The circle method from the perspective of higher order Fourier analysis

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- This will be an expository talk, rather than a talk on current research.
- My objective will be to try to convey the basic principles of higher order Fourier analysis, by revisiting the classical circle method and viewing it from this more modern viewpoint.
- The focus here will be on the "big picture", and we will be imprecise in the technical details.

- The classical circle method of Hardy and Littlewood allows one to asymptotically count solutions to many systems of equations in number theory.
- A typical question that can be answered by this method: *How many arithmetic progressions of length three are there in the prime numbers up to a large threshold N?*
- Answer (van der Corput, 1939): $(\mathfrak{S} + o(1))(\frac{N^2}{\log^3 N})$, where $\mathfrak{S}$ is the singular series

$$\mathfrak{S} = \prod_{p \geq 3} \left(1 - \frac{p^2 - 4p + 1}{(p-1)^4}\right) \approx 1.0481.$$

Traditionally, the circle method proceeds as follows.

- First, one expresses the question one wishes to solve as a certain multilinear sum. For instance, van der Corput's result is equivalent to the assertion

$$\sum_{n,r\in\mathbf{Z}} f(n)f(n+r)f(n+2r) = (\mathfrak{S}+o(1))N^2,$$

where $f(n) := \Lambda(n)1_{n\leq N}$ and Λ is the von Mangoldt function.

- Then, one uses Fourier-analytic identities to rewrite this sum in terms of exponential sums. For instance, we can write

$$\sum_{n,r \in \mathbf{Z}} f(n)f(n+r)f(n+2r) = \int_0^1 S(\alpha)^2 S(-2\alpha)\, d\alpha$$

where $S(\alpha)$ is the exponential sum

$$S(\alpha) := \sum_{n \leq N} \Lambda(n) e(-n\alpha)$$

and $e(\alpha) := e^{2\pi i \alpha}$.

$$\sum_{n,r \in \mathbf{Z}} f(n)f(n+r)f(n+2r) = \int_0^1 S(\alpha)^2 S(-2\alpha)\, d\alpha$$

- One often divides the region of integration $[0, 1]$ of α into major arcs, where α is close to a rational $\frac{a}{q}$ with small denominator q, and minor arcs, which comprise all other α.
- In favorable situations, the contribution of minor arcs is negligible, and the contribution of major arcs can be efficiently estimated, for instance by the methods of multiplicative number theory.
- Putting together all the estimates then (hopefully) gives an asymptotic for the desired problem.

- However, the circle method does not always work.
- For instance, sometimes the error bounds on the minor arcs dominates the main terms coming from the major arcs. (This is for instance the case in the twin primes conjecture, or even Goldbach conjecture.)
- For some problems, such as counting arithmetic progressions of length four or more, Fourier expansion does not produce an integral over a single frequency α, but rather over multiple frequencies $\alpha_1, \alpha_2, \ldots$, making it even less likely for major arc contributions to dominate.

- From the modern perspective of higher order Fourier analysis, given any counting problem in additive combinatorics, the first step is to ascertain the complexity of the problem, which roughly speaking describes which contributions to the count are "non-negligible".
- The circle method tends to work for problems of complexity one - in which contributions coming from Fourier phases such as $n \mapsto e(\alpha n)$ are the non-negligible ones. (**Example:** how many length three arithmetic progressions in the primes up to N?)
- They are particularly effective for complexity zero problems, in which it is the contributions from major arc Fourier phases, such as the constant phase $n \mapsto 1$, which are the non-negligible ones. (**Example:** how many primes up to N differ by a perfect square?)

- Higher order Fourier analysis can treat problems of arbitrary finite complexity, in which the non-negligible contributions are coming from higher order analogues of Fourier phases, such as polynomial phases $n \mapsto e(P(n))$ or nilsequences $n \mapsto F(g(n)\Gamma)$.
- For instance, the problem of counting progressions of length four turns out to have complexity two, and can be handled by quadratic Fourier analysis.
- Unfortunately, some problems are infinite complexity - they have a huge number of non-negligible contributions - and are out of reach of even higher order Fourier analysis. (**Example:** how many twin primes up to N?)
- We now have a pretty good understanding of what the "true" complexity of a problem is, though there is still some ongoing research in this area (e.g., Altman 2021).

