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THE INTRODUCTION OF AL-RISĀLA AL-MUHĪTĪYYA: AN ENGLISH TRANSLATION

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Abstract: One of the most significant mathematical achievements of Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī is his *al-Risāla al-Muhītīyya* ("The Treatise on the Circumference") which he had completed in July 1424 (Sha'bān 827 A.H.L.). In this paper we present an English translation of the introduction of this treatise. Also, we comment on the errors of al-Būzjānī and al-Bīrūnī as indicated by al-Kāshī.

AMS Subject Classification: 01A30
Key Words: Al-Kāshī, al-Kāshānī, Kāshānī, Kāshī, *Kāshānī nāmeh, al-Risāla al-Muhītīyya, Muhītīyya,* Abu'l-Wafā' al-Būzjānī, Abū Rayhān al-Bīrūnī, Archimedes, *On the Measurement of the Circle, Sphere and Cylinder, Taksīr al-Dā'irā, Archimedes' Division of the Circle, Kitāb al-Kāmil, Qānoūn al-Mas'ūdī,* Nasīr al-Dīn al-Tūsī, *Majmū' al-Rasā'il,* Ptolemy's *Almagest.*

1. Preliminaries

One of the most significant mathematical achievements of Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī is *al-Risāla al-Muhītīyya* ("The Treatise on the Circumference"), which is also known as *Risāla-i muhītīyya,* or just *Muhītīyya.* This masterpiece of computational techniques, written longhand in Arabic by al-Kāshī himself, was completed in Samarqand in July 1424 (Sha'bān 827 A.H.L.). This original manuscript of *al-Risāla al-Muhītīyya* [1] is Number 162 of the mathematics collection (Number 5389 of the general collection) of the central library of the Āstāne Qudse Razawī, Mashhad, Iran. At the conclusion of *al-Risāla*

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al-Muhītīyya, al-Kāshī wrote, *katabahoo muallefohoo*, which means "written by its [the] author". Several other sources confirm that [1] is indeed the authentic manuscript of *al-Risāla al-Muhītīyya*, including Abu'l-Qāsim Qurbānī [9, p. 130] and the staff at the central library of the Āstāne Qudse Razawī. Also, on the second page of [1], the authenticity of this copy has been certified by Bahā al-Dīn al-'Āmilī (1532-1610)[1], who had this copy of *al-Risāla al-Muhītīyya* in his possession for some time. Moreover, E.S. Kennedy in the review of [7] [MR0055264 (14, 1051a)], stated that the Mashhad Shrine Library copy was written in al-Kāshī's hand. Based on this original manuscript, A. Qurbānī [9] has written a Persian translation of the introduction of *al-Risāla al-Muhītīyya.* Paul L. Luckey [7] has written a German translation with a detailed commentary of the introduction of this treatise as well. Also, Boris A. Rosenfeld and Adolf P. Youschkevitch have written a Russian translation of the introduction of *al-Risāla al-Muhītīyya* with commentaries [13].

In the introduction of *al-Risāla al-Muhītīyya,* al-Kāshī stated that the calculations of π by Archimedes (c. 287-212 BC), Abu'l-Wafā' al-Būzjānī (940-998), and Abū Rayhān al-Bīrūnī (973-after 1050) motivated him to write this treatise on improving the estimation of π by these three renowned mathematicians. Al-Kāshī did not criticize Archimedes' calculation of π, he just discussed Archimedes' method and his result. However, he not only stated al-Būzjānī's and al-Bīrūnī's methods as well as their results, but he also indicated their errors. In the case of al-Būzjānī we will see that al-Kāshī's judgment was premature. Finally, he stated that in using his method to find the perimeter of a circle in which its diameter is six hundred thousand times the diameter of the sphere Earth, the difference between his calculations and the actual value will not reach the breadth of a horse hair.

Our goal in this paper is to present, for the first time, an English translation of the introduction of *al-Risāla al-Muhītīyya* based on Qurbānī's Persian translation (Figure 1). Also, we will comment on the errors of al-Būzjānī and al-Bīrūnī as noted by al-Kāshī.

