac{\Delta(q)^4}{\sqrt{\Delta_e(q)}}$.*

The square root and the fraction make sense because $\Delta_e(q)$ has only even exponents and $\sqrt{\Delta_e(q)}$ has a 1 term since $\Delta_e(q)$ has a 1 term (0 is an even triangular number).

Theorem 15. $\overline{\Sigma}_7(q) = q^7 T(q)^{16}$.

Proof. We begin by combining Lemma 9 and Lemma 10 (iv):

$$\overline{\Sigma}(q) = 1 + D(q) + D(q)^3 + \overline{\Sigma}_7(q). \tag{5}$$

Applying part (i) of Lemma 11 to Definition 3.4,

$$D(q)^7 \overline{\Sigma}(q)^8 = q^7 V(q)^8,$$

which, using part (ii) of Lemma 11, becomes

$$\left(q\Delta(q)^8\right)^7 \overline{\Sigma}(q)^8 = q^7 V(q)^8.$$

Finally, a bit of simplification yields

$$V(q) = \Delta(q)^7 \overline{\Sigma}(q). \tag{6}$$

If we substitute Lemma 14 into (6),

$$V(q) = \frac{\Delta(q)^8}{\Delta_e(q)},$$

which is exactly $T(q)^2$ on the right-hand side. Substituting into Definition 3.4, we have our proof. □

Theorem 16. $\delta(\overline{\Sigma}) = \delta(\overline{\Sigma}_7)$

Proof of Theorem 16. To show that $\delta(\overline{\Sigma}) = \delta(\overline{\Sigma}_7)$, we recall

$$\overline{\Sigma} = \overline{\Sigma}_0 \cup \overline{\Sigma}_1 \cup \overline{\Sigma}_3 \cup \overline{\Sigma}_7.$$

Since each $\overline{\Sigma}_k$ is disjoint, $\delta(\overline{\Sigma}) = \delta(\overline{\Sigma}_0) + \delta(\overline{\Sigma}_1) + \delta(\overline{\Sigma}_3) + \delta(\overline{\Sigma}_7)$. Because $\overline{\Sigma}_0 = \{0\}$ and $\overline{\Sigma}_1 = \left\{(2k+1)^2 \mid k \in \mathbb{N}, k \geq 0\right\}$, $\delta(\overline{\Sigma}_0) + \delta(\overline{\Sigma}_1) = 0$. We devote the following section to describing $\overline{\Sigma}_3$ and computing its density. As seen later, $\delta(\overline{\Sigma}_3) = 0$. □

Corollary 17. $0 \leq \delta(\overline{\Sigma}) \leq 1/16$.

Proof. We begin with Theorem 15, $\overline{\Sigma}_7(q) = q^7 T(q)^{16}$. By the Children's Binomial Theorem, if n is a monomial exponent of q on the right-hand side of the preceding equation, then $n \equiv 7 \pmod{16}$. (Also, for $n \equiv 15 \pmod{16}$, $n \notin \overline{\Sigma}_7$.) Thus, $0 \leq \delta\left(\overline{\Sigma}_7\right) \leq 1/16$ and by Theorem 16, $0 \leq \delta\left(\overline{\Sigma}\right) \leq 1/16$. □

Conjecture 18. $\delta(\overline{\Sigma}) = 1/32$.

4 The $3 \pmod 8$ Case

We note a few well-known facts which will be used in the course of the proof.

Proposition 19. $\mathbb{Z}[\sqrt{-2}]$ *is a unique factorization domain.*

Proposition 20. *An odd prime p can be written as $p = x^2 + 2y^2$ for integers x and y if and only if $p \equiv 1$ or $3 \pmod 8$.*

For proofs of the preceding propositions, we refer the reader to common texts on number theory [5, 8].

Proposition 21. *For an integer $n \equiv 3 \pmod 8$, the factorizations of n as $(a + b\sqrt{-2})(a - b\sqrt{-2})$ with $a, b > 0$ are in bijection with the representations of n as $c^2 + 2d^2$ for some odd $c, d > 0$.*

Proof. It is clear that, if $n = c^2 + 2d^2$, then c and d must be odd, since $n \equiv 3 \pmod 8$ implies $c^2 \equiv 1 \pmod 8$ and $d^2 \equiv 1 \pmod 8$. If $n = (a + b\sqrt{-2})(a - b\sqrt{-2})$ for $a, b > 0$, then $n = a^2 + 2b^2$; by the preceding sentence, a and b are odd and positive. Conversely, if $n = a^2 + 2b^2$ for $a, b > 0$, then $n = (a + b\sqrt{-2})(a - b\sqrt{-2})$. □

