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Combinatorial results for certain semigroups of partial isometries of a finite chain*

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Abstract

Let $\mathcal{I}_n$ be the symmetric inverse semigroup on $X_n = \{1, 2, \ldots, n\}$ under composition of maps and let $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ be its subsemigroups of partial isometries and of order-preserving partial isometries of X_n, respectively. In this paper we investigate the cardinalities of some equivalences on $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ which lead naturally to obtaining the orders of these semigroups.

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1 Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$ and $\mathcal{I}_n$ be the partial one-to-one transformation semigroup on X_n under composition of mappings. Then $\mathcal{I}_n$ is an *inverse* semigroup (that is, for all $\alpha \in \mathcal{I}_n$ there exists a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). The importance of $\mathcal{I}_n$ (more commonly known as the symmetric inverse semigroup or monoid) to inverse semigroup theory may be likened to that of the symmetric group $\mathcal{S}_n$ to group theory. Every finite inverse semigroup S is embeddable in $\mathcal{I}_n$, the analogue of Cayley's theorem for finite groups, and to the regular representation of finite semigroups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of $\mathcal{I}_n$; see for example [2, 4, 5, 7, 15].
A transformation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving (order-reversing)* if $(\forall x, y \in \operatorname{Dom}\alpha)\ x \leq y \implies x\alpha \leq y\alpha\ (x\alpha \geq y\alpha)$ and, an *isometry (or distance-preserving)* if $(\forall x, y \in \operatorname{Dom}\alpha)\ | x - y |=| x\alpha - y\alpha |$. We shall denote by $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, the semigroups of partial isometries and of order-preserving partial isometries of an $n-$chain, respectively. Eventhough semigroups of partial isometries on more restrictive but richer mathematical structures have been studied by Wallen [17], and Bracci and Picasso [3] the study of corresponding semigroups on chains was only initiated recently by Al-Kharousi et al. [1]. This paper investigates the combinatorial properties of $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, thereby complementing the results in Al-Kharousi et al. [1] which dealt mainly with the algebraic and rank properties of these semigroups.

In this section we introduce basic terminologies and quote some elementary results from Section 1 of Al-Kharousi et al. [1] that will be needed in this paper. In Section 2 we obtain the cardinalities of two equivalences defined on $\mathcal{DP}_n$ and $\mathcal{ODP}_n$. These equivalences lead to formulae for the orders of $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ as well as new triangles of numbers that were as a result of this work recently recorded in [14]. Lastly, in Section 3 we computed the number of Green's $\mathcal{D}$-classes in $\mathcal{DP}_n$.

For standard concepts in semigroup and symmetric inverse semigroup theory, see for example [9, 11]. Let

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\forall x, y \in \operatorname{Dom}\alpha)\ | x - y |=| x\alpha - y\alpha |\} \tag{1}$$

be the subsemigroup of $\mathcal{I}_n$ consisting of all partial isometries of X_n. Also let

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\forall x, y \in \operatorname{Dom}\alpha)\, x \leq y \Longrightarrow x\alpha \leq y\alpha\} \tag{2}$$

be the subsemigroup of $\mathcal{DP}_n$ consisting of all order-preserving partial isometries of X_n. It is clear that if $\alpha \in \mathcal{DP}_n$ ($\alpha \in \mathcal{ODP}_n$) then $\alpha^{-1} \in \mathcal{DP}_n$ ($\alpha^{-1} \in \mathcal{ODP}_n$) also. Thus we have the following result.

