THE DISTRIBUTION OF VALUES OF THE DIVISOR FUNCTION $d(n)$

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1. Throughout this note the letters p, q will be reserved for primes, and p_ν will denote the νth prime; $c_1, c_2, \ldots$ are to stand for absolute positive constants.

Let $d(n)$ denote, as usual, the number of positive divisors of n, and $D(x)$ the number of *distinct* values assumed by $d(n)$ in the range $1 \leqslant n \leqslant x$. Our principal object is to estimate the order of magnitude of $D(x)$ for large values of x.

The argument will be based on a result concerning *A-numbers*, which are defined as integers having the form

$$p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k},$$

where $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k$ and k is arbitrary. If $A(x)$ denotes the number of A-numbers not exceeding x, then, as was shown by Hardy and Ramanujan,†

$$\log A(x) \sim \frac{2\pi}{\sqrt{3}} \left(\frac{\log x}{\log\log x} \right)^{\frac{1}{2}} \quad (x \to \infty). \tag{1.1}$$

We shall be led to consider a certain subclass of the A-numbers, namely the *B-numbers*, defined as integers having the form

$$p_1^{q_1 - 1} p_2^{q_2 - 1} \ldots p_k^{q_k - 1},$$

where $q_1 \geqslant q_2 \geqslant \ldots \geqslant q_k$ and k is arbitrary. Making use of the one-one correspondence between A-numbers and B-numbers specified by the scheme

$$p_1^{a_1} \ldots p_k^{a_k} \longleftrightarrow p_1^{p_{a_1} - 1} \ldots p_1^{p_{a_k} - 1} \quad (a_1 \geqslant \ldots \geqslant a_k)$$

we shall obtain the following estimate for $B(x)$, the number of B-numbers not exceeding x.

Theorem I. *As* $x \to \infty$,

$$\log B(x) \sim \frac{2\pi\sqrt{2}}{\sqrt{3}} \frac{(\log x)^{\frac{1}{2}}}{\log\log x}.$$

This result will, in turn, lead to

Theorem II. *As* $x \to \infty$,

$$\log D(x) \sim \frac{2\pi\sqrt{2}}{\sqrt{3}} \frac{(\log x)^{\frac{1}{2}}}{\log\log x}.$$

† G. H. Hardy and S. Ramanujan, 'Asymptotic formulae for the distribution of integers of various types', *Proc. London Math. Soc.* (2), 16 (1917), 112–32. Reprinted in *Collected Papers of S. Ramanujan* (Cambridge, 1927), 245–61.

The main idea of the proof is as follows. Let m be called a *D-number* if $d(n) \neq d(m)$ for $0 < n < m$.† Then $D(x)$ is evidently the number of D-numbers not exceeding x. We shall show that every D-number is either a B-number or else does not differ greatly from a B-number. This will imply a relation between $B(x)$ and $D(x)$ which will enable us to infer Theorem II from Theorem I.

The problem of finding asymptotic formulae for $B(x)$ and $D(x)$ seems difficult. We are, however, able to obtain some results concerning the relative behaviour of $B(x)$ and $D(x)$ for large values of x. Even this is not trivial, for the relation between B and D—the sets of B-numbers and D-numbers respectively—is complicated. Thus there exist infinitely many m such that $m \in B$, $m \ni D$,‡ and infinitely many n such that $n \in D$, $n \ni B$.§ Some information on the relation between $B(x)$ and $D(x)$ is provided by

THEOREM III. *For all sufficiently large values of* x

$$D(x) - B(x) > c_1 \operatorname{logloglog} x.$$

A more interesting fact is that $B(x)$ and $D(x)$ have the same asymptotic behaviour. We shall, in fact, establish the following result.

THEOREM IV. *As* $x \to \infty$

$$\frac{D(x)}{B(x)} = 1 + O\left\{\frac{(\operatorname{loglog} x)^2}{(\log x)^{\frac{1}{2}}}\right\}.$$

A number of further questions involving $d(n)$ naturally suggest themselves. Thus, let $F(x)$ denote the greatest integer k having the property that there exists a run of k consecutive integers, say $n+1$, $n+2$,..., $n+k$, such that $n+k \leqslant x$ and $d(n+1)$, $d(n+2)$,..., $d(n+k)$ are all distinct. We shall prove

THEOREM V. *For all sufficiently large values of* x

$$F(x) > c_2 \frac{(\log x)^{\frac{1}{2}}}{\operatorname{loglog} x}.$$

As regards an upper bound for $F(x)$ we can at present prove nothing better than

$$F(x) < \exp\left\{c_3 \frac{(\log x)^{\frac{1}{2}}}{\operatorname{loglog} x}\right\},$$

an estimate which follows trivially from Theorem II. We conjecture that the true order of magnitude of $F(x)$ is $(\log x)^{c_4}$.

† It is almost obvious that a D-number is necessarily an A-number.

‡ The symbol $a \in X$ means that a is a member of the set X; $a \ni X$ means the contrary.

§ For let $k \geqslant 3$. If $m_k = p_1 \ldots p_k$, then $m_k \in B$, $m_k \ni D$. Also, if $n_k \in D$, $d(n_k) = 2^k$, then $n_k \ni B$.

A related problem consists in the estimation of the longest run of consecutive integers $\leqslant x$ all of which have the same number of divisors. This problem seems to be one of exceptional difficulty, and we have not been able to make any progress with it. We are not even able to prove that there exist infinitely many integers n for which $d(n) = d(n+1)$.

Let $\lambda(x)$ denote the least positive integer which does *not* occur among the numbers $d(n)$, $1 \leqslant n \leqslant x$. We shall conclude our note with the proof of the following result.