- In principle, higher order Fourier analysis is a generalisation of the classical circle method.
- But in practice, the two methods can look rather different at first.
- One reason for this is that many of the Fourier analytic identities that are so prominent in the circle method (e.g., Parseval/Plancherel identity, or the Fourier inversion formula) do **not**, as far as we know, have a useful analogue in higher degrees.
- So we instead rely on other tools than identities, such as inverse theorems, transference principles, and equidistribution theorems, to proceed in the higher order case.

- These non-identity-based tools of higher order Fourier analysis do not give as precise answers as the identity-based tools of the classical circle method.
- However, they are often still strong enough to ensure that the "error term" in one's counts are lower order than the "main term", which is often the most important question to resolve.
- In the rest of this talk I will focus mostly on the complexity one theory that one traditionally attacks via the circle method, but from the higher order perspective that generalises to higher complexities as well.
- In particular we downplay the role of Fourier identities as much as possible.

- The first step in using higher order Fourier analysis is to determine the complexity of the problem being studied.
- In most cases the counting problem boils down to evaluating a certain multilinear form involving some finite number of functions.
- For instance, to count the number of length three arithmetic progressions in a set $A \subset \mathbf{Z}$ is $\Lambda_3^{\mathbf{Z}}(1_A, 1_A, 1_A)$, where $\Lambda_3^{\mathbf{Z}}$ is the trilinear form

$$\Lambda_3^{\mathbf{Z}}(f, g, h) := \sum_{n \in \mathbf{Z}} \sum_{r=1}^{\infty} f(n) g(n+r) h(n+2r).$$

- So the next step is to understand the nature of the multilinear form.

- To avoid some minor technicalities let us work in a cyclic group $\mathbf{Z}/N\mathbf{Z}$ of odd order, and study the normalized trilinear form

$$\Lambda_3(f,g,h) := \mathbb{E}_{n,r\in\mathbf{Z}/N\mathbf{Z}} f(n)g(n+r)h(n+2r)$$

  using the averaging notation $\mathbb{E}_{n\in A} f(n) := \frac{1}{|A|}\sum_{n\in A} f(n)$.
- This form counts length three progressions in subsets of $\mathbf{Z}/N\mathbf{Z}$.
- With this normalisation we have $\Lambda_3(1,1,1) = 1$, and hence $|\Lambda_3(f,g,h)| \leq 1$ whenever f, g, h are 1-bounded (i.e., $|f|, |g|, |h| \leq 1$).
- Let us informally call a function f negligible for Λ_3 if $\Lambda_3(f,g,h)$ is small whenever g, h are 1-bounded, as well as for permutations.

- For instance, from the triangle inequality we have

$$|\Lambda_3(f,g,h)| \leq \mathbb{E}_{n \in \mathbf{Z}/N\mathbf{Z}} |f(n)| =: \|f\|_{L^1(\mathbf{Z}/N\mathbf{Z})}$$

  whenever g, h are 1-bounded; similarly for permutations. Thus any function small in L^1 is negligible.
- On the other hand, for any integer multiple α of $\frac{1}{N}$, the identity

$$\alpha n - 2\alpha(n+r) + \alpha(n+2r) = 0$$

  (reflecting the fact that the second derivative of $x \mapsto \alpha x$ vanishes) ensures that the function $f(n) := e(\alpha n)$ is **not** negligible, since on taking $g(n) := e(-2\alpha n)$ and $h(n) := e(\alpha n)$ we have $\Lambda_3(f,g,h) = 1$.
- More generally, if $g(n) = e(-2\alpha n)$ and $h(n) = e(\alpha n)$, then

$$\Lambda_3(f,g,h) = \mathbb{E}_{n \in \mathbf{Z}/N\mathbf{Z}} f(n) e(-\alpha n),$$

  thus f will be non-negligible if it has large inner product with a linear phase $e(\alpha n)$.

- It turns out that the converse is also true: if f is a 1-bounded function that is non-negligible for Λ_3, then it has a large inner product with a linear phase $e(\alpha n)$. This is the inverse theorem for Λ_3.
- It is because of this that this form Λ_3 has complexity 1, and is therefore amenable to the circle method.

