Part II

Investments

"This next song's about spreading risk in a volatile market by diversification."

There is a long version and short version of the Investments part of the book. You are looking at the long version. The short version will appear in a dedicated corporate finance version of the book, and can already be downloaded from the website.

This introduction appears in the Survey text only.

Transition

You still do not know where the cost of capital, the $\mathcal{E}(\tilde{r})$, in the present value formula comes from. We just assumed you knew it.

In the real world, the cost of capital is determined by your investors—and your investors have many choices. You have to understand their mindset. They can put money in bonds, stocks, projects, etc.—or into your own firm's projects. If investors very much like the alternatives, then your firm's cost of capital should be high. Think of your corporate cost of capital as your investors' opportunity cost of capital.

To learn more about where the cost of capital comes from, we have to study not just our own project, but more generally how investors think—what they like and dislike.

Of course, it is nice that we are often ourselves investors, so learning how to best invest money in the financial markets has some nice side benefits.

SIDE NOTE

In the Investments part of the book, we will use the terms "stock" and "security" very loosely. Our concern is really the choice among many investment opportunities, which includes bonds, options, futures, real estate, etc. It is just more convenient to use the phrase stock, rather than "any possible investment opportunity."

What You Want to Learn in this Part

The goal of this part of the book is to teach you how public investors think. This is important not only when you are an investor who wants to decide how to allocate money among the thousands of possible financial investments, but also for the corporate manager who wants to get you and other investors to part with their money in order to finance new corporate projects.

- Chapter 7 gives a short tour of historical rates of returns to whet your appetite, and explains some of the setup of equity markets.

  Typical questions: Did stocks, bonds, or cash perform better over the last thirty years? How safe were stocks compared to bonds or cash?

- Chapter 8 describes how you can trade equity securities (a.k.a. "stocks") , and how securities can be combined into portfolios.

  Typical questions: what was the rate of return on a portfolio that invested money into the 30 stocks that form the **Dow-Jones Industrial Average** (**DJIA**) index?

- Chapter 9 shows how to measure the risk and reward of securities.

  Typical questions: if you held a portfolio of the 30 stocks that form the DJIA, what would your risk be?

- Chapter 10 shows how to measure the risk and reward of portfolios.

  Typical questions: How would the risk change if you rearrange how much money you put into each of the 30 stocks of the Dow Jones?

- Chapter 11 shows how holding many stocks in your portfolio can reduce your risk.

  Typical questions: How much lower would your risk be if you held all 30 stocks in the DJIA instead of just a single stock in the DJIA?

- Chapter 12 shows how you can determine the *best* portfolio.

  Typical questions: Can you invest better than just into the 30 stocks in the DJIA? How much risk does PepsiCo contribute to your portfolio? Would it be better if you shifted more money into PepsiCo stock, or should you stick to your current portfolio?

- Chapter 13 shows what expected rates of return securities have to offer *if* investors make smart investment decisions. It gives the formula (cookbook) version of the "Capital Asset Pricing Model" (CAPM) model, which answers this question.

  Typical questions: What is a fair reward for PepsiCo's stock investors, given how much risk it contributes to the overall market portfolio?

- Chapter 14 explains more of the theory behind the CAPM.

- Chapter 16 expands on the efficient markets concept, first mentioned in the introduction. It also explains the difference between arbitrage and good bets.

  Typical questions: what kind of information can you use to beat the market?

CHAPTER 8[§]

Securities and Portfolios

Applications

This chapter appears in the Survey text only.

THIS chapter first explains where stocks come from and where they are traded. It then explains the process of going long (buying an asset to speculate that it will go up) and going short (selling an asset to speculate that it will go down). Finally, it explains portfolios and indexes.

8·1 A Brief Overview of Equities Market Institutions and Vehicles

Let's look into the institutional arrangements for equity trading. After all, from a corporate perspective, stocks are more interesting than many other financial instruments, such as foreign government bonds, even if there is more money in foreign government bonds than in corporate equity. It is the equity holders who finance most of the risks of corporate projects. Moreover, although there is more money in non equity financial markets, the subject area of investments also tends to focus on equities (stocks), because retail investors find it easy to participate and the data on stocks are relatively easy to come by. So it makes sense to describe a few institutional details as to how investors and stocks "connect"—exchange cash for claims, and vice versa.

8·1.A. Brokers

Retail brokers execute trades and keep track of portfolios. (They also arrange shorts.)

Most individuals place their orders to buy or sell stocks with a **retail broker**, such as *Ameritrade* (a "deep-discount" broker), *Charles Schwab* (a discount broker), or *Merrill Lynch* (a "full-service" broker). Discount brokers may charge only about $10 commission per trade, but they often receive "rebate" payments back from the market maker [see below] to which they route your order. This is called "payment for order flow." The market maker in turn recoups this payment to the broker by executing your trade at a price that is less favorable. Although the purpose of such an arrangement seems deceptive, the evidence suggests that discount brokers are still often cheaper in facilitating investor trades—especially small investor trades—even after taking this hidden payment into account. They just are not as (relatively) cheap as they want to make you believe. Investors can place either **market orders**, which ask for execution at the current price, or **limit orders**, which ask for execution if the price is above or below a limit that the investor can specify. (There are also many other types of orders, e.g., *stop-loss* orders [which instruct a broker to sell a security if it has lost a certain amount of money], *good-til-canceled* orders, and *fill-or-kill* orders.) The first function of retail brokers then is to handle the execution of trades. They usually do so by routing investors' orders to a centralized trading location (e.g., a particular stock exchange), the choice of which is typically at the retail broker's discretion, as is the particular agent (e.g., floor broker) engaged to execute the trade. The second function of retail brokers is to keep track of investors' holdings, to facilitate purchasing **on margin** (whereby investors can borrow money to purchase stock, allowing them to purchase more securities than they could afford on a pure cash basis), and to facilitate selling securities "short," which allows investors to speculate that a stock will go down.

Prime brokers leave execution to the client investor.

↦ *Unknown 8·3.B on Page 177*

Many larger and institutional investors , such as **funds** (described in Section 8·3.B), break the two functions apart: The investor can employ its own traders, while the broker takes care only of the bookkeeping of the investor's portfolio, margin provisions, and shorting provisions. Such limited brokers are called **prime brokers**.

How Shorting Stocks Works

Shorting is like borrowing and then issuing securities. Proceed interest may be earned by the broker or by the client (or be shared).

↦ *Bond Shorting, Subsection b on Page 105*

If you want to speculate that a stock goes down, you would want to short it. This shorting would be arranged by your broker. It is important enough to deserve an explanation:

- You find an investor in the market who is willing to lend you the shares. In a perfect market, this does not cost a penny. In the real world, the broker has to find a willing lender. Both the broker and lender usually earn a few basis points per year for doing you the favor of facilitating your short sale.
- After you have borrowed the shares, you sell them into the market to someone else who wanted to buy the shares. In a perfect market, you would keep the proceeds and earn interest on them. In the real world, your broker may force you to put these proceeds into low-yield safe bonds. If you are a small retail investor, your brokerage firm may even keep the interest proceeds altogether to himself.
- When you want to "unwind" your short, you repurchase the shares and return them to your lender.

For example, if you borrowed the shares when they were trading for $50, and the shares now sell for $30 each, you can repurchase them for $20 less than what you sold them into the market for. This $20 is your profit. In an ideal world, you can think of your role effectively as the same as that of the company—you can issue shares and use the proceeds to fund your investments. In the real world, you have to take transaction costs into account. (Shorting has become so common that there are now exchange-traded futures on stocks that make this even easier. Futures are explained in Section **??**.)

↦ *Futures and Forwards, Section **??** on Page **??***

Solve Now!

Q 8.1 What are the two main functions of brokerage firms?

Q 8.2 How does a prime broker differ from a retail broker?

Q 8.3 Is your rate of return higher if you short a stock in the perfect world or in the real world? Why?

8·1.B. Exchanges and Non-Exchanges

The two big stock exchanges are the NYSE and NASDAQ. The NYSE is a hybrid market. The NASDAQ is only electronic.