Lemma 1.1 *$\mathcal{DP}_n$ and $\mathcal{ODP}_n$ are inverse subsemigroups of $\mathcal{I}_n$.*

Next, let α be in $\mathcal{I}_n$. The *height* of α is $h(\alpha) = |\operatorname{Im}\alpha|$ and *fix* of α is denoted by $f(\alpha)$, and defined by $f(\alpha) = |F(\alpha)|$, where

$$F(\alpha) = \{x \in \operatorname{Dom}\alpha : x\alpha = x\}.$$

Lemma 1.2 *[1, Lemma 1.7] Let $\alpha \in \mathcal{DP}_n$ be such that $h(\alpha) = p$. Then $f(\alpha) = 0$ or 1 or p.*

Corollary 1.3 *[1, Corollary 1.8] Let $\alpha \in \mathcal{DP}_n$. If $f(\alpha) = p > 1$ then $f(\alpha) = h(\alpha)$. Equivalently, if $f(\alpha) > 1$ then α is a partial identity.*

Lemma 1.4 *[1, Lemma 1.3] Let $\alpha \in \mathcal{DP}_n$. Then α is either order-preserving or order-reversing.*

Lemma 1.5 *[1, Lemma 1.5] Let $\alpha \in \mathcal{DP}_n$. For $1 < i < n$, if $F(\alpha) = \{i\}$ then, for all $x \in \operatorname{Dom}\alpha$, we have $x + x\alpha = 2i$.*

For convenience we shall henceforth denote the set of order-reversing isometries of X_n by $\mathcal{DP}_n^*$.

Lemma 1.6 *[1, Lemma 1.6] Let $\alpha \in \mathcal{DP}_n^*$. Then $x + x\alpha = y + y\alpha$ for all $x, y \in \operatorname{Dom}\alpha$.*

Lemma 1.7 *[1, Lemma 1.10] Let $\alpha \in \mathcal{ODP}_n$ and $f(\alpha) \geq 1$. Then α is a partial identity.*

Lemma 1.8 *[1, Lemma 1.11] Let $\alpha \in \mathcal{ODP}_n$. Then α is either strictly order-decreasing or strictly order-increasing or a partial identity.*

2 Combinatorial results

Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. Many numbers and triangle of numbers regarded as combinatorial gems like the Stirling numbers [9, pp. 42 & 96], the factorial [12, 15], the binomial [7], the Fibonacci number [8], Catalan numbers [6], Lah numbers [6, 10], etc., have all featured in these enumeration problems. For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups we refer the reader to Umar [16]. These enumeration problems lead to many numbers in Sloane's encyclopaedia of integer sequences [14] but there are also others that are not yet or have just been recorded in [14].
As in Umar [16], for natural numbers $n \geq p \geq m \geq 0$ we define

$$F(n;p) =\mid \{\alpha \in S : h(\alpha) =\mid \operatorname{Im}\alpha \mid = p\} \mid, \tag{3}$$

$$F(n;m) =\mid \{\alpha \in S : f(\alpha) = m\} \mid \tag{4}$$

where S is any subsemigroup of $\mathcal{I}_n$. Also, let $i = a_i = a$, for all $a \in \{p, m\}$, and $0 \leq i \leq n$.

Lemma 2.1 *Let $S = \mathcal{ODP}_n$. Then $F(n;p_1) = n^2$ and $F(n;p_n) = 1$, for all $n \geq 2$.*

Proof. Since all partial injections of height 1 are vacuously partial isometries, the first statement of the lemma follows immediately. For the second statement, it is not difficult to see that there is exactly one order-preserving partial isometry of height n: $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ (the identity). □

Lemma 2.2 *Let $S = \mathcal{ODP}_n$. Then $F(n;p_2) = \frac{1}{6}n(n-1)(2n-1)$, for all $n \geq 2$.*

Proof. First, we say that 2-subsets of X_n (that is, subsets of size 2), say $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, are of the same *type* if $|a_1 - a_2| = |b_1 - b_2|$. Now observe that if $|a_1 - a_2| = i$ $(1 \leq i \leq n-1)$ then there are $n - i$ subsets of this type. However, for order-preserving partial isometries, once we choose a 2-subset as a domain then the possible image sets must be of the same type and there is only one possible order-preserving bijection between any two 2-subsets of the same type. It is now clear that $F(n;p_2) = F(n;2) = \sum_{i=1}^{n-1}(n-i)^2 = \frac{1}{6}n(n-1)(2n-1)$, as required. □