Theorem 22. *If $n \equiv 3 \pmod 8$, then $n \in \overline{\Sigma}_3$ if and only if we can write $n = p^e k^2$ for some prime $p \equiv 3 \pmod 8$, $e \equiv 1 \pmod 4$, k an odd integer, and $p \nmid k$.*

Proof. Lemma 10 showed us that $\overline{\Sigma}_3(q) = D(q)D(q)^2$. Using the Children's Binomial Theorem, we see that $n \in \overline{\Sigma}_3$ if and only if the number of representations of n as $a^2 + 2b^2$, a and b positive, is odd. Let $n = a^2 + 2b^2$ be one such representation. Then, $n \equiv 3 \pmod 8$ if and only if a and b are odd. Suppose n can be factored over $\mathbb{Z}$ as

$$n = p_1^{e_1} \cdots p_r^{e_r} q_1^{f_1} \cdots q_s^{f_s}$$

with $p_j \equiv 1$ or $3 \pmod 8$ for each $j \in [r]$ and $q_j \equiv 5$ or $7 \pmod 8$. If $p_j \equiv 1, 3 \pmod 8$, then factoring p over $R = \mathbb{Z}[\sqrt{-2}]$ yields $p_j = (a_j + b_j\sqrt{-2})(a_j + b_j\sqrt{-2})$ for some R-primes $a_j \pm b_j\sqrt{-2}$. This factorization is unique up to multiplication by units and switching the order of the factors. However, $a^2 + 2b^2 = 1$ has only the solutions $(a, b) = (\pm 1, 0)$, so we may simply assume that a and b are positive to ensure uniqueness. (a cannot be 0, since p is an odd prime.) Then the factorization of n over R has the form

$$n = \prod_{j=1}^{r} (a_j + b_j\sqrt{-2})^{e_j} (a_j - b_j\sqrt{-2})^{e_j} \prod_{j=1}^{s} q_j^{f_j}.$$

Grouping terms on the right-hand side yields a product
$n = (a + b\sqrt{-2})(a - b\sqrt{2})$, where

$$a + b\sqrt{-2} = \prod_{j=1}^{r} (a_j \pm b_j\sqrt{-2})^{A_j} \prod_{j=1}^{r} p_j^{(e_j - A_j)/2} \prod_{j=1}^{s} q_j^{f_j/2}$$

for some $(A_1, \dots, A_r) \in \{0, \dots, e_1\} \times \cdots \times \{0, \dots, e_r\}$. Since each such product gives a positive b if and only if the same product with all the $\pm$'s switched with $\mp$'s gives a negative b, the number of representations of n as $a^2 + 2b^2$, a and b positive, is half the number of choices for the A_j's and the $\pm$'s. If $r =$ the number of p_j's, $s =$ the number of even e_j's, and $t =$ the number of nonzero A_j's among those j so that e_j is even, then the number of representations of n is found with the nested sums

$$R(n) = \sum_{\{j_1,\dots,j_t\} \subseteq [s]} \sum_{\substack{A_{j_1} = e_{j_1}, e_{j_1}-2, \dots, 2 \\ \vdots \\ A_{j_t} = e_{j_t}, e_{j_t}-2, \dots, 2}} \sum_{\substack{A_{s+1} = e_{s+1}, e_{s+1}-2, \dots, 1 \\ \vdots \\ A_r = e_r, e_r - 2, \dots, 1}} 2^{r-s+t-1}.$$

For $R(n)$ to be odd, $t = 1 - (r - s)$. Clearly, $r - s$, the number of odd e_j's, is either 0 or 1. If $r - s$ is 0, then n is a square which is a contradiction because $n \equiv 3 \pmod 8$. The fact that $r - s = 1$ implies that n has only one p_j such that e_j is odd. Without loss of generality, we choose it to be p_1. Clearly $p_1 \equiv 3 \pmod 8$ because $n \equiv 3 \pmod 8$ and the other factors of n produce a square term. This can be used to show that

$$R(n) = \sum_{A_1 = e_1, e_1 - 2, \dots, 1} 1$$

which reduces to

$$R(n) = \left\lceil \frac{e_1}{2} \right\rceil .$$

Now, $R(n) \equiv 1 \pmod 2$ only when $e_1 \equiv 1 \pmod 4$. □

Corollary 23. $\delta(\overline{\Sigma}_3) = 0$.