A retail broker would route your transaction to a centralized trading location. The most prominent are exchanges. An **exchanges** is a centralized trading locations where financial securities are traded. The two most important stock exchanges in the United States are the **New York Stock Exchange** (**NYSE**, also nicknamed the **Big Board**) and **NASDAQ** (originally an acronym for “National Association of Securities Dealers Automated Quotation” System). The NYSE used to be exclusively an **auction market**, in which one designated **specialist** (assigned for each stock) managed the auction process by trading with individual brokers on the floor of the exchange. This specialist was often a monopolist. However, even the NYSE now conducts much trading electronically. In contrast to the NYSE’s hybrid human-electronic process primarily in one physical location on Wall Street, NASDAQ has always been a purely electronic exchange without specialists. (For security reasons, its location—well, the location of its computer systems—is secret.) For each NASDAQ stock, there is at least one **market maker**, a broker-dealer who has agreed to continuously stand by to offer to buy or sell shares, electronically of course, thereby creating a liquid and immediate market for the general public. Moreover, market makers are paid for providing liquidity: they receive additional rebates from the exchange when they post a bid or an ask that is executed. Most NASDAQ stocks have multiple market makers, drawn from a pool of about 500 trading firms (such as J.P. Morgan or ETrade), which compete to offer the best price. Market makers have one advantage over the general public: They can see the **limit order book**, which contains as-yet-unexecuted orders from investors to purchase or sell if the stock price changes—giving them a good idea at which price a lot of buying or selling activity will happen. The NYSE is the older exchange, and for historical reasons, is the biggest exchange for trading most “blue chip” stocks. (“Blue chip” now means “well established and serious”; ironically, the term itself came from poker, where the highest-denomination chips were blue.) In 2006, the NYSE listed just under 3,000 companies worth about $25 trillion. (This is about twice the annual U.S. GDP.) NASDAQ tends to trade smaller and high-technology firms, lists about as many firms, and has more trading activity than the NYSE. Some stocks are traded on both exchanges.

Auction markets, popular in other countries, have lower execution costs, but also lower execution speed.

Continuous trading—trading at any moment an investor wants to execute—relies on the presence of the standby intermediaries (specialists or market makers), who are willing to absorb shares when no one else is available. This is risky business, and thus any intermediary must earn a good rate of return to be willing to do so. To avoid this cost, some countries have organized their exchanges into non continuous auction systems, which match buy and sell orders a couple of times each day. The disadvantage is that you cannot execute orders immediately but have to delay until a whole range of buy orders and sell orders have accumulated. The advantage is that this eliminates the risk that an (expensive) intermediary would otherwise have to bear. Thus, auctions generally offer lower trading costs but slower execution.

New alternative trading institutions: electronic communication networks (ECNs).

Even in the United States, innovation and change are everywhere. For example, **electronic communications networks** (**ECNs**) have recently made big inroads into the trading business, replacing exchanges, especially for large institutional trades. (They can trade the same stocks that exchanges are trading, and compete with exchanges in terms of cost and speed of execution.) An ECN cuts out the specialist, allowing investors to post price-contingent orders themselves. ECNs may specialize in lower execution costs, higher broker kickbacks, or faster execution. The biggest ECNs are *Archipelago* and *Instinet*. In 2005, the NYSE merged with Archipelago, and NASDAQ purchased Instinet. (It is hard to keep track of the most recent trading arrangements. For example, in 2006, the NYSE also merged with ArcaEx, yet another electronic trading system, and merged with Euronext, a pan-European stock exchange based in Paris. As of this writing, it is now officially called **NYSE Euronext**. In addition, the NYSE converted from a mutual company owned by its traders into a publicly traded for-profit company itself.)

And crossing networks and more...

An even more interesting method to buy and trade stocks is that of **crossing systems**, such as *ITG*'s *POSIT*. ITG focuses primarily on matching large institutional trades with one another in an auction-like manner. If no match on the other side is found, the order may simply not be executed. But if a match is made, by cutting out the specialist or market maker, the execution is a lot cheaper than it would have been on an exchange. Recently, even more novel trading places have sprung up. For example, **Liquidnet** uses peer-to-peer networking—like the original Napster—to match buyers and sellers in real time. ECNs or electronic limit order books are now the dominant trading systems for equities worldwide, with only the U.S. exchange floors as holdouts. Similar exchanges and computer programs are also used to trade futures, derivatives, currencies, and even some bonds.

There are also informal financial markets, especially OTC (over-the-counter).

There are many other financial markets, too. There are financial exchanges handling stock options, commodities, insurance contracts, and so on. A huge segment is the **over-the-counter** (**OTC**) markets. Over-the-counter means "call around, usually to a set of traders well known to trade in the asset, until you find someone willing to buy or sell at a price you like." Though undergoing rapid institutional change, most bond transactions are still over-the-counter. Although OTC markets handle significantly more bond trading in terms of transaction dollar amounts than exchanges, their transaction costs are prohibitively high for retail investors—if you call without knowing the market in great detail, the person on the other end of the line will be happy to quote you a shamelessly high price, hoping that you do not know any better. The **NASD** (National Association of Securities Dealers) also operates a semi-OTC market for the stocks of smaller firms, which are listed on the so-called **pink sheets**. Foreign securities trade on their local national exchanges, but the costs for U.S. retail investors are again often too high to make direct participation worthwhile.

Solve Now!

Q 8.4 How does a crossing system differ from an electronic exchange?

Q 8.5 What is a specialist? What is a market maker? When trading, what advantage do the two have over you?

Q 8.6 Describe some alternatives to trading on the main stock exchanges.

8·1.C. Investment Companies and Vehicles

The SEC regulates many investment vehicles that are active in the U.S. financial markets. Under the Investment Company Act of 1940, there are three types of **investment companies**: open-end funds, closed-end funds, and unit investment trusts (UITs).

The "open end" feature allows investors to redeem their shares. It forces the fund's shares to trade for close to the value of its holdings.

In the United States, open-end fund is a synonym for mutual fund. (Elsewhere, mutual funds can include other classes.) Being **open end** means that the fund can create shares at will. Investors can also redeem their fund shares at the end of each trading day in exchange for the **net asset value** (**NAV**), which must be posted daily. This gives investors little reason to sell their fund shares to other investors—thus, mutual funds do not trade on any exchanges. The redemption right gives the law of one price a lot of bite—fund shares are almost always worth nearly exactly what their underlying holdings are worth. If an open-end fund's share price were to fall much below the value of its holdings, an arbitrageur could buy up the fund shares, redeem them, and thereby earn free money. (One discrepancy is due to some odd tax complications: the fund's capital gains and losses are passed through to the fund investors at the end of every year, but they may not be what every investor experienced.) Interestingly, in the United States, there are now more mutual funds than there are stocks in the financial market.

Closed-end funds do not allow shares to be redeemed. This is useful for funds which are investing in illiquid assets.

In a **closed-end fund**, there is one big initial primary offering of fund shares, and investors cannot redeem their fund shares for the underlying value. The advantage of a closed-end fund is that it can itself invest in assets that are less liquid. After all, it may not be forced to sell its holdings on the whims of its own investors. Many closed-end funds are exchange traded, so that if a closed-end fund investor needs cash, she can resell her shares. The disadvantage of the closed-end scheme is that the law of one price has much less bite. On average, closed-end funds trade persistently below the value of their underlying holdings, roughly in line with the (often high) fees that the managers of many of these closed-end funds are charging.

Mutual funds are open-ended, actively traded investment vehicles.

Both mutual funds and closed-end fund managers are allowed to trade their fund holdings quite actively—and many do so. Although some funds specialize in imitating common stock market indexes, many more try to guess the markets or try to be more "boutique." Most funds are classified into a category based on their general trading motivation (such as "market timing," or "growth" or "value," or "income" or "capital appreciation").

UITs are passive "basket" investment vehicles.

A **unit investment trust** (**UIT**) is sort of closed end in its creation (usually through one big primary offering) and sort of open end in its redemption policies (usually accepting investor redemption requests on demand). Moreover, SEC rules forbid UITs to trade actively (although this is about to change), and UITs must have a fixed termination date (even if it is 50 years in the future). UITs can be listed on a stock exchange, which makes it easy for retail investors to buy and sell them. Some early **exchange-traded funds** (**ETFs**) were structured as UITs, although this required some additional legal contortions that allowed them to create more shares on demand. This is why ETFs are nowadays usually structured as open-end funds.

ADRs are investment vehicles, too. Many ADRs (though not all) are regulated by the SEC under different rules.