Lemma 2.3 *Let $S = \mathcal{ODP}_n$. Then $F(n;p) = F(n-1;p-1) + F(n-1;p)$, for all $n \geq p \geq 3$.*

Proof. (Bijective proof.) Let $\alpha \in \mathcal{ODP}_n$ and $h(\alpha) = p$. Then it is clear that $F(n;p) = |A| + |B|$, where $A = \{\alpha \in \mathcal{ODP}_n : h(\alpha) = p \text{ and } n \notin \operatorname{Dom}\alpha \cup \operatorname{Im}\alpha\}$ and $B = \{\alpha \in \mathcal{ODP}_n : h(\alpha) = p \text{ and } n \in \operatorname{Dom}\alpha \cup \operatorname{Im}\alpha\}$. Define a map $\theta : \{\alpha \in \mathcal{ODP}_{n-1} : h(\alpha) = p\} \to A$ by $(\alpha)\theta = \alpha'$ where $x\alpha' = x\alpha\,(x \in \operatorname{Dom}\alpha)$. This is clearly a bijection since $n \notin \operatorname{Dom}\alpha \cup \operatorname{Im}\alpha$. Next, define the *right (left) waist* of α as $\varpi^+(\alpha) = \max(\operatorname{Dom}\alpha)$ $(w^+(\alpha) = \max(\operatorname{Im}\alpha))$ and a map $\Phi : \{\alpha \in \mathcal{ODP}_{n-1} : h(\alpha) = p-1\} \to B$ by $(\alpha)\Phi = \alpha'$ where

(i) $x\alpha' = x\alpha\,(x \in \operatorname{Dom}\alpha)$ and $n\alpha' = n$ (if $\varpi^+(\alpha) = w^+(\alpha)$);

(ii) $x\alpha' = x\alpha\,(x \in \operatorname{Dom}\alpha)$ and $n\alpha' = n - \varpi^+(\alpha) + w^+(\alpha) < n$ (if $\varpi^+(\alpha) > w^+(\alpha)$);

(iii) $x(\alpha')^{-1} = x\alpha^{-1}\,(x \in \operatorname{Im}\alpha)$ and $n(\alpha')^{-1} = n - \varpi^+(\alpha^{-1}) + w^+(\alpha^{-1}) < n$ (if $\varpi^+(\alpha) < w^+(\alpha)$).

In all cases $h(\alpha') = p$, and case (i) coincides with $n \in \operatorname{Dom}\alpha' \cap \operatorname{Im}\alpha'$; case (ii) coincides with $n \in \operatorname{Dom}\alpha' \setminus \operatorname{Im}\alpha'$; and case (iii) coincides with $n \in \operatorname{Im}\alpha' \setminus \operatorname{Dom}\alpha'$. Thus Φ is onto. Moreover, it is not difficult to see that α' in (i) is idempotent; α' in (ii) is (strictly) order-decreasing $(x\alpha' < x)$; and α' in (iii) is (strictly) order-increasing $(x\alpha' > x)$. Thus, Φ is one-to-one. Hence Φ is a bijection, as required. This establishes the statement of the lemma. □

Proposition 2.4 *Let $S = \mathcal{ODP}_n$. Then $F(n;p) = \frac{(2n-p+1)}{p+1}\binom{n}{p}$, where $n \geq p \geq 1$.*