THEOREM VI. *For $x \geqslant 6$, $\lambda(x)$ is equal to the least prime q satisfying the inequality $2^{q-1} > x$.*

2. The notation to be used below is as follows:

The letter ϵ denotes an arbitrarily small positive number; x denotes a sufficiently large number, i.e. a number exceeding a suitable absolute constant.†

The O-notation and the asymptotic formulae refer to the passage $x \to \infty$.

As usual $\pi(x)$ stands for the number of primes not exceeding x, and $\vartheta(x)$ for the sum $\sum_{p \leqslant x} \log p$.

The set of A-numbers will be denoted by A.

Given any k, there obviously exists a unique $m \in D$ such that $d(m) = k$. Moreover, there exists a unique $m^* \in B$ such that‡ $d(m^*) = k$. Hence a one–one correspondence can be set up between B and D, specified by the conditions

$$d(m) = d(m^*), \qquad m \in D, \qquad m^* \in B.$$

If $m \in D$, then m^* will invariably denote the B-number corresponding to m. It is, of course, obvious that $m^* \geqslant m$.

If
$$n = p_1^{a_1} \dots p_k^{a_k}, \tag{2.1}$$
we shall write
$$\{n\} = p_1^{a_1'} \dots p_k^{a_k'},$$
where $a_1', \dots, a_k'$ is a permutation of $a_1, \dots, a_k$ such that $a_1' \geqslant \dots \geqslant a_k'$. Evidently $\{n\} \leqslant n$.

If the canonical representation of an integer n is written in the form (2.1), it will be referred to as its *expansion*. We shall also speak of the expansion of n with exponents $> l$, of the expansion of n with primes $> z$, and so on, when referring to the *parts* of the expansion of n having these properties.

3. We shall make frequent use of two simple lemmas.

† Whenever x is required to exceed a bound depending on ϵ, that fact will be stated explicitly.

‡ For, if $k = q_1 \dots q_s$, where $q_1 \geqslant \dots \geqslant q_s$, then $m^* = p_1^{q_1-1} \dots p_s^{q_s-1}$.

Lemma 1. *Let n, t be positive integers. If $N(n,t)$ denotes the number of sets of integers $u_1, \ldots, u_n$ such that*

$$t \geqslant u_1 \geqslant \ldots \geqslant u_n \geqslant 0, \tag{3.1}$$

then

$$N(n,t) \leqslant (2n)^{2t}.$$

Consider the sets of integers $k_0, k_1, \ldots, k_t$ such that

$$k_0+\ldots+k_t = n; \qquad k_0, \ldots, k_t \geqslant 0. \tag{3.2}$$

A one–one correspondence between the sets $u_1, \ldots, u_n$ satisfying (3.1) and the sets $k_0, \ldots, k_t$ satisfying (3.2) may be established by the requirement that k_ν should be the number of u's having the value ν. Hence $N(n,t)$ is equal to the number of sets of k's satisfying (3.2), and therefore

$$N(n,t) \leqslant (n+1)^{t+1} \leqslant (2n)^{2t}.$$

Lemma 2. *If $m = p_1^{a_1} \ldots p_k^{a_k} \in D$ and, for some i, $a_i+1 = tt'$, where $t \geqslant t' \geqslant 2$, then*

$$p_i^t \leqslant p_{k+1}, \tag{3.3}$$

and, if m is sufficiently large,

$$t < 2 \log\log m, \qquad a_i+1 < 4(\log\log m)^2.$$

Write

$$m_1 = p_1^{a_1} \ldots p_{i-1}^{a_{i-1}} p_i^{t-1} p_{i+1}^{a_{i+1}} \ldots p_k^{a_k} p_{k+1}^{t'-1}.$$

Then $d(m_1) = d(m)$, and so $m_1 \geqslant m$. Hence (3.3) follows at once. Moreover, if m is sufficiently large,

$$p_k < \tfrac{3}{2} \log m, \qquad p_{k+1} < 2 \log m,$$

$$t \leqslant \frac{\log p_{k+1}}{\log p_i} < \frac{\log(2 \log m)}{\log 2} < 2 \log\log m,$$

$$a_i+1 \leqslant t^2 < 4(\log\log m)^2.$$

4. In this and the next section we give a proof of Theorem I. We shall write

$$y = x^{(2-\epsilon)/\log\log x},$$

and shall assume that $x > x_0(\epsilon)$.

Let $m \in A$, $m \leqslant y$, and suppose that $p > (\log x)^{\frac{1}{2}}/(\log\log x)^2$ is a prime factor of m. Then the exponent of p is $< 3(\log x)^{\frac{1}{2}} \log\log x$; for otherwise

$$y \geqslant m \geqslant \Big(\prod_{q \leqslant p} q\Big)^{3(\log x)^{\frac{1}{2}} \log\log x},$$

$$(2-\epsilon) \frac{\log x}{\log\log x} \geqslant 3(\log x)^{\frac{1}{2}} \log\log x \,.\, \vartheta(p)$$

$$\geqslant 3(\log x)^{\frac{1}{2}} \log\log x \,.\, \tfrac{3}{4} p$$

$$> 3(\log x)^{\frac{1}{2}} \log\log x \,.\, \frac{3}{4} \frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2} = \frac{9}{4} \frac{\log x}{\log\log x}.$$

Two A-numbers $\leqslant y$ will be said to belong to the same class if they coincide in their expansions with prime factors $> (\log x)^{\frac{1}{2}}/(\log\log x)^2$. In view of the remark made above the corresponding exponents must all be $< 3(\log x)^{\frac{1}{2}}\log\log x$.