Some other investment vehicles are regulated by the SEC under different rules. The most prominent may be certain kinds of **American Depositary Receipt** (**ADR**). An ADR is a passive investment vehicle that usually owns the stock of only one foreign security, held in escrow at a U.S. bank (usually the Bank of New York). The advantage of an ADR is that it makes it easier for U.S. retail investors to trade in the foreign security without incurring large transaction costs. ADRs are redeemable, which gives the law of one price great bite.

Other funds are entirely unregulated.

There are also funds that are structured so that they do not need to register with the SEC. This means that they cannot openly advertise for new investors and are limited to fewer than 100 investors. This includes most **hedge funds**, **venture capital funds**, and other **private equity funds**. Many **offshore funds** are set up to allow foreign investors to hold U.S. stocks not only without SEC regulation, but also without ever having to tread into the domain of the U.S. IRS.

Solve Now!

Q 8.7 What should happen if the holdings of an open-end fund are worth much more than what the shares of the fund are trading for? What should happen in a closed-end fund?

8·1.D. How Securities Appear and Disappear

Inflows

Most publicly traded equities appear on public exchanges, almost always NASDAQ, through **initial public offerings** (**IPOs**). This is an event in which a privately traded company first sells shares to ordinary retail and institutional investors. IPOs are usually executed by **underwriters** (investment bankers such as *Goldman Sachs* or *Merrill Lynch*), which are familiar with the complex legal and regulatory process and which have easy access to an investor client base to buy the newly issued shares. Shares in IPOs are typically sold at a fixed price—and for about 10% below the price at which they are likely to trade on the first day of after-market open trading. (Many IPO shares are allocated to the brokerage firm's favorite customers, and they can be an important source of profit.)

Firms first sell public shares in IPOs.

Usually, about a third of the company is sold in the IPO, and the typical IPO offers shares worth between \$20 million and \$100 million, although some are much larger (e.g., privatizations, like British Telecom). About two-thirds of all such IPO companies never amount to much or even die within a couple of years, but the remaining third soon thereafter offer more shares in **seasoned equity offerings** (**SEOs**). These days, however, much expansion in the number of shares in publicly traded companies, especially large companies, comes not from seasoned equity offerings but from employee stock option plans, which eventually become unrestricted publicly traded shares.

Money flows into the financial markets also through SEOs.

Because IPOs face complex legal regulation, an alternative of **reverse mergers** has recently become prominent. In a reverse merger, a large privately owned company that wants to go public merges with a small company that is already publicly traded. The owners of the big company receive newly issued shares in the combined entity. And, of course, any time a publicly traded company purchases assets, such as privately held companies, and issues more shares, capital is in effect being deployed from the private sector into the public markets.

A reverse merger has become another common way to enter the public financial markets.

In 1933/1934, Congress established the **Securities and Exchange Commission** (SEC) through the *Securities Exchange Acts*. It further regulated investment advisors through the *Investment Advisers Act of 1940*. (The details of these acts can be obtained at the SEC website.) Aside from regulating the IPO process, they also prescribe what publicly traded corporations must do. For example, publicly traded companies must regularly report their financials and other information to the SEC. Moreover, these acts prohibit **insider trading** on unreleased specific information, although more general trading by insiders is legal (and seems to be done fairly profitably). The SEC can only pursue civil fines. It is up to the states to pursue criminal sanctions, which they often do simultaneously. (Other regulations that publicly traded firms have to follow derive from some other federal laws, and, more importantly, state laws.)

Publicly traded companies must follow rules. For example, they must report their financials, and restrict insider trading.

Anecdote: Trading Volume in the Tech Bubble

During the tech bubble of 1999 and 2000, IPOs appreciated by 65% on their opening day *on average*. Getting an IPO share allocation was like getting free money. Of course, ordinary investors rarely received any such share allocations—only the underwriter's favorite clients did. This later sparked a number of lawsuits, one of which revealed that **Credit Suisse First Boston** (**CSFB**) allocated shares of IPOs to more than 100 customers who, in return for IPO allocations, funneled between 33% and 65% percent of their IPO profits back to CSFB in the form of excessive trading of other stocks (like Compaq and Disney) at inflated trading commissions.

How important was this "kickback" activity? In the aggregate, in 1999 and 2000, underwriters left about \$66 billion on the table for their first-day IPO buyers. If investors rebated 20% back to underwriters in the form of extra commissions, this would amount to \$13 billion in excessive underwriter profits. At an average commission of 10 cents per share, this would require 130 billion shares traded, or an average of 250 million shares per trading day. This figure suggests that kickback portfolio churning may have accounted for as much as 10 percent of all shares traded!

Source: Ritter-Welch (2002).

Outflows

Money flows out from the financial markets via dividends and share repurchases.

Capital flows out of the financial markets in a number of ways. The most important venues are capital distributions such as dividends and share repurchases. Many companies pay some of their earnings in **dividends** to investors. Dividends, of course, do not fall like manna from heaven. For example, a firm worth $100,000 may pay $1,000, and would therefore be worth $99,000 after the dividend distribution. If you own a share of $100, you would own (roughly) $99 in stock and $1 in dividends after the payment—still $100 in total, no better or worse. (If you have to pay some taxes on dividend receipts, you might come out for the worse.) Alternatively, firms may reduce their outstanding shares by paying out earnings in **share repurchases**. For example, the firm may dedicate the $1,000 to share repurchases, and you could ask the firm to dedicate $100 thereof to repurchasing your share. But even if you hold onto your share, you have not lost anything. Previously, you owned $\$100/\$100,000 = 0.1\%$ of a $100,000 company, for a net of $100. Now, you will own $\$100/\$99,000 = 1.0101\%$ of a $99,000 company—multiply this to find that your share is still worth $100. In either case, the value of outstanding public equity in the firm has shrunk from $100,000 to $99,000. To learn more about dividends and share repurchases, you should read a corporate finance text.

Shares can also shrink out of the financial markets in bankruptcies, liquidations, and delistings.

Firms can also exit the public financial markets entirely by delisting. Delistings usually occur either when a firm is purchased by another firm or when it runs into financial difficulties so bad that they fail to meet minimum listing requirements. Often, such financial difficulties lead to bankruptcy or liquidation. Some firms even voluntarily liquidate, determining that they can pay their shareholders more if they sell their assets and return the money to them. This is rare, because managers usually like to keep their jobs—even if continuation of the company is not in the interest of shareholders. More commonly, firms make bad investments and fall in value to the point where they are delisted from the exchange and/or go into bankruptcy. Fortunately, investors enjoy **limited liability**, which means that they can at most lose their investments and do not have to pay further for any sins of management.

Solve Now!

Q 8.8 What are the main mechanisms by which money flows from investors into firms?

Q 8.9 What are the institutional mechanisms by which funds disappear from the public financial markets back into the pockets of investors.

Q 8.10 How do shares disappear from the stock exchange?

8·2 Equities Transaction Costs

8·2.A. Going Long

The costs of buying shares depend on the stock, but typically are less than 0.5 percent (round-trip) for a large firm's stock.

The process of buying stocks is familiar to almost everyone: you call up your broker to purchase 100 shares of a stock (say PepsiCo) with cash sitting in your account, and the shares appear in your account and the cash disappears from your account. When you want to sell your shares, you call your broker again to sell the shares and the appropriate value of the shares returns as cash into your account. There are some transaction costs in the process: the broker collects a commission (typically ranging from about $8 at a discount broker to $100 at a full-service broker); and you are most likely to buy your shares at the **ask price**, which is higher than the **bid price**, at which you can sell the shares. For a stock like PepsiCo, trading around $50, the "bid-ask spread" may be 10 to 20 cents or about 0.2 percent. Buying and then immediately selling 1,000 shares of PepsiCo ($50,000), a **round-trip transaction**, might cost you **transaction costs** of around $100 to $200 (lost to the bid-ask spread) plus $16 to $200 (lost to your broker). Your $50,000 would have turned into about $49,600 to $49,900.

8·2.B. Going Short: The Academic Fiction

Idealized shorting, as used in academia, gives shortsellers interest on the stock that is being sold.

But, what if you want to speculate that a stock will be going down rather than up? This is called **shorting** a stock. (In optional Section b, we have already discussed shorting in the context of Treasury securities and apples.) Optimally, you would want to do the same thing that the PepsiCo company does: give other investors who want to buy shares in PEP the exact same payoffs (including dividends!) that PEP will provide in exchange for them giving you $50. If the share price declines to $30, upon termination of the short, you would have received their $50 up front and only repaid them $30—you would have earned $20. In addition, you could have earned interest on the $50. This is the idealized world of theoretical finance and of this book, in which borrowing and lending can be done without friction. The upper half of Table 8.1 shows such an example of a particular portfolio that involves idealized, frictionless shorting.