Proof. (The proof is by double induction).
Basis step: First, note that the formulae for $F(n;p_1)$, $F(n;p_n)$ and $F(n;p_2)$ are true by Lemmas 2.1 and 2.2.
Inductive step: Suppose $F(n-1;p)$ is true for all $n-1 \geq p$. (This is the induction hypothesis.) Now using Lemma 2.3, we see that

$$\begin{aligned}
F(n;p) &= F(n-1;p-1) + F(n-1;p) \\
&= \frac{(2n-p)}{p}\binom{n-1}{p-1} + \frac{(2n-p-1)}{p+1}\binom{n-1}{p} \text{ (by ind. hyp.)} \\
&= \frac{(2n-p)}{p}\frac{p}{n}\binom{n}{p} + \frac{(2n-p-1)}{p+1}\frac{(n-p)}{n}\binom{n}{p} \\
&= \frac{(2n-p)(p+1) + (2n-p-1)(n-p)}{n(p+1)}\binom{n}{p} \\
&= \frac{(2n^2-np+n)}{n(p+1)}\binom{n}{p} = \frac{(2n-p+1)}{p+1}\binom{n}{p},
\end{aligned}$$

as required. □

To find the order of $\mathcal{ODP}_n$ the next lemma seems indispensable.

Lemma 2.5 *For any integer $n \geq 2$, we have*

$$\sum_{p=1}^{n} \frac{2n-p+1}{p+1}\binom{n}{p} = 3 \cdot 2^n - 2n - 3.$$

Proof. It is enough to observe that $2n-p+1 = 2(n-p) + (p+1)$. □

Theorem 2.6 *Let $S = \mathcal{ODP}_n$. Then*

$$|\ \mathcal{ODP}_n\ | = 3 \cdot 2^n - 2(n+1).$$

Proof. This follows from Proposition 2.4, Lemma 2.5, and some algebraic manipulation. □

Lemma 2.7 *Let $S = \mathcal{ODP}_n$. Then $F(n;m) = \binom{n}{m}$, for all $n \geq m \geq 1$.*

Proof. This follows directly from Lemma 1.5. □

Proposition 2.8 *Let $S = \mathcal{ODP}_n$. Then $F(n;m_0) = 2^{n+1} - (2n+1)$.*

Proof. This follows from Theorem 2.6, Lemma 2.5 and the fact that $|\mathcal{ODP}_n| = \sum_{m=0}^{n} F(n;m)$. □

Remark 2.9 *The triangles of numbers $F(n;p)$ and $F(n;m)$, the sequence $F(n;m_0)$ have as a result of this work just been recorded in Sloane [14]. However, $|\mathcal{ODP}_n|$ is [14, A097813].*

Remark 2.10 *For $p = 0, 1$ the concepts of order-preserving and order-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for $p \geq 2$; see [5, page 2, last paragraph].*

We now use Remark 2.10 and Lemma 1.6 to deduce corresponding combinatorial results for $\mathcal{DP}_n$ from those of $\mathcal{ODP}_n$ above.

Lemma 2.11 *Let $S = \mathcal{DP}_n$. Then $F(n;p_1) = F(n;1) = n^2$ and $F(n;p_n) = F(n;n) = 2$, for all $n \geq 2$.*

Lemma 2.12 *Let $S = \mathcal{DP}_n$. Then $F(n;p_2) = F(n;2) = \frac{1}{3}n(n-1)(2n-1)$, for all $n \geq 2$.*

Lemma 2.13 *Let $S = \mathcal{DP}_n$. Then $F(n;p) = F(n-1;p-1) + F(n-1;p)$, for all $n \geq p \geq 3$.*

Proposition 2.14 *Let $S = \mathcal{DP}_n$. Then $F(n;p) = \frac{2(2n-p+1)}{p+1}\binom{n}{p}$, where $n \geq p \geq 2$.*

Theorem 2.15 *Let $S = \mathcal{DP}_n$. Then*

$$\mid \mathcal{DP}_n \mid = 3 \cdot 2^{n+1} - (n+2)^2 - 1.$$

Proof. This follows from Proposition 2.14, Lemma 2.11 and some algebraic manipulation. ☐

Lemma 2.16 *Let $S = \mathcal{DP}_n$. Then $F(n;m) = \binom{n}{m}$, for all $n \geq m \geq 2$.*