2. Notes

Note 2.1. In both the Arabic and the Persian languages there is not a word which is equivalent to the word "circumference". In both of these languages we use the word *muhīt* for both the circumference of a circle as well as the perimeter of a geometric figure. Thus, we see where the word *al-Muhītīyya* ("on

[1]بسم الله الرحمن الرحیم

۲۳۴. «[2]ستایش خداوندی را سزد که از نسبت قطر به محیط آگاه است[۱] و اندازهٔ هر مرکب و بسیط را می شناسد و آفرینندهٔ زمین و آسمانها و قرار دهندهٔ نور در تاریکی است.[3] و درود و سلام بر محمد مصطفی که مرکز دایرهٔ رسالت و محیط اقطار رهنمایی و دادگری است و بر خاندان و یاران پاك او باد.[۲]

[4]اما بعد نیازمندترین بندگان خدای تعالی به آمرزش وی جمشید پسر مسعود پسر محمود، طبیب کاشانی ملقب به غیاث که خداوند احوال او را نیکو گرداند می گوید:[5] ارشمیدس ثابت کرده است که محیط (دایره) از سه برابر قطرش به اندازهٔ کمتر از $\frac{۱}{۷}$ و بیشتر از $\frac{۱۰}{۷۱}$ قطر، بزرگتر است.[۳] [6]پس تفاوت بین این دو مقدار $\frac{۱}{۴۹۷}$ (قطر) است.[7] پس دایره‌ای که قطرش ۴۹۷ ذراع یا قصب[۴] یا فرسنگ باشد مقدار محیطش در حدود یك ذراع یا قصب یا فرسنگ مجهول و مشکوك است و[8] دایرهٔ عظیمه‌ای که بر کرهٔ زمین واقع باشد محیطش در حدود پنج فرسنگ مجهول است[9] زیرا قطر آن برحسب فرسنگ تقریباً پنج برابر مقدار مذکور می باشد[۵] و[10] در فلك البروج (در محیط...) در حدود بسیار بیش از صد هزار فرسنگ مجهول است[۶]، و این مقادیر که در محیطها (این اندازه) زیاد هستند در مساحت (ها) چه خواهند بود[۷]؟[11] این به علت آن است که وی (= ارشمیدس) محیط ۹۶ ضلعی (منتظم) محاط در دایره را استخراج کرده است و آن از محیط دایره کوچکتر می باشد[12] زیرا هر ضلع آن از قوس روبروی آن کوچکتر است و مجموع اضلاع آن از محیط دایره کوچکتر می باشد[13] و (ارشمیدس) محیط چند ضلعی دیگری را که مشابه با اولی و محیط بر (همان) دایره است استخراج کرده و به مدد قضیهٔ اول نخستین مقالهٔ کتاب خود به ثبوت رسانیده است که آن از محیط دایرهٔ مذکور بزرگتر است و تفاوت بین آنها (= در محیط) همان است که گفته شد.»[۱]

۲۳۵. «[14]و اما ابوالوفای بوزجانی وتر نصف $\frac{۱}{۳۶۰}$ محیط دایره (= وتر قوس نیم درجه) را به فرض آنکه قطر دایره ۱۲۰ باشد به حساب تقریبی به دست آورده و آن را در ۷۲۰ ضرب کرده تا محیط ۷۲۰ ضلعی (منتظم) محاط در دایره حاصل شود و[15] (نیز) محیط ۷۲۰ ضلعی محیطی مشابه با اولی را استخراج کرده و گفته است که اگر قطر ۱۲۰ باشد محیط آن ۳۷۶ و کسری می شود و این کسر از (۵۹, ۱۰, ۵۹;۰) بیشتر[۲] و از (۱۲, ۵۴, ۲۳, ۵۹;۰) کمتر است و تفاوت بین این دو مقدار (۱۲, ۵۵, ۱۲, ۰;۰) می باشد.[16] و این (تفاوت) در دایرهٔ عظیمه‌ای که بر کرهٔ زمین واقع باشد تقریباً به هزار ذراع بالغ می شود.[17] با این حال او (= بوزجانی) در مقدار وتر (قوس) نیم درجه اشتباه کرده است،[18] زیرا او این (وتر) را

(۰;۳۱, ۲۴, ۵۵, ۵۴, ۵۵)

Figure 1: Introduction of al-Risāla al-Muhītīyya

the circumference") comes from. However some Persian scientists, including al-Bīrūnī, used the word *dowr* which means "around" instead of *muhīt* for the circumference.