- In the classical circle method, the inverse theorem from Λ_3 is a quick corollary of the Fourier identity

$$\Lambda_3(f,g,h) = \sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} \hat{f}(\xi)\hat{g}(-2\xi)\hat{h}(\xi)$$

where $\hat{f}(\xi) := \mathbb{E}_{n \in \mathbf{Z}/N\mathbf{Z}} f(n) e(-\xi n/N)$ are the Fourier coefficients of f, combined with the Plancherel identity

$$\|f\|_{L^2(\mathbf{Z}/N\mathbf{Z})}^2 = \mathbb{E}_{n \in \mathbf{Z}/N\mathbf{Z}} |f(n)|^2 = \sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} |\hat{f}(\xi)|^2.$$

- An alternate way to proceed is to first apply the Cauchy–Schwarz inequality (or Weyl differencing or the van der Corput inequality) twice to obtain the generalised von Neumann inequality

$$|\Lambda_3(f, g, h)| \leq \|f\|_{U^2(\mathbf{Z}/N\mathbf{Z})}$$

for any 1-bounded g, h, where the Gowers uniformity norm $U^2(\mathbf{Z}/N\mathbf{Z})$ is defined by

$$\|f\|_{U^2(\mathbf{Z}/N\mathbf{Z})}^4 := \mathbb{E}_{x,h_1,h_2 \in \mathbf{Z}/N\mathbf{Z}} f(x)\overline{f}(x+h_1)\overline{f}(x+h_2)f(x+h_1+h_2).$$

- The inverse theorem for Λ then follows from an inverse theorem for U^2: If a 1-bounded function f has large $U^2(\mathbf{Z}/N\mathbf{Z})$ norm, then it has large inner product with a linear phase $e(\alpha n)$.

- The U^2 inverse theorem can be deduced from the Fourier identity

$$\|f\|^4_{U^2(\mathbf{Z}/N\mathbf{Z})} = \sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} |\hat{f}(\xi)|^4$$

  or from spectral theory or representation theory.
- This seems to be the one place in the entire theory of higher order Fourier analysis where some sort of identity (either a Fourier identity, a trace identity, or a representation-theoretic identity) is needed.
- The U^2 norm is part of a hierarchy of Gowers uniformity norms U^k that determine complexity: a multilinear form Λ has complexity s if it is controlled by the U^{s+1} norm.

Now that we know what functions are negligible for Λ_3 (or U^2), the next step is an appropriate structure theorem to decompose f. Here is an example of such a theorem (ignoring technical quantitative issues):

Informal structure theorem

Let f be a 1-bounded function. Then we can split

$$f = f_{\mathrm{str}} + f_{\mathrm{unf}}$$

where

- f_{str} is 1-bounded and is a linear combination (with bounded coefficients) of a bounded number of linear phases $n \mapsto e(\alpha n)$.
- f_{unf} is Fourier-uniform (small inner products with all linear phases $n \mapsto e(\alpha n)$), hence negligible for U^2 (and and thus also Λ_3).

Furthermore, if f is non-negative, so is f_{str}.

- In practice, such a structure theorem allows one to restrict attention to the contribution of the structured part f_{str} when studying complexity one problems:

$$\Lambda_3(f, g, h) \approx \Lambda_3(f_{\mathrm{str}}, g_{\mathrm{str}}, h_{\mathrm{str}}).$$

- Similar structure theorems are available for other multilinear forms, so long as one has a good inverse theorem for that form (or for an associated Gowers uniformity norm).

- Morally speaking, such decompositions would follow from the Fourier inversion formula

$$f(n) = \sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} \hat{f}(\xi) e(n\xi/N)$$

  by taking f_{str} to be the contribution of the "large" Fourier coefficients and f_{unf} the contribution of the "small" Fourier coefficients.
- But then f_{str} need not be 1-bounded (or non-negative when f is non-negative).

- One can proceed instead through the machinery of conditional expectation. Roughly speaking, f_{str} is the orthogonal projection to the Hilbert space generated by the large Fourier modes. Can be made quantitative using the energy increment argument.
- Nowadays we have slick proofs of structure theorems based on the Hahn–Banach theorem (Gowers 2010, Reingold–Trevisan–Tulsiani–Vadhan 2008).
- Thanks to the transference principle (Green-T. 2004), which has a slick modern proof using the technique of densification (Conlon–Fox–Zhao 2014), we also have structure theorems for many unbounded functions f, such as the von Mangoldt function.