Proof. This follows from Corollary 1.3. ☐

Proposition 2.17 *Let $S = \mathcal{DP}_n$. Then $F(2n;m_1) = \frac{2(2^{2n}-1)}{3}$ and $F(2n-1;m_1) = \frac{2(2^{2n-2}-1)}{3} + 2^{2n-2}$, for all $n \geq 1$.*

Proof. Let $F(\alpha) = \{i\}$. Then by Lemma 1.5, for any $x \in \operatorname{Dom}\alpha$ we have $x + x\alpha = 2i$. Thus there $2i-1$ possible elements for $\operatorname{Dom}\alpha : (x, x\alpha) \in \{(1, 2i-1), (2, 2i-2), \ldots, (2i-1, 1)\}$. However, (excluding (i,i)) we see that there are $\sum_{j=0}^{2i-2}\binom{2i-2}{j} = 2^{2i-2}$, possible partial isometries with $F(\alpha) = \{i\}$, where $2i-1 \leq n$ (equivalently, $i \leq (n+1)/2$). Moreover, by symmetry we see that $F(\alpha) = \{i\}$ and $F(\alpha) = \{n-i+1\}$ give rise to equal numbers of partial isometries. Note that if n is odd (even) the equation $i = n-i+1$ has one (no) solution. Hence, if $n = 2a-1$ we have

$$2\sum_{i=1}^{a-1} 2^{2i-2} + 2^{2a-2} = \frac{2(2^{2a-2}-1)}{3} + 2^{2a-2}$$

partial isometries with exactly one fixed point; if $n = 2a$ we have

$$2\sum_{i=1}^{a} 2^{2i-2} = \frac{2(2^{2a}-1)}{3}$$

partial isometries with exactly one fixed point. $\square$

Proposition 2.18 *Let $S = \mathcal{DP}_n$. Then*

$$F(n; m_0) = \begin{cases} \frac{13\cdot 2^n-(3n^2+9n+10)}{3}, & \text{if } n \text{ is even;} \\ \frac{25\cdot 2^{n-1}-(3n^2+9n+10)}{3}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows from Theorem 2.15, Lemma 2.16, Proposition 2.17 and the fact that $|\mathcal{DP}_n| = \sum_{m=0}^{n} F(n; m)$. $\square$

Remark 2.19 *The triangles of numbers $F(n; p)$ and $F(n; m)$ and, the sequences $|\mathcal{DP}_n|$ and $F(n; m_0)$, have as a result of this work been recorded recently in Sloane [14]. However, $F(n; m_1)$ is [14, A061547].*

3 Number of $\mathcal{D}$-classes

First, notice that from [1, Lemma 2.1] we deduce that number of $\mathcal{L}$-classes in $K(n, p) = \{\alpha \in \mathcal{DP}_n : h(\alpha) = p\}$ (as well as the number of $\mathcal{R}$-classes there) is $\binom{n}{p}$. To count the number of $\mathcal{D}$-classes in $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, first we recall (from [1]) that the *gap* and *reverse gap of the image set of α (with $h(\alpha) = p$) are ordered $(p-1)$-tuples* defined as follows:

$$g(\operatorname{Im}\alpha) = (|a_2\alpha - a_1\alpha|, |a_3\alpha - a_2\alpha|, \ldots, |a_p\alpha - a_{p-1}\alpha|)$$

and

$$g^R(\operatorname{Im}\alpha) = (|a_p\alpha - a_{p-1}\alpha|), \ldots, |a_3\alpha - a_2\alpha|, |a_2\alpha - a_1\alpha|),$$

where $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix}$ with $1 \leq a_1 < a_2 < \cdots < a_p \leq n$. Further, let $d_i = |a_{i+1}\alpha - a_i\alpha|$ for $i = 1, 2, \ldots, p-1$. Then

$$g(\operatorname{Im}\alpha) = (d_1, d_2, \ldots, d_{p-1}) \text{ and } g^R(\operatorname{Im}\alpha) = (d_{p-1}, d_{p-2}, \ldots, d_1).$$