گرفته و این درست نیست و صحیح آن چنین است

(۳۶, ۳۸, ۵۶, ۲۴, ۳۱; ۰)

و دلیل این را بعداً خواهیم گفت.»[۱]

۲۳۶. «[19]و اما ابوریحان بیرونی وتر دو جزء از ۳۶۰ قسمت محیط (= وتر قوس دو درجه) را حساب کرده و محیط ۱۸۰ ضلعی (منتظم) محاطی را مساوی با

(۰, ۴۸, ۱۰, ۵۹, ۱۶; ۶)

و محیط ۱۸۰ ضلعی (منتظم) محیطی مشابه با آن را مساوی با

(۶, ۱۹, ۵۸, ۱, ۱۷; ۶)

حساب کرده و[20] نصف مجموع این دو را محیط دایره گرفته است و آن را به فرض آنکه قطر دایره[۲] [21] واحد باشد با ارقام هندی به کسری که مخرج آن چند رقمی است تبدیل کرده است. و این در دایره‌ای که مساوی با دایرهٔ عظیمهٔ واقع بر کرهٔ زمین باشد تقریباً به یك فرسنگ می‌رسد. [22] با این حال وی (= بیرونی) در وتر (قوس) دو درجه اشتباه کرده است[23] زیرا او این وتر را

(۳۶, ۴۳, ۳۹, ۵, ۲; ۰)

گرفته در صورتی که باید آن را

(۲۲, ۲۶, ۳۹, ۵, ۲; ۰)

گرفته باشد. و[24] باید دانست که او (= بیرونی) جیب قوس یك درجه را که نصف وتر قوس دو درجه است در جدول جیب (کتاب) *قانون مسعودی* خود

(۴۳, ۴۹, ۲, ۱; ۰)

ثبت کرده و این درست است اما دو برابر آن را غلط حساب کرده است.»[۳]

۲۳۷. «[25]چون این اعمال (= محاسبات) مختل بود خواستیم محیط دایره را به فرض معلوم بودن قطر آن بر حسب واحد معینی چنان استخراج کنیم که بر ما یقین حاصل شود که در دایره‌ای که قطرش ششصد هزار برابر قطر زمین باشد تفاوت بین نتیجهٔ حساب ما و آنچه حق است (= مقدار واقعی محیط) به یك مو نرسد، [26]مویی که ضخامتش یك ششم عرض یك دانه جو متوسط است، و آنچه کوچکتر از آن (= تفاوتی که کمتر از یك مو) باشد قابل اهمیت نیست. و[27] این رساله را مشتمل بر استخراج محیط در ده فصل و یك خاتمه نوشتم و آن را محیطیه نامیدم در حالی که از خداوند عزیز وهاب یاری می‌طلبم و اوست هدایت کنندهٔ به راه راست.» (پایان)

Figure 1: Continuation: Introduction of al-Risāla al-Muhīṭīyya

Note 2.2. The fractional part of numbers in *al-Risāla al-Muhīṭīyya* are written in sexagesimal system. Sexagesimal system is the first place-value system in history. To write a number in sexagesimal system we use modern 'comma and semi-colon' notations. That is, in the sexagesimal system the digits are separated by commas, and the integral and fractional parts by semicolons.

Note 2.3. Al-Kāshī used the following units of measurements in *al-Risāla al-Muhītīyya*:

1 *farsakh* $\approx$ 12000 *adru'* $\approx$ 6 km

1 *dirā'* $\approx$ 24 *qasabāt* $\approx$ 50 cm

1 *isba* = 1 *finger* $\approx$ 6 times the width of a medium-size grain of *barley*

Width of a medium-size grain of *barley* $\approx$ 6 *sha'rāt* (plural of sha'ra)

1 *sha'ra* = The breadth of a horse (mane) hair

Adru' is the plural of *dirā'* $\approx$ 50 cm and it is different from the Iranian *zar'* $\approx$ 104 cm. We note that the value of *dirā'* varies from country to country. For example, in Syria it is about 68 cm while in Egypt it is 58 cm.