- For some explicit choices of functions f, such as (suitable truncations of) the von Mangoldt function Λ, the inner products of f with such functions as linear phases $e(\alpha n)$ can be computed (for both "major arc" and "minor arc" α) by various techniques in analytic number theory.
- This can allow one to extract the structured portion of f_{str} in such cases. For instance, the structured component of Λ (when restricted to $\{1, \ldots, N\}$) can basically be taken to be the function $n \mapsto \frac{W}{\phi(W)} 1_{(n,W)=1}$, for $W = \prod_{p \leq w} p$ and w growing slowly with N.
- This is good enough to answer many questions about linear equations in primes.

- For more general functions f, the structured component f_{str} might involve some bounded number of linear phases $e(\alpha_1 n), \ldots, e(\alpha_k n)$ where the frequencies $\alpha_1, \ldots, \alpha_k$ are unknown.
- The computation of quantities such as $\Lambda_3(f, g, h)$ then hinges on the joint distribution of the fractional parts of the $\alpha_j n$, that is to say the distribution of

$$(\alpha_1, \ldots, \alpha_k)n \bmod \mathbf{Z}^k$$

  in the torus $\mathbf{R}^k/\mathbf{Z}^k$. This is achieved through equidistribution theorems.
- In applications we need quite quantitative (and "single-scale") information about this distribution, but for this high-level talk I will only discuss distribution at a qualitative asymptotic level.

One of the oldest and most influential equidistribution theorems is

Equidistribution theorem (Bohl 1909, Weyl 1910, Sierpiński 1910)

If α is an irrational real number, then $\alpha n \bmod 1$ is equidistributed in $\mathbf{R}/\mathbf{Z}$. (That is to say, $\mathbb{E}_{n \in [N]} f(\alpha n) = \int_{\mathbf{R}/\mathbf{Z}} f + o(1)$ as $N \to \infty$ for any continuous $f : \mathbf{R}/\mathbf{Z} \to \mathbf{C}$.)

Of course, if $\alpha = \frac{a}{q}$ is rational, then $\alpha n \bmod 1$ is instead equidistributed in the finite subgroup $\frac{1}{q}\mathbf{Z}/\mathbf{Z}$ of $\mathbf{Z}$. Easily established by estimation of the exponential sums $\mathbb{E}_{n \in [N]} e(k\alpha n)$ for $k \in \mathbf{Z}$.

There is a higher-dimensional version:

Higher-dimensional equidistribution

If $\alpha_1, \ldots, \alpha_k$ are real numbers, then $(\alpha_1, \ldots, \alpha_k)n \bmod \mathbf{Z}^k$ is equidistributed in some closed subgroup of $\mathbf{R}^k/\mathbf{Z}^k$. That is to say, there exists a closed subtorus T of $\mathbf{R}^k/\mathbf{Z}^k$ and a finite subgroup H of $\mathbf{R}^k/\mathbf{Z}^k$ such that

$$\mathbb{E}_{n \in [N]} f(\alpha_1 n, \ldots, \alpha_k n) = \int_{T+H} f + o(1)$$

for any continuous $f : \mathbf{R}^k/\mathbf{Z}^k \to \mathbf{C}$, using Haar probability measure on $T + H$.

For instance, if $\alpha_1 = \sqrt{2}$ and $\alpha_2 = 2\sqrt{2} + \frac{1}{2}$, one has $T = \{(x, 2x) : x \in \mathbf{R}/\mathbf{Z}\}$ and $H = \{(0,0), (0, 1/2)\}$.

- This equidistribution theorem is a simple example of the deeper phenomenon of measure rigidity: sequences generated in some way by a nilpotent (or unipotent) group should be asymptotically equidistributed with respect to a Haar measure associated to a closed group (as opposed to, say, a Cantor measure).
- The most famous measure rigidity result the celebrated Ratner's theorem (Ratner, 1990).
- In principle, such equidistribution theorems reduce counting problems for structured functions to questions in (Lie) group theory.