An A-number

$$m = p_1^{a_1}...p_k^{a_k} \leqslant y \quad (a_1 \geqslant ... \geqslant a_k)$$

will be called *restricted* if $a_1 \leqslant 3(\log x)^{\frac{1}{2}}\log\log x$. The number of such numbers will be denoted by $A_R(x)$.

We note at once that each class contains at least one restricted number.† Hence $A_R(x) \geqslant C(x)$, where $C(x)$ is the number of classes. If $K_i(x)$ denotes the number of numbers in the ith class, then

$$A(y) = K_1(x)+K_2(x)+...+K_{C(x)}(x).$$

But each class contains fewer than

$$\left(\frac{\log x}{\log 2}\right)^{3(\log x)^{\frac{1}{2}}/(\log\log x)^3} < \exp\left\{4\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right\}$$

numbers. For the number of primes $\leqslant (\log x)^{\frac{1}{2}}/(\log\log x)^2$ is less than $3(\log x)^{\frac{1}{2}}/(\log\log x)^3$, and each exponent is $\leqslant \log x/\log 2$. Hence

$$A(y) < C(x)\exp\left\{4\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right\},$$

$$A_R(x) \geqslant C(x) > A(y)\exp\left\{-4\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right\},$$

and so, by (1.1), $$\log A_R(x) > \frac{2\pi\sqrt{(2-2\epsilon)}}{\sqrt{3}}\frac{(\log x)^{\frac{1}{2}}}{\log\log x}. \tag{4.1}$$

Now let $m = p_1^{a_1}...p_k^{a_k}$ be any restricted A-number ($\leqslant y$). We associate with it the unique B-number $\bar{m}$ defined as

$$\bar{m} = p_1^{p_{a_1}-1}...p_k^{p_{a_k}-1}.$$

Different A-numbers have clearly different B-numbers associated with them.

Now, by the prime number theorem, we have, for $a_i > \alpha = \alpha(\epsilon)$,

$$p_{a_i}-1 < (1+\tfrac{1}{8}\epsilon)a_i\log a_i.$$

Since m is restricted, $a_i \leqslant 3(\log x)^{\frac{1}{2}}\log\log x$. Hence, for $a_i > \alpha$,

$$p_{a_i}-1 < (1+\tfrac{1}{8}\epsilon)(\tfrac{1}{2}+\tfrac{1}{8}\epsilon)\log\log x\,.\,a_i$$
$$< (\tfrac{1}{2}+\tfrac{1}{4}\epsilon)\log\log x\,.\,a_i.$$

† For take any A-number $\leqslant y$. If it is not restricted, we can make it so by replacing each exponent which exceeds $3(\log x)^{\frac{1}{2}}\log\log x$ by $[3(\log x)^{\frac{1}{2}}\log\log x]$.

When $a_i \leqslant \alpha$ this relation is, of course, trivially true. We therefore have

$$\bar{m} < (p_1^{a_1} \dots p_k^{a_k})^{(\frac{1}{2}+\frac{1}{4}\epsilon)\log\log x} = m^{(\frac{1}{2}+\frac{1}{4}\epsilon)\log\log x}$$

$$\leqslant y^{(\frac{1}{2}+\frac{1}{4}\epsilon)\log\log x} < x.$$

Hence $$B(x) \geqslant A_R(x),$$

and so, by (4.1), $$\log B(x) > \frac{2\pi\sqrt{(2-2\epsilon)}}{\sqrt{3}} \frac{(\log x)^{\frac{1}{2}}}{\log\log x}. \tag{4.2}$$

5. Consider next the B-numbers $\leqslant x$. Two such numbers will be regarded as belonging to the same class if they coincide in their expansions with exponents $> (\log x)^{\frac{1}{2}}/(\log\log x)^3$.

All prime factors of a B-number $\leqslant x$ are $< 2\log x$. Hence, by virtue of Lemma 1, each class contains at most

$$(4\log x)^{2(\log x)^{\frac{1}{2}}/(\log\log x)} < \exp\left\{3\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right\}$$

numbers.

Let a B-number $\leqslant x$ be called *restricted* if all its exponents are greater than $(\log x)^{\frac{1}{2}}/(\log\log x)^3$, and denote the number of such numbers by $B_R(x)$. Then, for a given x, every class of B-numbers contains precisely one restricted number, and so $B_R(x) = \bar{C}(x)$, where $\bar{C}(x)$ denotes the number of classes. Hence, if $\bar{K}_i(x)$ is the number of numbers in the ith class, then

$$B(x) = \bar{K}_1(x) + \dots + \bar{K}_{\bar{C}(x)}(x),$$

and therefore

$$B(x) < \bar{C}(x)\exp\left\{3\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right\}$$

$$= B_R(x)\exp\left\{3\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right\}. \tag{5.1}$$

With every restricted B-number $m = p_1^{a_1} \dots p_k^{a_k}$ we associate the unique A-number

$$\bar{m} = p_1^{b_1} \dots p_k^{b_k},$$

where $a_i + 1 = p_{b_i}$ $(1 \leqslant i \leqslant k)$. It is then clear that $m_1 \neq m_2$ implies $\bar{m}_1 \neq \bar{m}_2$.

By the prime number theorem we have

$$b_i = \pi(a_i+1) < (1+\epsilon)\frac{a_i}{\log a_i}.$$

But, since m is restricted,

$$a_i > \frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^3}.$$

Therefore $$b_i < (2+3\epsilon)\frac{a_i}{\operatorname{loglog} x},$$

and so $$\bar{m} < (p_1^{a_1}...p_k^{a_k})^{(2+3\epsilon)/\operatorname{loglog} x} \leqslant x^{(2+3\epsilon)/\operatorname{loglog} x}.$$

This implies $$B_R(x) \leqslant A(x^{(2+3\epsilon)/\operatorname{loglog} x});$$

therefore, by (5.1) and (1.1),

$$\log B(x) < \frac{2\pi\sqrt{(2+4\epsilon)}}{\sqrt{3}}\frac{(\log x)^{\frac{1}{2}}}{\operatorname{loglog} x}. \tag{5.2}$$

Theorem I now follows at once from (4.2) and (5.2).