Qasabāt is the plural of *qasaba* $\approx$ 3.55 m. In old Persian it used to be called *qasab* (or *nāb*). However, this measurement is no longer in use in modern day Iran.

Farsakh $\approx$ 6 km and its plural is *farāsikh* in Arabic and *farsakhha* in Persian. In Persian *farsakh* is also called *farsang* or *parasang,* and in Arabic *farsakh* is also called *farsak.*

3. Translation

In this section we present a complete English translation of the introduction of *al-Risāla al-Muhītīyya.* The source of our translation will be Qurbānī's Persian translation of the introduction [9, pp. 134-137] of the original Arabic manuscript of *al-Risāla al-Muhītīyya* [1]. However, to make sure that nothing is lost in translating from Persian to English, we have consulted the original Arabic manuscript as well. To make the translation more legible and more suitable in English, we paraphrase some sentences and we break up al-Kāshī's (and hence Qurbānī's) long sentences into shorter sentences. We insert a superscript number in the Persian manuscript to indicate the beginning of a sentence in the English translation. The superscript number at the beginning of a sentence in the English translation corresponds to the same number in the beginning of the corresponding sentence in the Persian manuscript. We must note that a superscript number at the end or somewhere in the middle of an English sentence will indicate a footnote only. We do not feel it is necessary to include the English translation of Qurbānī's footnotes. The translation starts with the superscript 1 in Figure 1 and ends at the end of the sentence that started with the superscript 27 in Figure 1. Extra words or phrases in square brackets have been inserted by the author to clarify a word or sentence.

> [1]In the Name of God, Most Gracious, Most Merciful
>
> [2]Thanks be to the deserving God Who is aware of the ratio of the diameter to the circumference[2] and Who knows the measurements of every compound and [every] trivial, and Who is the creator of the Earth and heavens, and He is Who [is able to] place light in darkness. [3]And praise and peace be upon Muhammad Mustafā who is the center of the circle of prophethood and the circumference[s] of the diameters of guidance and righteousness, and [praise and peace be upon] his family and his virtuous companions.
>
> [4]Now, the most needy servant of the Exalted God's forgiveness Jamshīd son of Mas'ūd son of Mahmūd Tabīb Kāshānī [physician from Kāshān] known as Ghiyāth who asks for good circumstances from God says: [5]Archimedes has proven that the circumference [of a circle] is smaller than three times its diameter by less than $\frac{1}{7}$ of the diameter and larger than three times of the diameter by less than $\frac{10}{71}$ of the diameter[3]. [6]Hence, the difference of these two quantities is $\frac{1}{497}$ [of the diameter]. [7]Therefore, the circle that its diameter is 497 *adru'*, or *qasabāt,* or *farāsikh*, its circumference is about one *dirā'*, one *qasaba,* or one *farsakh, majhool* or *mashkook*[4]. [8]And the circumference of the greatest circle of the sphere Earth[5] is about five *farāsikh majhool*. [9]Because its diameter [the diameter of the greatest circle of the Earth] in terms of *farāsīkh* is almost five times as much as the mentioned quantity[6] [the diameter of the above circle]. [10]And in the region of *falak al-burooj* [Zodiac] [the circumference] is much more than one hundred thousand *farāsikh majhool*[7]*,* and these amounts that are [this much] large for the circumferences what are they going to be in [the calculation of the] area[s][8]? [11]This is because he [Archimedes] calculated the perimeter of the inscribed [regular] polygon with 96 sides which is less than the circumference of the circle. [12]Because each side of this [regular polygon] is smaller than the length of its corresponding arc of the circle and the sum of all sides is less than the circumference of the circle. [13]And [Archimedes] calculated the perimeter of the similar circumscribed polygon and with the help of the First Figure [Proposition] of the First Article [Book I] of his book [see Comment 4.1] he has proven that this [latter perimeter] is larger than the circum-

ference of the mentioned circle and their difference is as we stated above [$\frac{1}{497}$].