8·2.C. Going Short: The Real World

Real-world shorting requires more than just offering the same cash flows as the stock

In the real world, shorting is not so easy. First, there are rules and regulations that the SEC imposes on short-selling that you have to follow. Second, you need to credibly guarantee that you can give the share purchaser all the cash flows that PepsiCo shares offer. (What have you committed to if the share price triples? Remember that you have unlimited liability as a short!) Third, a real investor in PepsiCo also receives the accounting statements of PepsiCo in the mail and can vote at the annual meetings. How do you offer this service? The answer is that you need to find an investor who already owns the shares and who is willing to lend them to you, so that you can sell the shares—real physical shares—to someone on the exchange. You then owe shares to this lending investor, rather than to the person buying the shares on the exchange.

DIG DEEPER

> The most important SEC regulation concerning shorting is that the broker must borrow the shares from a willing owner and then resell them to a third party. It is not enough for another investor to be willing to take the other side of the short trade: instead, the shares have to be actually physically found from an investor holding them. For some smaller stocks, it can occasionally be difficult to find someone willing to lend the shares, which can make shorting difficult or impossible. In addition, there is a second SEC rule that states that shares can only be shorted on an up-tick. The intent is to reduce further short-selling during a stock market crash, when up-ticks are rare.

Real-world shorting loses the use of (some) proceeds. This is a "friction."

All of the details necessary to execute a short can be arranged by your broker. Unlike buying shares long, execution of a short is often not instantaneous. But more importantly, the broker's service comes at a price. The broker usually does not return to you the $100 paid by the person buying the shares, so that you can invest the proceeds in bonds. That is, if the stock price declines to $90, you still made $10, but the interest on the $100 is earned by your broker, not by you. In addition, as with a purchase of shares, the broker earns commissions and the bid-ask spread goes against you. The lower panel in Table 8.1 contrasts the idealized version of shorting (used in this book) to the grittier real-world version of shorting.

Large fund investors can short at more favorable rates than ordinary investors.

Large clients can usually negotiate to receive at least some of the interest earned on the $100, at least for large, liquid stocks. Hypothetically, if such a large investor were both short one share of a firm and long one share of the same firm, she would lose about 100 to 300 basis points per year. On a $100 share, the cost of being long one share and being short one share would typically be $1 to $3 per share per year. This money is shared between the brokerage firm and the investor willing to loan out shares to you for shorting (so that you can sell them to someone else). Nowadays in the real-world, large stock index funds earn most of their profits through lending out shares to shortsellers.

Our theories will assume that shorting is possible—a model simplification that is often but not always acceptable.

Shorting is not ideal in the real world—but it is a whole lot more ideal in financial markets than in non-financial markets. Consider the large long exposure risk that a house purchaser suffers. If the house value drops by 20%, the owner could easily lose more than all his equity stake in the house. To hedge against drops in the value of the house, it would make sense for this purchaser to go short on equivalent housing in the same neighborhood. This way, if real estate prices were to go down, the short position in the neighborhood would mitigate the own-house loss. For all practical purposes, this is unfortunately impossible. (I have myself failed to figure out how to do this.) In effect, the costs of shorting can be almost infinitely high. When you use the situation in real estate as your benchmark, it indeed seems reasonable to assume no transaction costs to shorting equities, after all, at least for our academic purposes.

Solve Now!

Q 8.11 What are the main differences between academic, theoretical, perfect shorting and real-world, practical shorting?

Q 8.12 If you simultaneously buy and short $5,000 of IBM at the beginning of the year, and you terminate these two positions at the end of the year, how much would it cost you in the real world?

Q 8.13 Assume you believe that stock in KO will go up by 12% and stock in PEP will go up by 15% over the next year. The current risk-free interest rate is 2% per year. You have $300,000 to invest, and your broker allows you to go short up to $100,000.

(a) How much could you go long in PEP?

(b) If your forecast comes true, how much money would you earn in a fictional world? What would your rate of return be?

(c) If your forecast comes true, how much money would you earn in the real world?

Anecdote: Eternal Shorts?

The short must make good on all promises that the underlying firms make. There are some rare instances in which this can cause unexpected problems. For example, when Heartland Industrial Partners acquired Mascotech for about $2 billion in 2000, the latter promised to dispose of some non-operating assets and distribute the proceeds to the original shareholders. As of 2005, that has not yet happened. Anyone having written a short on Mascotech—including myself—still has an escrowed obligation as of 2005 that cannot be closed out.

Table 8.1: Shorting in an Idealized and in the Real World

Idealized Shorting Example

Your Wealth: \$200.

You can sell \$100 worth of KO shares (or an equivalent promise) to another investor, who wants to hold KO shares. This gives you \$200 + \$100 = \$300 of cash, which you can invest into Pepsico.

Portfolio P: $w_{KO} = -\$100,\ w_{PEP} = \$300, \quad \Longrightarrow w_{KO} = -50\%,\ w_{PEP} = 150\%.$

Hypothetical Rates of Return: KO = −10%, PEP = +15% .

$\Longrightarrow$ Portfolio Rate of Return: $r_P = -50\% \cdot (-10\%) + 150\% \cdot (+15\%) = +27.5\%.$

\$100 KO shares borrowed became a liability of \$90, for a gain of \$10;

\$300 PEP shares invested became an asset of \$345, for a gain of \$45.

$\Longrightarrow$ Your net portfolio gain is \$55 on an original investment of assets worth \$200, which comes to a +27.5% rate of return.

Real World Retail Investor Shorting Example

Your Wealth: \$200.

The broker finds another investor to borrow shares from and sells the shares (on your behalf) for \$100 to another investor, who wants to hold KO shares. The broker keeps \$100, because in our example, the retail investor is assumed to receive absolutely no shorting proceeds. (Institutional investors can typically receive some, but not all of the shorting proceeds.) You still have \$200 in cash (\$100 less than in the idealized case), which you can invest into Pepsico.

Portfolio P: $w_{KO} = -\$100,\ w_{PEP} = \$200, \quad \Longrightarrow w_{KO} = -50\%,\ w_{PEP} = 100\%.$

Hypothetical Rates of Return: KO = −10%, PEP = +15% .

$\Longrightarrow$ Portfolio Rate of Return: $r_P = -50\% \cdot (-10\%) + 100\% \cdot (+15\%) = +20.0\%.$

\$100 KO shares borrowed became a liability of \$90, for a gain of \$10;

\$200 PEP shares invested became an asset of \$230, for a gain of \$30.

$\Longrightarrow$ The net portfolio gain is \$40 on an original investment of assets worth \$200, which comes to a +20% rate of return.

Both examples ignore trading costs incurred in buying and selling securities.

8·3 Portfolios and Indexes

8·3.A. Portfolio Returns

What exactly is a portfolio? Is it a set of returns? No. The portfolio is a set of investment weights. When these weights are multiplied by their asset returns, you obtain your overall portfolio return.

IMPORTANT: A **portfolio** is a set of investment weights.

Overall portfolio returns are the investment-weighted average returns of its constituents.