6. We shall next deduce Theorem II. Let $m = p_1^{\nu_1}...p_k^{\nu_k}$ $(\nu_1,..., \nu_k > 0)$ be a sufficiently large number and suppose that $\nu_1+1,..., \nu_k+1$ are not all primes. Consider, for definiteness, the smallest composite ν_i+1 and write

$$\nu_i+1 = tq,$$

where q is the least prime factor of ν_i+1; then $t \geqslant q \geqslant 2$. Put

$$n = p_1^{\nu_1}...p_{i-1}^{\nu_{i-1}}p_i^{t-1}p_{i+1}^{\nu_{i+1}}...p_k^{\nu_k}p_{k+1}^{q-1},$$

and $m' = \{n\}$. Then $$m' < m\exp\{(\log m)^{\frac{1}{2}}\}. \tag{6.1}$$

This inequality is established by almost the same argument as Lemma 2. Suppose first that $n \geqslant m$. Since

$$\frac{n}{m} = \left(\frac{p_{k+1}}{p_i^t}\right)^{q-1}$$

this implies $$p_i^t \leqslant p_{k+1} < 2\log m,$$

$$q \leqslant t < 2\operatorname{loglog} m,$$

$$m' \leqslant n < mp_{k+1}^{q-1} < m(2\log m)^{2\operatorname{loglog} m} < m\exp\{(\log m)^{\frac{1}{2}}\}.$$

Hence (6.1) holds when $n \geqslant m$; when $n < m$ it holds, of course, trivially.

Now consider a sufficiently large number m such that $m \in D$, $m \ni B$, $m \leqslant x$. We shall then obtain an upper bound for m^*. First construct m' by the process described above; if $m' \ni B$, construct $(m')' = m''$; if $m'' \ni B$ construct $(m'')' = m'''$, and so on. After a suitable number of steps we shall obtain a number $m^{(r)}$ such that $m^{(r)} \in B$, $m^{(r-1)} \ni B$. Since, moreover, $d(m^{(r)}) = d(m)$ it follows that $m^* = m^{(r)}$. In view of (6.1) we have

$$m^{(\nu)} < m^{(\nu-1)}\exp\{(\log m^{(\nu-1)})^{\frac{1}{2}}\} \quad (1 \leqslant \nu \leqslant r), \tag{6.2}$$

where $m^{(0)} = m$.

To estimate r we note that, if $m = p_1^{a_1}...p_k^{a_k}$ $(\in D)$, then

$$r = \sum_{\kappa=1}^{k}\{\Omega(a_\kappa+1)-1\},$$

where $\Omega(u)$ denotes the total number of prime factors of u, multiple ones being counted multiply. If $\Omega(a_\kappa+1) > 1$ (i.e. if $a_\kappa+1$ is composite), then, by Lemma 2,

$$a_\kappa+1 < 4(\log\log m)^2,$$
$$p_\kappa \leqslant p_{k+1}^{\frac{1}{2}} < 2(\log m)^{\frac{1}{2}},$$
$$\kappa < 2(\log m)^{\frac{1}{2}}.$$

Hence

$$r \leqslant \sum_{\substack{1\leqslant\kappa\leqslant k\\ \Omega(a_\kappa+1)>1}} (a_\kappa+1) < 4(\log\log m)^2 \sum_{\substack{1\leqslant\kappa\leqslant k\\ \Omega(a_\kappa+1)>1}} 1$$
$$< 8(\log m)^{\frac{1}{2}}(\log\log m)^2,$$

and so

$$r < (\log x)^{\frac{2}{3}}. \tag{6.3}$$

Let now the numbers $m_0, m_1,\ldots, m_r$ be defined by the relations

$$m_0 = m,$$
$$m_\nu = m_{\nu-1}\exp\{(\log m_{\nu-1})^{\frac{1}{4}}\} \quad (1 \leqslant \nu \leqslant r),$$

so that $m < m_1 < \ldots < m_r$.

If $\frac{1}{2}x \leqslant m \leqslant x$, then

$$\log m_\nu = \log m_{\nu-1} + (\log m_{\nu-1})^{\frac{1}{4}}$$
$$< \log m_{\nu-1}.\{1+(\log \tfrac{1}{2}x)^{-\frac{3}{4}}\},$$

and so, by (6.3),

$$\log m_r < \log m.\{1+(\log \tfrac{1}{2}x)^{-\frac{3}{4}}\}^r$$
$$< \log x.\{1+(\log \tfrac{1}{2}x)^{-\frac{3}{4}}\}^{(\log x)^{\frac{2}{3}}},$$
$$m_r < x^{1+\epsilon},$$

provided $x > x_0(\epsilon)$. This estimate is still true for $m < \frac{1}{2}x$, for in that case, putting $\mu = \frac{1}{2}x$, we obtain $m_r \leqslant \mu_r < x^{1+\epsilon}$.

Thus, for all sufficiently large m such that $m \in D$, $m \ni B$, $m \leqslant x$ we have, using (6.2),

$$m^{(r)} < m_r < x^{1+\epsilon},$$

and therefore

$$m^* < x^{1+\epsilon} \quad (x > x_0).$$

If $m \in D$, $m \in B$, $m \leqslant x$, then, of course, $m^* = m$ and the above inequality is still true. Finally, it is obviously true when m is small. Hence

$$D(x) \leqslant B(x^{1+\epsilon}) \quad (x > x_0). \tag{6.4}$$

Moreover, since $m \leqslant m^*$, we have

$$B(x) \leqslant D(x), \tag{6.5}$$

and Theorem II now follows by (6.4), (6.5), and Theorem I.

