PC13-6

Sieves

PROBLEM 43 A-H

The familiar Sieve of Eratosthenes yields prime numbers by the following algorithm:

Write down the consecutive integers from 2 to N. Circle the 2 and cross off every second integer following. Circle the next remaining integer (3) and cross off every third integer following. Continue this process: circle the next remaining integer (K) and cross off every Kth integer following (which will include integers previously crossed off). For the first few integers, the procedure then produces:

A40

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11
~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~ ~~21~~

The resulting list of (circled) prime numbers is correct to the square of the last K used.

Listed below are some other schemes for sieving the integers. In each case, the Problem is to determine the 1000th circled number.

1. Apply the same scheme as Eratosthenes, but cross off every Kth remaining number. The resulting sequence begins 2, 3, 5, 7, 11, 13, 17, 23, 25, 29, 37,... A3309

2. Apply the scheme of No. 1, but begin the series of integers with 3, rather than 2. The resulting sequence begins 3, 4, 5, 7, 8, 11, 13, 17, 19, 20, 26,... A3310

3. Using the integers from 3 to N, when K is circled, cross off subsequent integers with the value KX+1. For example, when 11 is circled, cross off 12, 23, 34, 45, 56, and so on. A3311, A100464

4. Using the integers from 7 to N, when K is circled, cross off subsequent integers with the value KX-1. For example, when 7 is circled, cross off 13, 20, 27, 34, 41, 48, and so on. The resulting sequence begins 7, 8, 9, 10, 11, 12, 14, 16, 18, 22, 25,... A100562

5. Using the integers from 3 to N, when K is circled, cross off every 3rd remaining number. The resulting sequence begins 3, 4, 5, 7, 10, 14, 20, 29,... A3312

6. Using the integers from 3 to N, when K is circled, cross off every Mth remaining number, where M is 2, 4, or 5. A52548, A100585, A100586

7. Using the integers from 2 to N, circle the 2 and cross off every second number (that is, all even numbers). Circle the first remaining number, 3, and

cross off every third of all remaining numbers (that is, cross off 5, 11, 17, 23, 29, and so on). Circle the first remaining number, 7, and cross off every seventh of all remaining numbers (that is, cross off 19, 39, 61, 81, 103, and so on). Those numbers finally remaining (including 1) form the sequence (1, 2, 3, 7, 9, 13, 15,...) that Ulam named Lucky Numbers.

A959

8. Another of Ulam's sequences is

A2858

(1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26,...)

in which each new member can be formed in one and only one way by adding two different earlier numbers. Thus, numbers like 12 and 15 do not appear because they can be formed in more than one way (8+4, 11+1, for example), and 33 will not appear because it cannot be formed at all.

The original scheme of Eratosthenes for locating prime numbers is not, at first glance, a practical notion, since it seems to say "write down all the numbers in the desired range and then eliminate those that are not prime." Offhand, this corresponds to the advice to young sculptors on how to carve a beautiful statue of a horse: "Take this block of marble and cut away the parts that don't look like a horse." The process suddenly becomes practical with D. H. Lehmer's observation that the numbers themselves are not necessary to the scheme, but only the positions of the numbers, and a computer contains lots of positions; namely, bits. For example, a block of 1000 words (on a 32-bit word machine) can represent 32,000 consecutive numbers. Starting with these bits all set to zero, ones can be stored at every second position after position 2; at every third position after position 3, and so on, following the pattern laid down by Eratosthenes. The bit positions that are still zero after sifting, represent prime numbers. This was the procedure used in 1959 to validate the table of 6,000,000 prime numbers which had been calculated by a less efficient scheme. The entire check calculation took 21 minutes on an IBM 7094, where the original calculation, in 1957, consumed 120 hours on an IBM 704.

For contest purposes (such as the offer to undergraduates made in PC12) these 8 problems are subject to the following rules. (1) At least 4 of the sequences must be explored, and (2) the computer printout should show the first 100 numbers of the sequence, the 1000th number, and the limit on the value of N that was used in the sieve.