You would usually own not just one security but form a portfolio consisting of many holdings. Your ultimate goal—and the subject of the area of investments—is to select good portfolios with high rates of return. But how do you compute your portfolio rate of return? For example, say you hold \$500 in PEP, \$300 in KO, and \$200 in CSG. Your total investment is \$1,000, and your portfolio investment weights are 50% in PEP, 30% in KO, and 20% in CSG. If the rate of return on PEP is 5%, the rate of return on KO is 2%, and the rate of return on CSG is −4%, then the rate of return on your overall portfolio (P) is

$$
\begin{aligned}
\tilde{r}_{P} &= \left(\frac{\$500}{\$1,000}\right)\cdot(+5\%) + \left(\frac{\$300}{\$1,000}\right)\cdot(+2\%) + \left(\frac{\$200}{\$1,000}\right)\cdot(-4\%) \\
& \quad (50\%)\cdot(+5\%) + (30\%)\cdot(+2\%) + (20\%)\cdot(-4\%) = +2.3\% \\
&= w_{PEP}\cdot r_{PEP} + w_{KO}\cdot r_{KO} + w_{CSG}\cdot r_{CSG}
\end{aligned}
$$

When you own multiple assets, your overall investment rate of return is the investment-weighted rate of return on each investment, with the weights being the relative investment proportions. Let us check this. A \$500 investment in PEP at a 5% rate of return gave \$25. A \$300 investment in KO at a 2% rate of return gave \$6. A \$200 investment in CSG at a −4% rate of return gave −\$8. The net dollar return on the \$1,000 was therefore \$25 + \$6 − \$8 = \$23. The portfolio P rate of return was $(r_{P,t=1} - r_{P,t=0})/r_{P,t=0} = \$23/\$1,000 - 1 = 2.3\%$. This also clarifies that the portfolio formula also works with absolute dollar investments instead of relative percentage investments:

$$
\begin{aligned}
r_{P} &= (\$500)\cdot(+5\%) + (\$300)\cdot(+2\%) + (\$200)\cdot(-4\%) = +\$23 \\
&= w_{PEP}\cdot r_{PEP} + w_{KO}\cdot r_{KO} + w_{CSG}\cdot r_{CSG}
\end{aligned}
$$

SIDE NOTE

When a security has additional payouts (such as dividends) over the measurement period, its rate of return should really be written as

$$
r_{t-1,t} = \frac{(P_t + \text{Dividends}_{t-1,t}) - P_{t-1}}{P_{t-1}}
$$

Alternatively, you could quote a net price at the end of the period, which includes dividends. Our discussions will mostly just ignore dividends and stock splits. That is, when I write about returns, I usually mean rates of returns that take into account all payments to the investor, but I sometimes abbreviate this as $(P_t - P_{t-1})/P_{t-1}$ for convenience.

Notation: number investments. A portfolio is a set of known weights.

The goal of the subject of investments is to evaluate all possible investment choices in order to determine the best portfolio. We need to come up with good notation that does not make discussing this task too cumbersome. Let us use R and r as our designated letters for “rate of return.” But with thousands of possible investment choices, it is rather inconvenient to work with ticker symbols (or even full stock names). It would also be tedious to write “average the returns over all possible stocks (ticker symbols) and other securities” and the name them all. Therefore, we often change the names of our securities to the numbers $1, 2, 3, \ldots, N$. We also usually use the letter P to name a portfolio (or, if we work with multiple portfolios, with a capital letter close by, such as Q or O). We call the investment weight in security i by the moniker w_i, where i is a number between 1 and N. Finally, we rely on “summation notation”: $\sum_{i=1}^{N} f(i)$

is the algebraic way of stating that we compute the sum $f(1) + f(2) + ...f(N)$. For example, $\sum_{i=2}^{4}\sqrt{i}$ is notation for $\sqrt{2}+\sqrt{3}+\sqrt{4} \approx 5.15$. (Appendix Chapter 1·1 reviews summations.) Yes, notation is a pain, but with the notation we have, we can now write the rate of return on a portfolio much more easily:

IMPORTANT: The rate of return R on a portfolio P that consists of N securities named 1 through N is

$$r_P = \sum_{i=1}^{N} w_i \cdot r_i = w_1 \cdot r_1 + w_2 \cdot r_2 + \cdots + w_N \cdot r_N \tag{8.1}$$

where w_i is the investment weight in the i-th security (from N choices). If weights are quoted as a fraction of the overall investment portfolio, their sum must add up to 100%,

$$\sum_{i=1}^{N} w_i = 100\%$$

Notation: Abbreviations

OK, we are cheating a little on notation: each rate of return R should really have three subscripts: one to name the financial security (e.g., PEP or i or 4), one for the beginning of the period (e.g., t), and one for the end of the period (e.g., $t+1$)—too many for my taste. When there is no danger of confusion—or the formula works no matter what periods we choose (as long as we choose the same period for all securities)—let us omit the time subscripts.

8·3.B. Funds and Net Holdings

Funds are themselves portfolios of other financial assets.

One can think of portfolios, consisting of stocks, the same ways as one can think of stocks themselves. Indeed, **funds** are firms which hold underlying stocks or other financial assets and are thus themselves de facto portfolio—and funds can be bought and sold just like any other stocks. Investors often like buying shares in funds because they believe that a professional manager can pick securities better than they can, plus funds in effect allow individual investors to purchase thousands of stocks, even if they only have a small amount of money to invest. Depending on their legal arrangements, funds may be called **exchange-traded funds** (bought and sold on a financial market), **mutual funds** (bought and sold by the general public, but not on an exchange), or **hedge funds** (not marketed to the broad public, and therefore not subject to SEC restrictions). Many mutual funds have prices that are listed daily in the *Wall Street Journal*. Like exchange funds, they can be purchased easily through most stock brokers. An **ADR** (**American Depositary Receipt**) is another common form of fund. It is a relatively easy way by which a large foreign company can trade shares on the New York Stock Exchange. Its domestic shares are put into an escrow, and the U.S. exchange trades the ADR. An ADR really operates like an open-end fund which holds shares only in this one company.

Anecdote: More funds or more stocks?

In 1999, U.S. equity funds managed roughly 3 trillion dollars of assets, or about one-third of U.S. stock market capitalization. More surprisingly, there were more U.S. equity funds than there were U.S. stocks: In 1999, there were 8,435 equities, but 11,882 equity funds. Source: Harry Mamaysky and Matthew Spiegel.

Open-ended versus closed-ended Mutual Funds.

Mutual funds come in two main forms. **Open-ended** funds allow any investor to exchange the fund shares for the underlying assets in the appropriate proportion. For example, if a fund has sold 50 shares, and used the money to purchase 200 shares of PepsiCo and 300 shares of Coca Cola, then each mutual fund share represents 4 PepsiCo shares and 6 Coca Cola shares. The mutual fund share holder can, at her will, exchange her fund share into 4 PepsiCo and 6 Coca Cola shares. This forces an **arbitrage** link between the price of the fund and the value of its assets: if the price of the fund drops too much relative to the underlying assets, then investors will redeem their mutual fund shares. In a **closed-end** mutual fund, redemption is not permitted. If the underlying fund assets are very illiquid (e.g., real-estate in emerging countries), an open-ended like redemption request would be very expensive or even impossible to satisfy. Closed-end funds often trade for substantially less than their underlying assets, and for significant periods of time. Among the explanations for this **closed end fund discount** are the significant fees collected by the fund managers. At the end of 2002, there were about 7,000 open-end mutual funds with $4 trillion in assets. There were only about 500 closed-end mutual funds with about $150 billion in assets, and another 4,500 hedge funds with assets of about $350 billion (most hedge funds are closed-end). Thus, open-ended funds controlled about ten times more money than closed-ended funds.

How to compute net underlying holdings.

Most mutual funds disclose their holdings on a quarterly basis to the SEC (semi-annual is mandatory), which makes it easy for investors to compute their net exposures (at least on the reporting day). For example, assume that Fund FA holds $500,000 of PepsiCo and $1,500,000 of Coca Cola. Assume that Fund FBholds $300,000 of Coca Cola and $700,000 of Cadbury Schweppes. What is your net portfolio if you put 60% of your wealth into Fund FA and 40% of into Fund FB? The funds have holdings of

$$\begin{array}{lrcrlrcrlrcr}
\text{Fund FA Holdings:} & w_{\text{FA,PEP}} & = & 0.25 & , & w_{\text{FA,KO}} & = & 0.75 & , & w_{\text{FA,CSG}} & = & 0 \\
\text{Fund FB Holdings:} & w_{\text{FB,PEP}} & = & 0 & , & w_{\text{FB,KO}} & = & 0.30 & , & w_{\text{FB,CSG}} & = & 0.70
\end{array}$$

You can compute your net exposures by computing the sum of your holdings multiplied by the fund holdings:

$$\begin{array}{rcrcrcr}
w_{\text{PEP}} & = & 60\% \cdot 0.25 & + & 40\% \cdot 0.00 & = & 15\% \\
w_{\text{KO}} & = & 60\% \cdot 0.75 & + & 40\% \cdot 0.30 & = & 57\% \\
w_{\text{CSG}} & = & 60\% \cdot 0.00 & + & 40\% \cdot 0.70 & = & 28\% \\
w_{\text{i}} & = & w_{\text{FA}} \cdot w_{\text{FA},i} & + & w_{\text{FB}} \cdot w_{\text{FB},i} & &
\end{array}$$

which adds up to 100%. For example, for a $2,000 investment, you own $300 of shares in PepsiCo, $1,140 of shares in Coca Cola, and $570 of shares in Cadbury Schweppes. In sum, you can think of funds and portfolios the same way you think of stocks: they are investment opportunities representing combinations of assets. You can always compute the underlying stock holdings represented by the funds. In fact, if you wish, you could even see ordinary firms as portfolios bundling underlying assets for you.