7. Theorem III can be established very rapidly. Let t and r be defined by the inequalities

$$\begin{aligned} p_1 \ldots p_t \leqslant x < p_1 \ldots p_t\, p_{t+1},\\ 2^{2^r-1} < p_t < 2^{2^{r+1}-1}, \end{aligned} \tag{7.1}$$

and write $$a_\nu = 2^{2^\nu-1}p_2 p_3 \ldots p_{t-1} \quad (\nu = 3, 4, \ldots, r).$$

Then $$a_\nu < p_2 \ldots p_t < x, \qquad d(a_\nu) = 2^{t+\nu-2}.$$

Denote, for $3 \leqslant \nu \leqslant r$, by m_ν the D-number satisfying

$$d(m_\nu) = 2^{t+\nu-2}. \tag{7.2}$$

Then clearly $m_\nu < x$ whilst, in view of (7.2) and (7.1), we have

$$m_\nu^* = p_1 \cdots p_{t+\nu-2} \geqslant p_1 \cdots p_{t+1} > x.$$

Thus there exist at least $r-2$ D-numbers $\leqslant x$ (namely $m_3, \ldots, m_r$) for which the corresponding B-numbers exceed x. Hence

$$D(x) - B(x) \geqslant r-2 > c_1 \operatorname{logloglog} x.$$

8. The next three sections contain the proof of Theorem IV. We have

$$D(x) = B(x) + D_1(x), \tag{8.1}$$

where $D_1(x)$ is the number of $m \in D$ with $m \leqslant x$, $m^* > x$. It is clear that $D_1(x)$ only enumerates D-numbers which are not B-numbers.

If $m \in D$, $m \ni B$, and $m = p_1^{a_1} \ldots p_k^{a_k}$, then at least one $a_i + 1$ is composite. In that case we call a_i a *critical exponent* and p_i a *critical prime* of m.

Let $D_2(x)$ be the number of numbers counted in $D_1(x)$ which possess at least one critical prime $< 2(\log x)^{\frac{1}{2}}$, and let $D_3(x)$ be the number of numbers counted in $D_1(x)$ all of whose critical primes are $\geqslant 2(\log x)^{\frac{1}{2}}$.

We first estimate $D_2(x)$. If $p_i < 2(\log x)^{\frac{1}{2}}$ is a critical prime of $m \in D$ and $a_i + 1 = tq$ $(t \geqslant q \geqslant 2)$, then, by Lemma 2,

$$a_i < t^2 < 4(\operatorname{loglog} x)^2. \tag{8.2}$$

A critical prime p_i can be chosen in at most $2 \log x$ ways, and, by (8.2), the corresponding critical exponent in at most $4(\operatorname{loglog} x)^2$ ways. The expansion of m preceding p_i can be chosen in at most

$$\left(\frac{\log x}{\log 2}\right)^{2(\log x)^{\frac{1}{2}}}$$

ways. Again, by Lemma 1 and (8.2), the expansion of m succeeding p_i can be chosen in at most

$$(4 \log x)^{8(\operatorname{loglog} x)^2}$$

ways. Hence

$$D_2(x) \leqslant 2 \log x \,.\, 4(\operatorname{loglog} x)^2 \left(\frac{\log x}{\log 2}\right)^{2(\log x)^{\frac{1}{2}}} (4 \log x)^{8(\operatorname{loglog} x)^2}$$

$$< \exp\{c_5 (\log x)^{\frac{1}{2}} \operatorname{loglog} x\}.$$

Hence, by Theorem I and (8.1),

$$D(x) < B(x) + (\log x)^{-1} B(x) + D_3(x).$$

In view of (6.5) the proof of Theorem IV will be complete if we can show that

$$D_3(x) < c_6 \frac{(\log\log x)^2}{(\log x)^{\frac{1}{3}}} B(x). \tag{8.3}$$

9. We shall denote by D_3 the set of integers counted in $D_3(x)$. We observe that every critical exponent of every $m \in D_3$ must be equal to 3. For if $a_i > 3$ were a critical exponent, then it would be possible to write

$$a_i+1 = tq,$$

where $t \geqslant 3$, $q \geqslant 2$; in view of Lemma 2 this would imply

$$p_i^3 \leqslant p_{k+1} < 2\log x, \qquad p_i < 2(\log x)^{\frac{1}{3}},$$

which is contrary to hypothesis. Thus every $m \in D_3$ has the form

$$m = \mu p_{r+1}^3 \ldots p_s^3 p_{s+1}^2 \ldots p_t^2 p_{t+1} \ldots p_u. \tag{9.1}$$

Here the squares or the first powers of primes, or both, might be missing. The letter μ denotes the expansion of m with exponents > 3, say

$$\mu = p_1^{a_1} \ldots p_r^{a_r},$$

where $a_1 \geqslant \ldots \geqslant a_r > 3$ and $a_1+1,\ldots, a_r+1$ are all primes. We shall continue to use μ in this sense throughout this and the next section and shall refer to it as the *kernel* of m.