- For instance, Roth's theorem on progressions of length three roughly speaking asserts that there is a lower bound $\Lambda_3(1_A, 1_A, 1_A) \gg 1$ whenever A is a subset of $\mathbf{Z}/N\mathbf{Z}$ of density $\gg 1$.
- Using structural decompositions, this theorem basically reduces to that of understanding the equidistribution of

$$(\alpha n, \alpha(n+r), \alpha(n+2r)) \in (\mathbf{R}/\mathbf{Z})^{3k}$$

  for an arbitrary vector $\alpha = (\alpha_1, \ldots, \alpha_k)$ of real numbers, and how it intersects Cartesian products like $E \times E \times E$ for some "bounded complexity" subset E of $(\mathbf{R}/\mathbf{Z})^k$ (think of a union of boundedly many cubes).
- The equidistribution theorem says that this tuple is equidistributed in a closed subgroup of $(\mathbf{R}/\mathbf{Z})^{3k}$. **Key point:** that this subgroup contains a diagonal group $\{(g, g, g) : g \in G\}$, where $G \leq (\mathbf{R}/\mathbf{Z})^k$ is the equidistribution group of αn.

- Now let us see what changes when we look at length four progressions in $\mathbf{Z}/N\mathbf{Z}$. We are now faced with a quartilinear form

$$\Lambda_4(f,g,h,k) = \mathbb{E}_{n,r\in\mathbf{Z}/N\mathbf{Z}} f(n)g(n+r)h(n+2r)k(n+3r).$$

- Whereas Λ_3 is controlled by the U^2 norm, Λ_4 is instead controlled by the U^3 norm

$$\|f\|_{U^3(\mathbf{Z}/N\mathbf{Z})}^8 = \mathbb{E}_{n,h_1,h_2,h_3} f(n)\overline{f}(n+h_1)\overline{f}(n+h_2)\overline{f}(n+h_3)$$

$$f(n+h_1+h_2)f(n+h_1+h_3)f(n+h_2+h_3)\overline{f}(n+h_1+h_2+h_3).$$

- Fourier identities are now rather unhelpful: in Fourier space one has

$$\Lambda_4(f,g,h,k) = \sum_{\xi,\eta \in \mathbf{Z}/N\mathbf{Z}} \hat f(\xi)\hat g(-2\xi+\eta)\hat h(\xi-2\eta)\hat k(\eta)$$

and

$$\|f\|_{U^3(\mathbf{Z}/N\mathbf{Z})} = N^{1/2}\|\hat f\|_{U^3(\mathbf{Z}/N\mathbf{Z})}.$$

- But the main reason why linear Fourier analysis is not the right tool here is that **linear phases are no longer the only source of non-neglibility**.

- The old linear identity

$$\alpha n - 2\alpha(n+r) + \alpha(n+2r) = 0$$

  still makes $n \mapsto e(\alpha n)$ non-negligible (take $f(n) = e(\alpha n)$, $g(n) = e(-2\alpha n)$, $h(n) = e(\alpha n)$, $k(n) = 1$).
- But we now also have a quadratic identity

$$\alpha n^2 - 3\alpha(n+r)^2 + 3\alpha(n+2r)^2 - \alpha(n+3r)^2 = 0$$

  (reflecting the fact that the third derivative of $x \mapsto \alpha x^2$ vanishes) which means that quadratic phases such as $n \mapsto e(\alpha n^2)$ are also non-negligible if α is an integer multiple of $1/N$.
- This is despite such quadratic phase functions typically have very small Fourier coefficients.

- To make matters worse, there are also generalisations of quadratic phases, known as bracket quadratic phases, which are also non-negligible.
- For instance, for generic integer multiples α, β of $1/N$, the identity

$$\lfloor \alpha n \rfloor \beta n - 3\lfloor \alpha(n+r) \rfloor \beta(n+r) + 3\lfloor \alpha(n+2r) \rfloor \beta(n+2r)$$
$$-\lfloor \alpha(n+3r) \rfloor \beta(n+3r) = 0$$

  holds a positive fraction of the time, basically because $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ holds for a positive fraction of reals x, y.
- Hence the bracket quadratic phase $n \mapsto e(\lfloor \alpha n \rfloor \beta n)$ is non-negligible, as is any function with a large inner product with such a bracket quadratic phase.