Proof. Let $\mathcal{O}_p$ denote the set of positive integers which are not divisible by p. By the Prime Number Theorem and Theorem 22, the number of elements of $\overline{\Sigma}_3$ less than or equal to n is

$$\begin{aligned}
\sum_{\substack{\text{prime } p \leq n \\ p \equiv 3 \pmod 8}} \sum_{k=1}^{\lfloor \sqrt{n/p} \rfloor} \chi(k \in \mathcal{O}_p) &\leq \sum_{\text{prime } p \leq n} \sum_{k=1}^{\lfloor \sqrt{n/p} \rfloor} 1 \\
&\leq \sum_{\text{prime } p \leq n} \sqrt{\frac{n}{p}} \\
&= \int_2^n \sqrt{\frac{n}{x}} \cdot \frac{1}{\log x} (1 + o(1))\, dx \\
&= O\left(\sqrt{n}\right) \int_2^n \frac{1}{\sqrt{x} \log x}\, dx \\
&= O\left(\sqrt{n}\right) \int_2^{\sqrt{n}} \frac{1}{\log u}\, du \quad \text{by letting } u = \sqrt{x} \\
&= O\left(\sqrt{n} \cdot \frac{\sqrt{n}}{\log n}\right) \\
&= O\left(\frac{n}{\log n}\right) = o(n).
\end{aligned}$$

□

5 Appendix

We conclude the paper with a few observations about the indices of elements of Σ corresponding to certain well-studied integer sequences.

Definition 5.1. *Let $c(n) = \lfloor \sqrt{n} \rfloor + \lfloor \sqrt{n/2} \rfloor$, the number of positive integers of the form k^2 or $2k^2$ less than or equal to n.*

Definition 5.2. *Let $\{\varsigma_n\}_{n=1}^{\infty}$ be the monotone increasing sequence comprised of all positive elements of Σ.*

Proposition 24. *For all $n \geq 1$, $c(\varsigma_n) = n$.*

Proof. For any $n \geq 1$, n is the number of elements of $\{\varsigma_n\}_{n=1}^{\infty}$ less than or equal to ς_n. Since $\{\varsigma_n\}_{n=1}^{\infty}$ is each positive integer of the form k^2 or $2k^2$ in monotone increasing order, $c(\varsigma_n) = n$. □

Definition 5.3. *Below, we define six (non-homogeneous) Beatty sequences for* $k \geq 1$*:*

i. Let w_k be the k-th winning positions in the 2-Wythoff game (OEIS A001954).

ii. Let $\alpha_k = \lfloor k(2+\sqrt{2}) \rfloor$ (OEIS A001952).

iii. Let $\beta_k = \lfloor k(2+\sqrt{2})/2 \rfloor$ (OEIS A003152).

iv. Let $\gamma_k = \lfloor (k-1/2)(2+2\sqrt{2}) \rfloor$ (OEIS A215247).

v. Let $\delta_k = \lfloor k(2+2\sqrt{2}) \rfloor$ (OEIS A197878).

vi. Let $\epsilon_k = \lfloor k(1+\sqrt{2}) \rfloor$ (OEIS A003151).

The following is a result in the 2-Wythoff winning positions [3].

Theorem 25. $w_k = \lfloor (k-1/2)(2+\sqrt{2}) \rfloor$.

Proposition 26. *Let* ς_n, β_n, α_n, δ_n, *and* γ_n *be defined as above. Then we have the following.*

i. If $n = (2k-1)^2$ *for some positive integer* k, *then* n *is the* w_k*-th term in* $\{\varsigma_n\}_{n=1}^{\infty}$.

ii. If $n = 4k^2$ *for some positive integer* k, *then* n *is the* α_k*-th term in* $\{\varsigma_n\}_{n=1}^{\infty}$.

iii. If $n = k^2$ *for some positive integer* k, *then* n *is the* β_k*-th term in* $\{\varsigma_n\}_{n=1}^{\infty}$.

iv. If $n = 2(2k-1)^2$ *for some positive integer* k, *then* n *is the* γ_k*-th term in* $\{\varsigma_n\}_{n=1}^{\infty}$.

v. If $n = 8k^2$ *for some positive integer* k, *then* n *is the* δ_k*-th term in* $\{\varsigma_n\}_{n=1}^{\infty}$.

vi. If $n = 2k^2$ *for some positive integer* k, *then* n *is the* ϵ_k*-th term in* $\{\varsigma_n\}_{n=1}^{\infty}$.