Since $m \in D_3$ we have $m \ni B$. Hence the sequence of numbers

$$m = m^{(0)}, m', m'', \ldots, m^{(i-1)}, m^{(i)} = m^*$$

can be constructed as in § 6. Now $m \leqslant x$, $m^* > x$ and therefore there exists a smallest value of k (depending on m) such that

$$m^{(k)} \leqslant x, \qquad m^{(k+1)} > x.$$

We shall write

$$m^{(k)} = \bar{m}.$$

It is then clear that $\bar{m}' > x$. The correspondence $m \to \bar{m}$ is unique and, since $d(\bar{m}) = d(m)$, it follows that $m_1 \neq m_2$ implies $\bar{m}_1 \neq \bar{m}_2$. The number of numbers in D_3 is therefore equal to the number of $\bar{m}$.

Now if m is given by (9.1), $\bar{m}$ will have the form

$$\bar{m} = \mu p_{r+1}^3 \ldots p_\sigma^3 p_{\sigma+1}^2 \ldots p_\tau^2 p_{\tau+1} \ldots p_\omega \quad (\sigma > r).$$

Here the squares or the first powers, or both, might be missing. By Lemma 2 (applied to m) we know that

$$p_s^2 \leqslant p_{u+1}. \tag{9.2}$$

But $\sigma \leqslant s$, and so

$$2(\log x)^{\frac{1}{3}} \leqslant p_{r+1} < \ldots < p_\sigma < 2(\log x)^{\frac{1}{2}}. \tag{9.3}$$

Let C_μ be the number of numbers in D_3 having kernel μ, i.e. the number of $\bar{m}$ with kernel μ. We shall show that

$$C_\mu < c_7 \log x. \tag{9.4}$$

A number $\bar{m}$ associated with μ may be of the following four types:

(i) $\mu p_{r+1}^3 \cdots p_\sigma^3 p_{\sigma+1}^2 \cdots p_\tau^2 p_{\tau+1} \cdots p_\omega \quad (\omega > \tau > \sigma > r)$;

(ii) $\mu p_{r+1}^3 \cdots p_\sigma^3 p_{\sigma+1} \cdots p_\tau \quad (\tau > \sigma > r)$;

(iii) $\mu p_{r+1}^3 \cdots p_\sigma^3 p_{\sigma+1}^2 \cdots p_\tau^2 \quad (\tau > \sigma > r)$;

(iv) $\mu p_{r+1}^3 \cdots p_\sigma^3 \quad (\sigma > r)$.

Let the contributions of these four types to C_μ be denoted by

$$C_\mu^{(\nu)} \quad (\nu = 1, 2, 3, 4).$$

We shall estimate these expressions in turn.

First let $\bar{m}$ be of type (i). It is then easily verified that†

$$\bar{m}' = \mu p_{r+1}^3 \cdots p_{\sigma-1}^3 p_\sigma^2 \cdots p_{\tau-1}^2 p_\tau \cdots p_\omega p_{\omega+1}.$$

Hence
$$\frac{\bar{m}'}{\bar{m}} = \frac{p_{\omega+1}}{p_\sigma p_\tau},$$

and since $\bar{m} \leqslant x < \bar{m}'$ this implies

$$p_\tau < \frac{p_{\omega+1}}{p_\sigma} < \frac{2\log x}{p_\sigma}. \tag{9.5}$$

Again, by (9.3),

$$\frac{x}{\bar{m}} < \frac{\bar{m}'}{\bar{m}} < \frac{p_{\omega+1}}{p_\sigma^2} < \frac{2\log x}{4(\log x)^{\frac{2}{3}}} = \tfrac{1}{2}(\log x)^{\frac{1}{3}},$$

and therefore
$$2x(\log x)^{-\frac{1}{3}} < \bar{m} \leqslant x. \tag{9.6}$$

In view of the construction of $\bar{m}$ it is clear that $\omega \geqslant u$. Hence, by (9.2) and (9.3),

$$p_{\omega+1} \geqslant p_{u+1} \geqslant p_{r+1}^2 \geqslant 4(\log x)^{\frac{2}{3}}. \tag{9.7}$$

Now $C_\mu^{(1)}$ is equal to the number of possible choices of σ, τ, ω. In view of (9.5) and (9.3) the number of choices of σ, τ does not exceed

$$\sum_{2(\log x)^{\frac{1}{3}} \leqslant p < 2(\log x)^{\frac{1}{2}}} \frac{2\log x}{p} < c_8 \log x.$$

To any given values of σ and τ corresponds at most one value of ω. For suppose, if possible, that $\bar{m}_1$ and $\bar{m}_2$ ($> \bar{m}_1$) have the same values of σ, τ but different values of ω. Then, by (9.7),

$$\frac{\bar{m}_2}{\bar{m}_1} \geqslant 4(\log x)^{\frac{2}{3}}, \tag{9.8}$$

and this is impossible since, by (9.6), both $\bar{m}_1$ and $\bar{m}_2$ lie in the interval $(2x(\log x)^{-\frac{1}{3}}, x)$. Thus we have

$$C_\mu^{(1)} < c_8 \log x. \tag{9.9}$$

Next, let $\bar{m}$ be of type (ii). We then have

$$\bar{m}' = \mu p_{r+1}^3 \cdots p_{\sigma-1}^3 p_\sigma p_{\sigma+1} \cdots p_\tau p_{\tau+1},$$

† If $\sigma = r+1$, then there are, in fact, no cubes in the expansion of $\bar{m}'$.

and $C_\mu^{(2)}$ is equal to the number of possible choices of σ, τ. Now σ can obviously be chosen in at most $2\log x$ ways, and to each σ there corresponds at most one τ. For we have

$$p_\sigma \geqslant 2(\log x)^{\frac{1}{3}},$$

and therefore

$$1 < \frac{\bar{m}'}{\bar{m}} = \frac{p_{\tau+1}}{p_\sigma^2} < \frac{2\log x}{4(\log x)^{\frac{2}{3}}} = \tfrac{1}{2}(\log x)^{\frac{1}{3}}.$$