It is a non-trivial fact that these are basically the **only** non-negligible functions for Λ_4, making the length four arithmetic progressions problem a “complexity two” problem, suitable for attack by quadratic Fourier analysis.

Informal inverse theorem (Green–T. 2008)

Let f be a 1-bounded function which is non-negligible for Λ_4 (or U^3). Then f has large inner product with a bracket quadratic

$$n \mapsto e(\lfloor \alpha_1 n \rfloor \beta_1 n + \cdots + \lfloor \alpha_k n \rfloor \beta_k n \rfloor + \gamma n)$$

for some integer multiples $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \gamma$ of $1/N$ and some bounded k.

- The proof of this theorem uses the U^2 inverse theorem and additive combinatorics, following the arguments of (Gowers, 1998).
- Higher order versions are also available (Green–T.–Ziegler 2012, Camarena–Szegedy 2010, Szegedy 2012, Candela–Szegedy 2017, Gutman–Manners–Varju 2018, Manners 2018). The proofs are intricate (usually involving an induction on the order k of the uniformity norm U^k), and many use the machinery of nonstandard analysis to allow one to apply theorems from continuous mathematics to this discrete setting.

- To use this theorem in applications one would like to combine it with an equidistribution theorem for bracket quadratics.
- This is **very messy** if done directly.
- It turns out that to get the cleanest equidistribution theory for quadratic and higher Fourier analysis, one should replace the bracket quadratic phases with an equivalent class of structured functions, namely the degree two nilsequences $F(g(n)\Gamma)$, where G/Γ is a degree 2 nilmanifold, $g : \mathbf{Z} \to G$ is a polynomial map, and $F : G/\Gamma \to \mathbf{C}$ is a Lipschitz function (we will not define these terms precisely here).

A basic example of a degree 2 nilmanifold is the Heisenberg nilmanifold

$$G/\Gamma := \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} \\ 0 & 1 & \mathbf{R} \\ 0 & 0 & 1 \end{pmatrix} / \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

If

$$g(n) := \begin{pmatrix} 1 & \alpha n & 0 \\ 0 & 1 & \beta n \\ 0 & 0 & 1 \end{pmatrix}$$

then one has

$$g(n)\Gamma = \begin{pmatrix} 1 & \{\alpha n\} & \{-\alpha n \lfloor \beta n \rfloor\} \\ 0 & 1 & \{\beta n\} \\ 0 & 0 & 1 \end{pmatrix} \Gamma$$

and one can begin to see the relation between bracket polynomials and nilmanifolds.

The following equidistribution theorem allows one (in principle at least) to calculate expressions involving nilsequences:

Informal nilmanifold equidistribution

If $g : \mathbf{Z} \to G$ is a polynomial map, then $g(n)\Gamma$ is equidistributed in a finite union of closed orbits $g_i H x_i$, for some closed subgroup H of G, some $g_i \in G$, and some rational $x_i \in G/\Gamma$.

The qualitative version of this theorem was proven in (Leibman 2005); a quantitative version in (Green–T. 2012). When combined with the other higher order Fourier analysis tools, has many applications (e.g., counting solutions to linear or (sometimes) polynomial equations in primes).

Some more recent developments in the subject:

- The technique of densification (Conlon–Fox–Zhao 2014) that can allow one to efficiently model sparse sets (or sparsely supported functions) by dense sets (or densely supported functions), in the presence of a pseudorandom majorant;
- Concatenation theorems (T.–Ziegler 2016; Peluse–Prendiville 2019) that allow one to “concatenate” local structure into global structure;
- The technique of degree lowering (Peluse–Prendiville 2019) that can make the true complexity of a pattern significantly lower than the naive complexity;

- Quantitative inverse theorems (Manners 2018, Gowers–Milićević 2017, 2020);
- Formulations using nonstandard analysis and “nilspaces” (Camarena–Szegedy 2010, Candela 2017, Gutman–Manners–Varju 2019, 2020);
- Quantitative equidistribution theorems for polynomials over finite fields (Milićević 2019, Janzer 2020, Cohen–Moshkowitz 2021, Adiprasito–Kazhdan–Ziegler 2021, Lampert–Ziegler 2021).

Thanks for listening!