For example, if

$$\alpha = \begin{pmatrix} 1 & 2 & 4 & 7 & 8 \\ 3 & 4 & 6 & 9 & 10 \end{pmatrix}, \ \beta = \begin{pmatrix} 2 & 4 & 7 & 8 \\ 10 & 8 & 5 & 4 \end{pmatrix} \in \mathcal{DP}_{10}$$

then $g(\operatorname{Im}\alpha) = (1, 2, 3, 1)$, $g(\operatorname{Im}\beta) = (2, 3, 1)$, $g^R(\operatorname{Im}\alpha) = (1, 3, 2, 1)$ and $g^R(\operatorname{Im}\beta) = (1, 3, 2)$.

Theorem 3.1 *[1, Theorem 2.4] Let $\mathcal{DP}_n$ be as defined in (1) and let $\alpha, \beta \in \mathcal{DP}_n$. Then*

$$(\alpha, \beta) \in \mathcal{D} \text{ if and only if } g(\operatorname{Im}\alpha) = g(\operatorname{Im}\beta) \text{ or } g^R(\operatorname{Im}\alpha) = g(\operatorname{Im}\beta).$$

Theorem 3.2 *[1, Theorem 2.5] Let $\mathcal{ODP}_n$ be as defined in (2) and let $\alpha, \beta \in \mathcal{ODP}_n$. Then*

$$(\alpha, \beta) \in \mathcal{D} \text{ if and only if } g(\operatorname{Im}\alpha) = g(\operatorname{Im}\beta).$$

Next we prove some preliminary results towards our goal.

Lemma 3.3 *Let $p - 1 \leq \sum_{i=1}^{p-1} d_i \leq n - 1$.*

Proof. The greatest lower bound is attained if $d_i = 1$ for all $i \in \{1, 2, \dots, p-1\}$ whilst the least upper bound is attained if α is order-preserving (order-reversing) and $a_1\alpha = 1, a_p\alpha = n$ $(a_1\alpha = n, a_p\alpha = 1)$. □

Let $d(n, p)$ be the number of distinct ordered p-tuples: $(d_1, d_2, \dots, d_p)$ with $\sum_{i=1}^{p} d_i = n$. This is clearly the number of *compositions* of n into p parts. Thus, we have

Lemma 3.4 *[13, p.151] $d(n, p) = \binom{n-1}{p-1}$.*

We shall henceforth use the following well-known binomial identity when needed:

$$\sum_{m=p}^{n} \binom{m}{p} = \binom{n+1}{p+1}.$$

An ordered p-tuple: $(d_1, d_2, \dots, d_p)$ is said to be *symmetric* if

$$(d_1, d_2, \dots, d_p) = (d_1, d_2, \dots, d_p)^R = (d_p, d_{p-1}, \dots, d_1).$$

Let $d_s(n, p)$ be the number of distinct symmetric ordered p-tuples: $(d_1, d_2, \dots, d_p)$ with $\sum_{i=1}^{p} d_i = n$. Then we have

Lemma 3.5 $d_s(n; p) = \begin{cases} 0, & \text{if } n \text{ is odd and } p \text{ is even;} \\ \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{p-1}{2} \rfloor}, & \text{otherwise.} \end{cases}$

Proof. First notice that for symmetric ordered p-tuples the right-half is the reverse of the left-half. Of course if p is odd then the left-half and right-half would exclude the middle term which must be even (odd) if n is even (odd), respectively. Next, observe that if p is even then

$$\sum_{i=1}^{p/2} d_i = \sum_{i=1}^{p/2} d_{p-i}$$

and so

$$\sum_{i=1}^{p} d_i \;=\; 2\sum_{i=1}^{p/2} d_i = n$$

which implies that n must be even. In other words, if n is odd and p is even then $d_s(n,p)=0$, as required. Otherwise, we have three cases:
(i) n and p are both odd.