Hence (9.6) continues to hold in the present case. Moreover, m must have the form

$$m = \mu p_{r+1}^3 \cdots p_s^3 p_{s+1} \cdots p_t.$$

Hence, by Lemma 2, $p_{t+1} \geqslant p_s^2$, and so

$$p_{\tau+1} \geqslant p_\sigma^2 \geqslant 4(\log x)^{\frac{2}{3}}.$$

Thus, if $\bar{m}_1$ and $\bar{m}_2$ ($> \bar{m}_1$) have the same value of σ but different values of τ, then (9.8) is still valid. But (9.6) and (9.8) are incompatible, and so our assumption is untenable. We therefore have

$$C_\mu^{(2)} < 2\log x. \tag{9.10}$$

If $\bar{m}$ is of type (iii), then

$$\bar{m}' = \mu p_{r+1}^3 \cdots p_{\sigma-1}^3 p_\sigma^2 \cdots p_{\tau-1}^2 p_\tau p_{\tau+1},$$

and so

$$1 < \frac{\bar{m}'}{\bar{m}} = \frac{p_{\tau+1}}{p_\sigma p_\tau}.$$

But $p_{\tau+1}/p_\tau < 2$, and therefore this inequality cannot be satisfied. Hence

$$C_\mu^{(3)} = 0. \tag{9.11}$$

Again, if $\bar{m}$ is of type (iv), then

$$\bar{m}' = \mu p_{r+1}^3 \cdots p_{\sigma-1}^3 p_\sigma p_{\sigma+1},$$

$$1 < \frac{\bar{m}'}{\bar{m}} = \frac{p_{\sigma+1}}{p_\sigma^2}.$$

But $p_{\sigma+1} < p_\sigma^2$, and so we have a contradiction. Thus

$$C_\mu^{(4)} = 0. \tag{9.12}$$

The relation (9.4) now follows by virtue of (9.9), (9.10), (9.11), and (9.12), and we therefore have

$$D_3(x) = \sum_\mu C_\mu \leqslant hc_7 \log x, \tag{9.13}$$

where h denotes the number of different kernels in D_3.

10. If μ is the kernel of a number in D_3, we shall denote by B_μ the number of B-numbers $\leqslant x$ with μ as their kernel. We shall show that

$$B_\mu \geqslant c_9 \frac{(\log x)^{\frac{1}{3}}}{(\log\log x)^2}. \tag{10.1}$$

Let μ be the kernel of a number $m \in D_3$ given by (9.1). Then, since

$$2(\log x)^{\frac{1}{4}} \leqslant p_{r+1} < 2(\log x)^{\frac{1}{2}},$$

we have, by Lemma 2, $\qquad p_{u+1} \geqslant 4(\log x)^{\frac{3}{4}}.$

Hence
$$2(\log x)^{\frac{3}{4}} \leqslant \vartheta(p_u) - \vartheta(p_r). \tag{10.2}$$

Write
$$\xi = \frac{(\log x)^{\frac{3}{4}}}{2 \operatorname{loglog} x},$$

and consider the number

$$n = \mu p_{r+1}^2 \cdots p_i^2 p_{i+1} \cdots p_j,$$

where $r < i < j \leqslant \xi$. Then $p_j \leqslant \frac{1}{2}(\log x)^{\frac{3}{4}}$, and

$$n \leqslant \mu \exp\{2\vartheta(p_j)\} \leqslant \mu \exp\{2(\log x)^{\frac{3}{4}}\}.$$

Hence, by (10.2),

$$n \leqslant \mu \exp\{\vartheta(p_u) - \vartheta(p_r)\} = \mu p_{r+1} \cdots p_u \leqslant m \leqslant x.$$

It follows that n is enumerated by B_μ, and therefore

$$B_\mu \geqslant \sum_{r<i<j\leqslant\xi} 1 \geqslant \tfrac{1}{3}\xi^2 = c_9 \frac{(\log x)^{\frac{3}{2}}}{(\operatorname{loglog} x)^2}.$$

This establishes (10.1) which, in turn, implies

$$B(x) \geqslant \sum_{\mu} B_\mu \geqslant h c_9 \frac{(\log x)^{\frac{3}{2}}}{(\operatorname{loglog} x)^2} \tag{10.3}$$

(the summation being extended over the kernels of numbers in D_3). The relation (8.3) now follows at once by (9.13) and (10.3), and the proof of Theorem IV is therefore complete.

11. In this section the letter r, as well as p and q, is reserved for primes. To prove Theorem V, let

$$k = \left[\frac{(\log x)^{\frac{1}{2}}}{2 \operatorname{loglog} x}\right]. \tag{11.1}$$

As previously, $p_1, \ldots, p_k$ denote the first k primes; $q_1, \ldots, q_k$ now denote, in that order, the first k primes exceeding $(\log x)^{\frac{1}{2}}$. We shall write $q = q_1$. Each p_i is less than $(\log x)^{\frac{1}{2}}$ and each q_i is less than $2(\log x)^{\frac{1}{2}}$. Hence

$$M = p_1^{q_1} \ldots p_k^{q_k} < x^{\frac{1}{2}}. \tag{11.2}$$

Let T be the set of positive integers t satisfying the system of congruences

$$t + \nu \equiv p_\nu^{q_\nu - 1} \pmod{p_\nu^{q_\nu}} \quad (\nu = 1, 2, \ldots, k)$$

and the inequality $t + k \leqslant x$. When $1 \leqslant \mu \leqslant k$ and $r \neq p_\mu$, let $T(\mu, r)$ denote† the set of those numbers $t \in T$ which satisfy the additional congruence

$$t + \mu \equiv 0 \pmod{r^{q-1}}.$$

† Whenever the symbol $T(\mu, r)$ is subsequently used, it will be understood that $1 \leqslant \mu \leqslant k$, $r \neq p_\mu$.