Proof. Let $n = (2k-1)^2$ be the k-th odd square. By Definition 5.1,

$$\begin{aligned} c(n) &= \lfloor \sqrt{(2k-1)^2} \rfloor + \lfloor \sqrt{(2k-1)^2/2} \rfloor \\ &= \lfloor 2k-1+(2k-1)/\sqrt{2} \rfloor \\ &= \lfloor (2k-1)(1+1/\sqrt{2}) \rfloor \\ &= \lfloor (k-1/2)(2+\sqrt{2}) \rfloor. \end{aligned}$$

Theorem 25 and Proposition 24 complete the proof of (i). Now, let $n = 4k^2$ be the k-th positive even square. By Definition 5.1,

$$\begin{aligned} c(n) &= \lfloor \sqrt{4k^2} \rfloor + \lfloor \sqrt{4k^2/2} \rfloor \\ &= \lfloor 2k + 2k/\sqrt{2}) \rfloor \\ &= \lfloor k(2+\sqrt{2}) \rfloor. \end{aligned}$$

Proposition 24 completes the proof of (ii). Let $n = k^2$ be the k-th positive square. By Definition 5.1,

$$\begin{aligned} c(n) &= \lfloor \sqrt{k^2} \rfloor + \lfloor \sqrt{k^2/2} \rfloor \\ &= \lfloor k + k/\sqrt{2}) \rfloor \\ &= \lfloor k(2+\sqrt{2})/2 \rfloor. \end{aligned}$$

Again, Proposition 24 completes the proof of (iii). Let $n = 2(2k-1)^2$ be the k-th positive twice odd square. By Definition 5.1,

$$\begin{aligned} c(n) &= \lfloor \sqrt{2(2k-1)^2} \rfloor + \lfloor \sqrt{2(2k-1)^2/2} \rfloor \\ &= \lfloor \sqrt{2}(2k-1) + 2k - 1 \rfloor \\ &= \lfloor (2k-1)(1+\sqrt{2}) \rfloor \\ &= \lfloor (k-1/2)(2+2\sqrt{2}) \rfloor. \end{aligned}$$

Proposition 24 completes the proof of (iv). Let $n = 8k^2$ be the k-th positive twice even square. By Definition 5.1,

$$\begin{aligned} c(n) &= \lfloor \sqrt{8k^2} \rfloor + \lfloor \sqrt{8k^2/2} \rfloor \\ &= \lfloor 2\sqrt{2}k + 2k \rfloor \\ &= \lfloor k(2+2\sqrt{2}) \rfloor. \end{aligned}$$

Proposition 24 completes the proof of (v). Finally, let $n = 2k^2$ be the k-th positive twice-square. By Definition 5.1,

$$\begin{aligned} c(n) &= \lfloor \sqrt{2k^2} \rfloor + \lfloor \sqrt{2k^2/2} \rfloor \\ &= \lfloor k\sqrt{2} + k \rfloor \\ &= \lfloor k(1+\sqrt{2}) \rfloor. \end{aligned}$$

Proposition 24 completes the proof of (vi). □

Proposition 26(i) may be interpreted in the following somewhat surprising way. Let $\mathbb{W}$ denote the positive natural numbers, i.e., the "whole numbers". Define a function $\mathcal{L} : \mathbb{W}^{\mathbb{W}} \to \mathbb{W}^{\mathbb{W}}$ as follows: given a function $f \in \mathbb{W}^{\mathbb{W}}$ which takes on infinitely many odd values, let $\mathcal{L}(f)$ be the function $g \in \mathbb{W}^{\mathbb{W}}$ so that $g(k)$ is the k-th smallest integer n so that $f(n)$ is odd, i.e., for $k \geq 1$,

$$g(k) = \min\{n : f(n) \equiv 1 \pmod 2 \text{ and } n > g(k-1)\},$$

where we take $g(0) = -\infty$ by convention. Then $\mathcal{L}(\mathcal{L}(\sigma)) = c$.

6 Acknowledgements

The authors wish to thank Mr. R. La Cross and Dr. G. Salazar of the South Carolina Governor's School for Science and Mathematics for their guidance and enthusiasm, and Mr. N. J. Smith for his facilitation of the research collaboration which led to the present manuscript.

References

[1] S. Ahlgren, Distribution of the parity of the partition function in arithmetic progressions, *Indagationes Math.* **10** (1999), 173–181.

[2] J. Bell, A summary of Euler's work on the pentagonal number theorem, *Archive for History of Exact Sciences.* **64** no. 3 (2010), 301–373.

[3] I. G. Connell, A generalization of Wythoff's game, *Canad. Math. Bull.* **2** (1959), 181–190.

[4] J. Cooper, D. Eichhorn, and K. O'Bryant, Reciprocals of binary series, *Int. J. Number Theory* **2** no. 4 (2006), 499–522.

[5] K. Ireland and M. Rosen, "A Classical Introduction to Modern Number Theory, Second Edition", *Springer-Verlag*, 1990.

[6] J.-L. Nicolas, I. Z. Ruzsa, and A. Sárközy, On the parity of additive representation functions, *J. Number Th.* **73** (1998), 292–317.

[7] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[8] G. H. Hardy, E. M. Wright, "An Introduction to the Theory of Numbers", *Oxford University Press*, Oxford, 2008.