[14]But, Abu'l-Wafā' al-Būzjānī obtained an approximation for the half of the chord of $\frac{1}{360}$ of the circumference of the circle[9] with the assumption that the diameter of the circle is 120 [units] and he multiplied this result by 720 to obtain the perimeter of the inscribed [regular] polygon with 720 sides. [15]Also, he calculated the perimeter of the similar circumscribed [regular] polygon with 720 sides and has said that if the diameter of the circle is 120 [units], then its circumference would be 376 units and a fraction[10], where this fraction is more than $0;59,10,59$ and less than $0;59,23,54,12$ and the difference of these two quantities [fractions] is $0;0,12,55,12$. [16]And [this] difference in the greatest circle of the sphere Earth approximately will amount to one thousand *adru'*. [17]However, in finding the value of the chord of half of a degree, he [al-Būzjānī] has erred [see Comment 4.2]. [18]Because, he has taken the value of this chord as $0;31,24,55,54,55$ which is not correct and its true value is $0;31,24,56,38,36$. Later [in the conclusion section] we will give a reason for this.

[19]However, Abū Rayhān al-Bīrūnī calculated the chord of two parts of the 360 parts of the circumference of the circle[11] and he calculated the perimeter of the inscribed [regular] polygon with 180 sides equal to $6;16,59,10,48,0$ and the perimeter of the similar circumscribed [regular] polygon with 180 sides equal to $6;17,1,58,19,6$. [20]He has taken the half of the sum of these two [perimeters] as the circumference of the circle and with the assumption that the diameter[12] of the circle is the unit he has converted this [the circumference] into a fraction in Indian [Arabic] numerals with multiple digits in its denominator. [21]And in a circle that is equal to the greatest circle of the sphere Earth this approximately will reach one *farsakh*. [22]However, he [al-Bīrūnī] has erred in the [calculation] of the chord of a two degree [arc]. [23]Because, he has taken this chord as $0;2,5,39,43,36$ while he should have taken it as $0;2,5,39,26,22$. [24]We must know that he [al-Bīrūnī] has recorded the *sine* of the arc of one degree which is half of the chord[13] of a two degree arc in the table of sines of his *Qānoūn al-Mas'ūdī* as $0;1,2,49,43$, which is correct, but he has calculated its double incorrectly [see Comment 4.3].

[25]Since these works [calculations] were confusing we [al-Kāshī]

> wanted to determine the circumference of the circle with the assumption that its diameter is known in terms of a certain measurement in such a way that we would be certain that in a circle that its diameter is six hundred thousand times the diameter of the sphere Earth, the difference between the result of our calculations and the actual value [of the perimeter] would not reach the breadth of a hair. [26]A hair that its breadth is one sixth of the width of an average grain of barley[14] and whatever is smaller than that [that is, the difference between his value and the actual value of the perimeter] is not important. [27]And while I am asking for help from the dear bestower God Who is the guidance to the right path, I wrote this treatise on the determination of the circumference consisting of ten sections and a conclusion, and I named [entitled] it *al-Muhītīyya*.

4. Comments

Comment 4.1. We know that the famous Archimedes' bounds for π is Proposition 3 of his treatise *On the Measurement of the Circle,* which is a small treatise with only three propositions, and there is no Book I in this treatise. Therefore, al-Kāshī must has been referring to a collection of Archimedes' work which contained *On the Measurement of the Circle*. Most likely al-Kāshī was referring to *Majmū' al-Rasā'il* ("Collection of Treatises") by Nasīr al-Dīn al-Tūsī that contains the Arabic translation of *On the Measurement of the Circle* as well as other Archimedes' treatises. For a complete list of Archimedes' work and their translations throughout the history including *Majmū' al-Rasā'il* the reader is referred to [5, pp. 230-231].

Comment 4.2. Al-Kāshī did not mention the work of al-Būzjānī that al-Kāshī used as his source for pointing out al-Būzjānī's error. According to Qurbānī [9, p. 137], al-Kāshī most likely was referring to statements made by Nasīr al-Dīn al-Tūsī (1201-1274) in his *Taksīr al-Dā'irā* which is an Arabic translation of *Archimedes' Division of the Circle*. In the commentary al-Tūsī stated:

> ...For example, let us assume that [the point] C is the center of a circle and the arc AB is equal to $\frac{1}{720}$ of the circumference of that circle. We connect [draw] the chord $\overline{AB}$. Based on the mentioned principles[15] and al-Būzjānī's calculations, the length of the chord

> $\overline{AB}$ is equal to $0;31,24,55,54,55$. Now, if we take the diameter as 120 parts, then this value would be the chord of half of a degree. Next, if we take this chord as one side of the inscribed [regular] polygon with 720 sides, then the perimeter of this polygon would be $376;59,10,59$...