Finally, let T^* be the set of those numbers $t \in T$ which do not belong to any $T(\mu, r)$.

The theorem will have been established if we can prove that T^* is not empty. For, if $t \in T^*$, then

$$p_\mu^{q_\mu - 1} \mid (t+\mu), \qquad p_\mu^{q_\mu} \nmid (t+\mu) \quad (1 \leqslant \mu \leqslant k),$$
$$r^{q-1} \nmid (t+\mu) \quad (1 \leqslant \mu \leqslant k;\, r \neq p_\mu).$$

Hence $$q_\mu \mid d(t+\mu), \qquad q_\nu \nmid d(t+\mu) \quad (1 \leqslant \mu, \nu \leqslant k;\, \mu \neq \nu).$$

The k numbers $d(t+1),\ldots, d(t+k)$ are therefore distinct and, since $t+k \leqslant x$, Theorem V follows by (11.1).

If S is any finite set, we shall denote by $|S|$ the number of its elements.

To estimate $|T(\mu, r)|$ we first note that, if $r \leqslant p_k$, say $r = p_i$, then $T(\mu, r)$ is empty. For otherwise

$$p_i^{q-1} \mid (i-\mu),$$

and so $2^{q-1} \leqslant k$, which is contrary to the definitions of k and q. It is also obvious that $T(\mu, r)$ is empty when $r^{q-1} > x$. On the other hand, if $r > p_k$, $r^{q-1} \leqslant x/M$, then

$$|T(\mu, r)| \leqslant \frac{x}{r^{q-1}M} + 1,$$

whilst, if $x/M < r^{q-1} \leqslant x$, then

$$|T(\mu, r)| \leqslant 1.$$

Hence

$$\sum_{\substack{1 \leqslant \mu \leqslant k \\ r}} |T(\mu, r)| \leqslant \sum_{\substack{1 \leqslant \mu \leqslant k \\ r^{q-1} \leqslant x/M}} \left(\frac{x}{r^{q-1}M} + 1\right) + \sum_{\substack{1 \leqslant \mu \leqslant k \\ r^{q-1} \leqslant x}} 1$$
$$\leqslant \frac{kx}{M} \sum_r \frac{1}{r^{q-1}} + 2kx^{1/(q-1)}.$$

But $$\sum_r \frac{1}{r^{q-1}} < \sum_{n=2}^{\infty} \frac{1}{n^{q-1}} < \frac{1}{2^{q-1}} + \int_2^{\infty} \frac{du}{u^{q-1}} < \frac{1}{2^{q-2}},$$

and therefore

$$\sum_{\substack{1 \leqslant \mu \leqslant k \\ r}} |T(\mu, r)| \leqslant \frac{kx}{M2^{q-2}} + 2kx^{1/(q-1)} < \frac{x}{M \log x} + x^{\frac{1}{3}}.$$

Hence, using (11.2), we obtain

$$|T^*| \geqslant |T| - \sum_{\substack{1 \leqslant \mu \leqslant k \\ r}} |T(\mu, r)| \geqslant \frac{x}{M} - 1 - \frac{x}{M \log x} - x^{\frac{1}{3}}$$
$$> x^{\frac{1}{2}}\left(1 - \frac{1}{\log x}\right) - 1 - x^{\frac{1}{3}} > 0.$$

This completes the proof.

12. Finally we prove Theorem VI. For $6 \leqslant x \leqslant 16$ the assertion is easily verified directly.† Assume next that $x > 16$. Denote by q' the prime preceding q, so that

$$2^{q'-1} \leqslant x < 2^{q-1}.$$

Let E be the set of numbers $d(n)$, $1 \leqslant n \leqslant x$. Then clearly $m \in E$ when $m \leqslant q'$, and $q \ni E$. It remains only to show that every number strictly between q' and q belongs to E. Let

$$q' < m < q,$$

and write
$$m = ab \quad (a \geqslant b \geqslant 2).$$

Since $d(2^{a-1}3^{b-1}) = ab$ it suffices to show that

$$2^{a-1}3^{b-1} \leqslant 2^{q'-1};$$

in other words that
$$a+(b-1)\tau \leqslant q', \tag{12.1}$$

where $\tau = \log 3/\log 2$.

Since $x > 16$ we have $q' \geqslant 5$ and so, by Bertrand's postulate,‡

$$q \leqslant 2q'-2.$$

For $b = 2$ this implies $2a < 2q'-2$, and so $a+2 \leqslant q'$. The inequality (12.1) is then evidently satisfied. When $b > 2$ and $5 \leqslant q' \leqslant 19$ the validity of (12.1) is easily verified directly. When $b > 2$ and $q' \geqslant 23$ we have

$$q' \geqslant 8\tau^2,$$

$$q' \geqslant 2\tau\sqrt{(2q')} \geqslant 2\tau b \geqslant \frac{(b-1)b}{b-2}\tau,$$

$$\frac{2}{b}q'+(b-1)\tau \leqslant q',$$

and (12.1) follows at once. Theorem VI is therefore proved.

Additional remark (30 June 1951). It may be worth mentioning that the ratio of two consecutive A-numbers tends to 1, and that the same result holds also for B-numbers and for D-numbers. For A-numbers and D-numbers this is obvious since D (and therefore A) contains the set of highly composite numbers; for B-numbers, on the other hand, the proof is a little more troublesome.

† We have, in fact, $\lambda(x) = 5$ for $6 \leqslant x < 16$, and $\lambda(16) = 7$.

‡ E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (Berlin and Leipzig, 1909), i, § 22.

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