Anecdote: The Worst of all Worlds: High Losses plus High Taxes

To prevent tax arbitrage—which are basically transactions that create fake losses to reduce taxable income—the IRS has instituted special tax rules for mutual funds. The precise treatment of the taxes is complex and beyond the scope of this book, but the most important aspect is simple: investors must absorb the underlying capital gains/losses and dividend payments received by the funds, as if they themselves had traded the underlying shares themselves.

In the second half of 2000, many mutual funds had lost significant amounts of money in the collapse of the technology bubble. Although their values had declined (and with them, the wealth of their clients), the funds had generally not yet realized these capital losses. But they had realized capital gains earlier in the year. The IRS requires these realized losses to be declared by fund investors as “pass-through” capital gains, which were therefore taxed. Thus, in 2000, many unlucky investors experienced high losses and still had to pay high taxes.

8·3.C. Some Common Indexes

An index is almost like a fund or portfolio, but it is not something that one can invest in *because an index is just a number.* (There are, however, funds that try to mimic the behavior of indexes.) Most commonly, an index is the figure obtained by computing a weighted sum of the prices of a predetermined basket of securities. It is intended to summarize the performance of a particular market or market segment. For example, the **Dow-Jones 30** index is a weighted average of the prices of 30 pre-selected "big" stocks. (Table b below lists them.) If you purchase a portfolio holding the same 30 stocks (or a fund holding the 30 stocks), your investment rate of return should be fairly close to the percentage change in the index—except for one difference. When stocks pay dividends, their stock prices decline by just about the amounts of dividends paid. (If they dropped less on average, you should purchase the stocks, collect the dividends, and then resell them for a profit. If they dropped more, you should short the stocks, pay the dividends, and then cover your shorts for a profit.) Your portfolio mimicking the Dow-Jones 30 should earn these dividends, even though the index would decline by the percent paid out in dividends. Therefore, in theory, you should be able to easily outperform an index. Unfortunately, in the real world, most portfolio managers fail to do so, primarily because of transaction costs and excessive trading.

Although an Index is a weighted average of stock prices, it is just a number, not a portfolio.

↦ *Table b on Page 234*

Most indexes, including the S&P 500 and the Dow-Jones 30, are not adjusted for dividends, although they are adjusted for **stock splits**. In a 3:1 stock split, a firm trading at $120 per old share would henceforth trade at $40 per new share. Each investor who held one old share would receive three new shares. The "guardians of the index" would adjust the index formulas by tripling the weight on the stock that split. In contrast to plain price indexes, there are also **total return indexes**. For example, the guardians of the formula for the German **Dax Performance Index** change the formula to reflect the return that a portfolio of Dax stocks would earn through dividends.

Some more detail: plain indexes versus total return indexes.

There are literally hundreds of indexes, created and published by hundreds of companies. The **Money&Investing** Section of the *Wall Street Journal* lists just a sampling. In the United States, the most prominent stock market indexes are the **S&P 500** (holding 500 large stocks), the aforementioned **Dow-Jones 30** (holding 30 big stocks, selected to cover different industries), and the **Nasdaq** index (holding the largest Nasdaq companies). The **Russell 2000** covers 2,000 small-firm stocks. There are also other asset class indexes. For example, **Lehman Brothers** publishes the **MBS** (Mortgage Bond Securities) index; **Dow Jones** also publishes a corporate bond index; and **Morgan Stanley** publishes a whole slew of country stock prices indexes (**MSCI EAFE**). Furthermore, each country with a stock market has its own domestic index. Some foreign stock market indexes are familiar even to casual investors: the **Financial Times Stock Exchange** Index, spelled **FTSE** and pronounced "foot-sy" for Great Britain; the **Nikkei**-225 Index for Japan; and the **DAX** index for Germany.

There are more indexes than one can count, and new ones get invented all the time.

Anecdote: The Presidential Election Market

The University of Iowa runs the Iowa Electronic Market, which are indexes measuring the likelihood for each presidential candidate to win the next presidential election. You can actually trade futures based on these indexes. This market tends to be a better forecaster of who the next president will be than the press.

8·3.D. Equal-Weighted and Value-Weighted Portfolios

An example of an equal-weighted and a value-weighted portfolio.

Two kinds of portfolios deserve special attention, the equal-weighted and the value-weighted market portfolio. To see the difference between the two, assume that there are only three securities in the market. The first is worth \$100 million, the second \$300 million, and the third \$600 million. An **equal-weighted** portfolio purchases an equal amount in each security. For example, if you had \$30 million, you would invest \$10 million into each security. Does it take trading to maintain an equal-weighted portfolio? Table 8.2 shows what happens when one stock's price changes: security $i = 1$ quadruples in value. If you do not trade, your portfolio holdings would be too much in security 1 relative to securities 2 and 3. To maintain an equal-weighted portfolio, you would have to rebalance. In the example, you would have to trade \$40 worth of stock.

Table 8.2: Maintaining an Equal-Weighted Portfolio

	Time 0			Time 1			
Security i	Market Value	Investor Pfio	Rate of Return	Market Value	Investor, No Trade	Investor, Desired	Necessary Trading
1	\$100	\$10	+300%	\$400	\$40	\$20	−\$20
2	\$300	\$10	0%	\$300	\$10	\$20	+\$10
3	\$600	\$10	0%	\$600	\$10	\$20	+\$10
Sum	\$1,000	\$30		\$1,300	\$60	\$40	\$0

Table 8.3: Maintaining a Value-Weighted Portfolio

	Time 0			Time 1			
Security i	Market Value	Investor Pfio	Rate of Return	Market Value	Investor, No Trade	Investor, Desired	Necessary Trading
1	\$100 (10%)	\$3	+300%	\$400 (31%)	\$12	\$12	\$0
2	\$300 (30%)	\$9	0%	\$300 (23%)	\$9	\$9	\$0
3	\$600 (60%)	\$18	0%	\$600 (46%)	\$18	\$18	\$0
Total	\$1,000 (100%)	\$30		\$1,300 (100%)	\$39	\$39	\$0

Value-weighted portfolios are easier to maintain.

A **value-weighted** portfolio purchases an amount proportional to the availability of each security. In the example, with \$30 million, a portfolio that invests \$3 million in the first security (weight: $\$2/\$20 = 10\%$), \$9 million in the second security (weight: 30%), and \$18 million in the third security (weight: 60%) is value-weighted. How difficult is it to maintain this value-weighted portfolio? In Table 8.3, the first security has again quadrupled in value, increasing in market capitalization to \$400 million. Without trading, your previously value-weighted portfolio has increased its holdings in this security from \$3 million to \$12 million. The portfolio weight in the first security would therefore have increased to $\$12/\$39 \approx 31\%$, the second security would have dropped to $\$9/\$39 \approx 23\%$, and the third security would have dropped to $\$18/\$39 \approx 46\%$. But these are exactly the weights that a value weighted portfolio of \$39 million, if initiated at time 1, would require! The portfolio weights require no adjustment because any changes in the market values of securities are reflected both in the overall market capitalizations and the weight of the securities in your portfolio. In contrast to the earlier equal-weighted portfolio, a value-weighted portfolio requires no trading. (The only exception are securities that enter and exit the market altogether.) Even though it may be easier at the beginning to select an equal-weighted portfolio (you do not need to know how much of each security is available), over time, it is easier to maintain a value-weighted portfolio.

IMPORTANT: To maintain an equal-weighted portfolio, continuous rebalancing is necessary. To maintain a value-weighted portfolio, no rebalancing is usually necessary.

It is possible for everyone in the economy to hold a value-weighted portfolio, but not an equal-weighted portfolio.