This is exactly what al-Kāshī attributed to al-Būzjānī in the introduction of *al-Risāla al-Muhītīyya.* However, Woepcke [14] has shown that the number that al-Tūsī has taken as the chord $\overline{AB}$ is in fact the *sine* of one half of one degree and not the chord of one half of one degree which al-Būzjānī himself obtained in his *Kitāb al-Kāmil* ("Complete Book"), which is a simplified version of Ptolemy's *Almagest.* Therefore, it was al-Kāshī who erred and not al-Būzjānī.

Comment 4.3. According to Qurbānī [9, p. 138] whatever al-Kāshī stated in the introduction of *al-Risāla al-Muhītīyya* about al-Bīrūnī and his calculation of the circumference of a circle was obtained from al-Bīrūnī's *Qānoūn al-Mas'ūdī.* In Chapter Five of Book III of *Qānoūn al-Mas'ūdī,* al-Bīrūnī recorded the chord of a two degree arc as $0;2,5,39,43,36$ which as al-Kāshī pointed out in the introduction of *al-Risāla al-Muhītīyya* is incorrect and is larger than the actual value of the chord of a two degree arc. This larger value of the chord of a two degree arc which al-Bīrūnī used as a side of a regular inscribed polygon with 180 sides resulted in a larger perimeter for this regular inscribed polygon. Also, since he used this value of the chord of a two degree arc to obtain the length of each side of a similar regular circumscribed polygon with 180 sides, again he obtained a larger value for the perimeter of the circumscribed polygon than its actual value. He used the perimeters of these inscribed and circumscribed regular polygons in a unit circle to find two approximations for π in sexagesimal system. Namely, $3;8,29,35,24$ and $3;8,30,59,10$, respectively. In the decimal system these values are 3.141742 and 3.141744 where both of these values are larger than 3.14166, the known value of π by Greeks and Indians at that time.

5. Footnotes

[1]Bahā al-Dīn al-'Āmilī was born in 1532 in Lebanon and lived there until he was 10 years old. Then, he spent the rest of his life in Iran where he died in 1610 in Isfahān, and upon his wishes he was buried in Mashhad next to the shrine of Imām Rezā.

[2]Here al-Kāshī refers to the fact that π is irrational and its exact value cannot be found.

[3]If d represents the diameter of a circle, then $3d + \frac{10}{71}d < circumference < 3d + \frac{1}{7}d$, and hence the ratio of the circumference of a circle to its diameter is less than $3\frac{1}{7}$ and larger than $3\frac{10}{71}$; that is, $3\frac{10}{71} < \pi < 3\frac{1}{7}$, the Archimedes' bounds for π.

[4]The words *majhool* and *mashkook* in both Arabic and Persian literally mean *unknown* and *suspicious,* respectively. However, here the author believes al-Kāshī meant that the error would one *dirā',* one *qasaba,* or one *farsakh.*

[5]In both the Arabic and the Persian languages, "sphere Earth" means "planet Earth".

[6]This indicates that al-Kāshī assumed that the diameter of Earth is about $5 \times 497 = 2485$ *farāsikh.*

[7]Therefore, al-Kāshī assumed that the diameter of the *falak al-burooj* is more than 49700000 *farāsikh.*

[8]That is, when the error in the calculations of the circumferences where we are just using the first power of the radius is this high, then in the calculations of areas and volumes where we use the second and the third powers of the radius will be much higher. This is for circles and spheres only, of course.

[9]That is, the chord corresponding to half of a degree.

[10]It was customary to write the integral part of a number in decimal system and the fractional part in sexagesimal system as we see here.

[11]That is, the chord of a two degree arc.

[12]According to *Qānoūn al-Mas'ūdī* this is half of the diameter (see Comment 4.3).

[13]This can be seen immediately from congruent right triangles.

[14]Here by a hair al-Kāshī meant a horse hair.

[15]That is, principles that are based on Ptolemy's *Almagest* for calculating the length of the chord of an arc of a circle, or other similar principles used by astronomers.

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