$$d_s(n,p)=\sum_{i\geq 1} d(\frac{n-2i+1}{2},\frac{p-1}{2})=\sum_{i\geq 1}\binom{\frac{n-2i-1}{2}}{\frac{p-3}{2}}=\binom{\frac{n-1}{2}}{\frac{p-1}{2}};$$

(ii) n and p are both even.

$$d_s(n,p)=d(\frac{n}{2},\frac{p}{2})=\binom{\frac{n-2}{2}}{\frac{p-2}{2}};$$

(iii) n is even and p is odd.

$$d_s(n,p)=\sum_{i\geq 1} d(\frac{n-2i}{2},\frac{p-1}{2})=\sum_{i\geq 1}\binom{\frac{n-2i-2}{2}}{\frac{p-3}{2}}=\binom{\frac{n-2}{2}}{\frac{p-1}{2}}.$$

□

Define an equivalence R on the class of ordered p-tuples: $(d_1,d_2,\ldots,d_p)$ with $\sum_{i=1}^{p} d_i = n$ by
$(a_1,a_2,\ldots,a_p)\,R\,(b_1,b_2,\ldots,b_p)$ if and only if
$(a_1,a_2,\ldots,a_p)=(b_1,b_2,\ldots,b_p)$ *or* $(a_1,a_2,\ldots,a_p)=(b_1,b_2,\ldots,b_p)^R$.

Let $e(n,p)$ be the number of these equivalence classes. Then we have

Lemma 3.6 $e(n;p)=\begin{cases}\frac{1}{2}\binom{n-1}{p-1}, & \textit{if } n \textit{ is odd and } p \textit{ is even;}\\ \frac{1}{2}[\binom{n-1}{p-1}+\binom{\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{p-1}{2}\rfloor}], & \textit{otherwise.}\end{cases}$

Proof. The result follows from Lemmas 3.4 and 3.5 and the observation that

$$e(n,p)=\frac{d(n,p)-d_s(n,p)}{2}+d_s(n,p)=\frac{d(n,p)+d_s(n,p)}{2}.$$

□

Now we have the main result of this section.

Theorem 3.7 *Let $B(n,p)$ be the number of $\mathcal{D}$-classes of height p in $\mathcal{DP}_n$. Then $B(n,0)=1$ and for $n\geq p\geq 1$ we have*

$$B(n,p)=\begin{cases}\frac{1}{2}[\binom{n-1}{p-1}+2\binom{\frac{n-1}{2}}{\frac{p}{2}}], & \textit{if } n \textit{ is odd and } p \textit{ is even;}\\ \frac{1}{2}[\binom{n-1}{p-1}+2\binom{\frac{n-2}{2}}{\frac{p}{2}}+\binom{\frac{n-2}{2}}{\frac{p-2}{2}}], & \textit{if } n,p \textit{ are both even;}\\ \frac{1}{2}[\binom{n-1}{p-1}+\binom{\lfloor\frac{n-2}{2}\rfloor}{\frac{p-1}{2}}], & \textit{otherwise.}\end{cases}$$

Proof. The result follows from Lemma 3.6 and the fact that
$B(n,p) = \sum_{i=p-1}^{n-1} e(i, p-1)$. □

Corollary 3.8 $\sum_{p=0}^{n} B(n,p) = \begin{cases} 2^{n-2} + 2^{\frac{n}{2}}, & \textit{if } n \textit{ is even;} \\ 2^{n-2} + 3 \cdot 2^{\frac{n-3}{2}}, & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. The result follows from Theorem 3.7. □

Remark 3.9 *The triangle of numbers $B(n;p)$ and seqence $\sum_{p\geq 0} B(n,p)$ have as at the time of submitting this paper not yet been recorded in Sloane [14]. However, $d_s(n;p)$ and $e(n,p)$ are [14, A051159] and [14, A034851], respectively.*

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