There is a second important feature of value-weighted portfolios: it is possible for everyone in the economy to hold a value-weighted portfolio, but not possible for everyone in the economy to hold an equal-weighted portfolio. Return to the example with \$1 billion in overall market capitalization and only two investors: the first has \$100 million in wealth, the second has \$900 million in wealth. Equal-weighted portfolios would have the first investor allocate \$33 million to the first security and have the second investor allocate \$300 million to the first security. In sum, they would want to purchase \$333 million in the first security—but there is only \$100 million worth of the first security to go around. The pie is just not big enough. In contrast, holding value-weighted portfolios, both investors could be fully satisfied with their slices. In the example, for the first security, the first investor would allocate \$10 million, the second investor would allocate \$90 million, and the sum-total would equal the \$100 million available in the economy.

IMPORTANT: It is possible for all investors in the economy to hold value-weighted portfolios. It is impossible for all investors in the economy to hold equal-weighted portfolios.

Time Convergence?

Three more points: First, over time, if you do not trade, even a non-value weighted portfolio becomes more and more value-weighted. The reason is that stocks that increase in market value turn into larger and larger fractions of your portfolio, and stocks that decline in market value turn into smaller and smaller fractions. Eventually, the largest firms in the economy will be the biggest component of your portfolio. Second, the most popular and important stock market indexes are more like value-weighted portfolios than equal-weighted portfolios. For example, the S&P500 index behaves much more like the value-weighted than like the equal-weighted market index. Third, over short time frames (say a month or even a year), broad stock market indexes within a country tend to be very highly correlated (say, above 95%), no matter whether they are equal-weighted, value-weighted, or arbitrary (e.g., the Dow-Jones 30 or the S&P500). Therefore, if the newspaper reports the return of the S&P500 yesterday, it is a pretty good estimator either for the return of broader portfolios (like a value-weighted overall stock market portfolio) or for the return of narrower portfolios (like the Dow-Jones 30). It is rare that one goes up dramatically, while the other goes down, and vice versa.

Solve Now!

Q 8.14 An investor's portfolio P consists of 40% of stock A and 60% of stock B. A has a rate of return of +4%, B has a rate of return of +6%. What is the overall portfolio rate of return?

Q 8.15 An investor owns \$40 in stock A and \$60 in stock B. A has a return of \$1.60, B has a return of +\$3.60. What is the overall portfolio return?

Q 8.16 An investor owns \$40 in stock A and \$60 in stock B. The first stock has a return of +4%, the second has a return of +6%. What is the overall portfolio rate of return?

Q 8.17 Write down the formula for the return of a portfolio, given individual security returns and their weights. First use summation notation, then write it out.

Q 8.18 A portfolio consists of \$200 invested in PEP, and \$600 invested in CSG. If the stock price per share on PEP increased from \$30 to \$33, and the stock price per share in CSG declined from \$40 to \$38 but CSG paid a dividend of \$1 per share, then what was the portfolio's return and rate of return?

Q 8.19 Fund FA holds $100,000 of PEP and $600,000 of KO, and $300,000 of CSG. Fund FBholds $5,000,000 of PEP, $1,000,000 of KO, and $4,000,000 of CSG. You have $500 to invest. Can you go long and short in the two funds to neutralize your exposure to PEP? (This means having a net zero exposure to PEP.) How much of each fund would you purchase? What are your de facto holdings of KO and CSG?

Q 8.20 What is $\sum_{j=1}^{5} j^2$?

Q 8.21 What is $\sum_{i=1}^{5} 2 \cdot i$?

Q 8.22 What is $\sum_{j=1}^{5} (j - 5)$?

Q 8.23 What is the difference between a hedge fund and a mutual fund?

Q 8.24 What is the difference between an open-ended mutual fund and a closed-ended mutual fund?

Q 8.25 You hold two funds. Fund FA has holdings in stocks 1, 2, and 3 of 0.15, 0.5, and 0.35, respectively. Fund FBhas holdings in stocks 1, 2, and 3, of 0.4, 0.2, and 0.4, respectively. You would like to have a portfolio that has a net investment weight of 30% on the first stock. If you have decided only to hold funds and not individual stocks, what would your exposure be on stocks 2 and 3?

Q 8.26 What is the difference between an index and a mutual fund?

Q 8.27 List a few prominent financial indexes.

Q 8.28 How does an index differ from a portfolio?

Q 8.29 Compute the value-weighted dollar investments of the two investors (with wealths $100 million and $900 million, respectively) for the second and third securities in the example on Page 181.

Q 8.30 Continuing with this example, what would be the dollar investments and relative investments if the first security were to double in value? Does this portfolio require rebalancing to remain value-weighted?

Q 8.31 There are two stocks: stock 1 has a market capitalization of $100 million, stock 2 has a market capitalization of $300 million.

(a) What are the investment weights of the equal-weighted portfolio?

(b) What are the investment weights of the value-weighted portfolio?

(c) There are 5 equally wealthy investors in this economy. How much of stock 1 would they demand if they all held the equal-weighted portfolio? If they held the value weighted portfolio?

(d) If the first stock appreciates by 10% and the second stock depreciates by 30%, how much trading would such an investor have to do to continue holding an equal-weighted portfolio?

(e) Repeat the previous question with a value-weighted portfolio.

Q 8.32 To maintain an equal-weighted portfolio, do you have to sell recent winner stocks or recent loser stocks? (Is this a bad or a good idea?)

Q 8.33 How different would the one-month performance of an investment in an S&P500 mimicking portfolio be, relative to the performance of the value-weighted market portfolio?

8·3.E. Quo Vadis? Random Returns on Portfolios

A Road map.

Most of your attention in the next few chapters will be devoted to the case where returns are not yet known: they are still "random variables," denoted with a tilde above the unknown quantity (e.g., $\tilde{r}$). The goal of investments is to select a portfolio P (that is, a set of N investment weights, $w_1, w_2, \ldots, w_N$) which offers the highest likely future performance with the least risk. Using both the tilde and our portfolio sum formula, we can write the uncertain future rate of return to our portfolio as

$$\tilde{r}_P = \sum_{i=1}^{N} w_i \cdot \tilde{r}_i = w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2 + \cdots + w_N \cdot \tilde{r}_N$$

We now need to find

1. a good measure for the reward (likely performance) of a portfolio;
2. and a good measure for the risk of a portfolio.

For this, you shall need statistics, the subject of the next chapter.

8·4 Summary

The chapter covered the following major points:

- Securities appear through initial public offerings (IPOs) on exchanges, and disappear through delistings.
- A round-trip transaction is one purchase and one sale of the same security. In the real world, trades incur both brokerage fees and the bid-ask spread. In addition, going short (selling without owning) incurs one extra cost—lack of full use of (interest earnings from) the short-sale proceeds.
- Portfolio returns are a weighted average of individual returns.
- Fund holdings can be deconstructed into individual underlying stock exposures.
- An index is usually computed as the weighted averages of its component price figures. The index is therefore just a number. In contrast, funds and portfolios are collections of underlying assets, the value of which are similarly computed as the weighted average of the underlying component values. Index funds attempt to mimic index percent changes by purchasing stocks similar to those used in the computation of the index.
- Maintaining an equal-weighted index requires constant rebalancing. Maintaining a value-weighted index requires no trading.
- Unlike other portfolios, the value-weighted market portfolio can be owned by each and every investor in equilibrium.

[No keyterm list for secpfios-g.]

End of Chapter Problems

Q 8.34 Explain the differences between a market order and a limit order.

Q 8.35 What extra functions do retail brokers handle that prime brokers do not?

Q 8.36 Describe the differences between the NYSE and NASDAQ.

Q 8.37 Roughly, how many firms are listed on the NYSE? How many are listed on NASDAQ? Then use the WWW to find an estimate of the current number.

Q 8.38 Is NASDAQ a crossing market?

Q 8.39 What is the OTC market?

Q 8.40 What are the three main types of investment companies as defined by the SEC? Which is the best deal in a perfect market?

Q 8.41 What are the two main mechanisms by which a privately held company can go public?

Q 8.42 When and under what circumstance was the SEC founded?

Q 8.43 Insider trading is a criminal offense. Does the SEC prosecute these charges?

Q 8.44 If a firm repurchases 1% of its shares, does this change the capitalization of the stock market on which it lists? If a firm pays 1% of its value in dividends, does this change the capitalization of the stock market on which it lists?

33 "Solve Now" Answers

1. Brokers execute orders and keep track of investors' portfolios. They also arrange for margin.
2. Prime brokers are usually used by larger investors. Prime brokers allow investors to employ their own traders to execute trades. (Like retail brokers, prime brokers provide portfolio accounting, margin, and securities borrowing.)
3. Your rate of return is higher if you short a stock in the perfect world, because you earn interest on the proceeds. In the real world, your broker may help himself to this interest.
4. A crossing system does not execute trades unless there is a counterparty. It also tries to cross orders a few times a day.
5. The specialist is often a monopolist who makes the market on the NYSE. The specialist buys and sells from his own inventory of a stock, thereby "making a market." market makers are the equivalent on NASDAQ, but there are usually many and they compete with one another. Unlike ordinary investors, both specialists and market makers can see the limit orders placed by other investors.
6. The alternatives are often electronic, and they often rely on matching trades—thus, they may not execute trades that they cannot match. Electronic Communication Networks are the dominant example of these. Another alternative is to execute the trade in the Over-The-Counter (OTC) market, which is a network of geographically dispersed dealers who are making markets in various securities.
7. In an open-ended fund, you should purchase fund shares and request redemption. (You could short the underlying holdings during the time you wait for the redemption in order not to suffer price risk.) In a closed-ended fund, you would have to oust the management to allow you to redeem your shares.
8. The main mechanisms by which money flows from investors into firms are first IPOs and SEOs, and second reverse mergers, which are then sold off to investors.
9. Funds disappear from the public financial markets back into the pockets of investors through dividends and share repurchases.

10. Shares can disappear in a delisting or a repurchase.

11. For academic shorting, you just promise the same cash that the shares themselves are paying. For real-world shorting, you ask your broker to find a holder of the shares, borrow them (i.e., and promise him the same payoffs), and then sell them to someone else. Most importantly, the broker who arranges this will not give you the cash obtained from selling the borrowed shares—that is, the broker will earn the interest on the cash, rather than you. Other important differences have to do with the fact that you can be called upon to terminate your short if the lender of shares wants to sell the shares (you have to return the borrowed shorts), and with the fact that you have to find someone willing to lend you the shares.

12. The bid-ask transaction round-trip costs (bid-ask spread and broker commissions) for either the long or the short would be around 30-60 basis points. In addition, you would have to provide the cash for the share purchase; the cash from the share short is most likely kept by the broker. The loss of proceeds would cost you another 300 to 500 basis points per year in lost interest proceeds, depending on who you are and whether you already have the money or whether you have to borrow the money. If the shares cost \$5,000, you would have "buy" transaction costs of around \$25 and "sell" transaction costs of around \$25, for total transaction costs of \$50; plus interest opportunity costs of around \$100 to \$250.

13.
    1. \$400,000.
    2. If you use the KO short proceeds to purchase stock in PEP, then $\$400,000 \cdot (+15\%) - \$100,000 \cdot (+12\%) = \$48,000$. On \$300,000 net investment, your rate of return would be 16%.
    3. $\$300,000 \cdot (+15\%) - \$100,000 \cdot (+12\%) = \$33,000$. On \$300,000 net investment, so your rate of return would be 11%. You would be better off forgetting about the shorting and earn the 15% on PEP.

14. $r_P = w_A \cdot r_A + w_B \cdot r_B = 40\% \cdot 4\% + 60\% \cdot 6\% = 5.2\%$.

15. $r_p = r_A + r_B = \$5.20$.

16. $r_P = w_A \cdot r_A + w_B \cdot r_B = \$40 \cdot 4\% + \$60 \cdot 6\% = \$5.20$. Note that the formula works with dollar investments, too.

17. See Formula 8.1 on Page 177.

18. $r_{PEP} = 10\%. r_{CSG} = (\$38 + \$1 - \$40)/\$40 = -2.5\%$. Therefore, the absolute return and the rate of return on the portfolio was

$$r_P = \$200 \cdot (10\%) + \$600 \cdot (-2.5\%) = +\$5$$
$$r_P = \left(\frac{\$200}{\$800}\right) \cdot (10\%) + \left(\frac{\$600}{\$800}\right) \cdot (-2.5\%) = +0.625\%$$

19. The PEP weights in the two funds are 10% and 50%, respectively. To have zero exposure, you solve

$$w_{FA} \cdot 10\% + (1 - w_{FA}) \cdot 50\% = 0 \quad w_{FA} = 1.25 \;, \quad w_{FB} = -0.25$$

Therefore, your net exposures are

$$w_{KO} = 1.25 \cdot 60\% + (-0.25) \cdot 10\% = 72.5\% w_{CSG} = 1.25 \cdot 30\% + (-0.25) \cdot 40\% = 27.5\%$$

If you have \$500 to invest, you would short \$125 in Fund FB, leaving you with \$625 to go long in Fund FA.

20. $\sum_{i=1}^{5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$. (Note that i is just a counter name, which can be replaced by any letter, so this answer is correct.)

21. $\sum_{j=1}^{5} (2 \cdot j) = 2 + 4 + 6 + 8 + 10 = 30$.

22. $\sum_{j=1}^{5} (j - 5) = -4 - 3 - 2 - 1 - 0 = -10$.

23. See Page 177.

24. See Page 177.

25.

$$w_{FA} \cdot 15\% + (1 - w_{FA}) \cdot 40\% = 30\% \quad w_{FA} = 40\%$$
$$w_{FA} \cdot w_{FA,1} + w_{FB} \cdot w_{FB,1} = w_1$$

If you purchase 40% in fund FA, your net holdings in each stock are

$$\text{Stock } 1: \quad 40\% \cdot 15\% + 60\% \cdot 40\% = 30\%$$
$$\text{Stock } 2: \quad 40\% \cdot 50\% + 60\% \cdot 20\% = 32\%$$
$$\text{Stock } 3: \quad 40\% \cdot 35\% + 60\% \cdot 40\% = 38\%$$
$$w_{FA} \cdot w_{FA,i} + w_{FB} \cdot w_{FB,i} = w_i$$

26. An index is a series of numbers; a mutual fund holds stocks. A mutual fund can hold stocks to mimic the return on an index.

27. See text for possible indexes to mention.

28. Indexes are numbers, not investments. Their percent change differs from a mimicking portfolio rate of return in that dividends are ignored.

29. In the second security: \$30 and \$270, for the first and second investor, respectively. In the third security: \$60 million and \$540 million, respectively.

30. As already computed in the text, the first security would increase from \$100 million to \$200 million, and thus total market capitalization would increase from \$1 billion to \$1.1 billion. Therefore, the weight of the three securities would be 18%, 27%, and 55%. (Moreover, the portfolio increased by 10% in value, which means that the first investor now has holdings worth \$110 million, and the second investor has holdings worth \$990 million.) The dollar investments are even simpler: the first investor started with \$10, \$30, and \$60 million, respectively, and now holds \$20, \$30, and \$60 million. The second investor now holds \$180, \$270, and \$540 million, respectively.

31.

   1. $w_1 = 50\%, w_2 = 50\%$.
   2. $w_1 = 25\%, w_2 = 75\%$.
   3. Each investor would own \$80 million worth of securities. If each investor held the equal-weighted portfolio, the total demand for stock 1 would be $\$40 \cdot 5 = \$200$. This is impossible. If each investor held the value-weighted portfolio, the total demand for stock 1 would be $\$20 \cdot 5 = \$100$. This is definitely possible.
   4. The equal-weighted portfolio would have started out at $w_1 = \$40m, w_2 = \$40m$. The returns would have made the portfolio $w_1 = \$44m, w_2 = \$28m$ for a total value of $\$72m$. The revised portfolio would have to be $w_1 = \$36m, w_2 = \$36m$. Therefore, the investor would have to sell $\$8m$ in security 1 in order to purchase $\$8m$ in security 2.
   5. The value-weighted investor portfolio started out at $w_1 = \$20m, w_2 = \$60m$, and without trading would have become $w_1 = \$22m, w_2 = 42m$ for a portfolio worth $\$64m$. The investor's weights would be $w_1 = 34.375\%$, $w_2 = 65.625\%$.
      In terms of the market, the first stock would have appreciated in value from \$100 million to \$110 million, while the second would have depreciated from \$300 million to \$210 million. The value-weighted market portfolio would therefore invest $w_1 = \$110/\$320 \approx 34.375\%$ and $w_2 = \$280/\$320 \approx 65.625\%$. Therefore, no trading is necessary.

32. Recent winners have to be sold, recent losers have to be bought. The answer to the question in parentheses is that if the stock market is competitive, past returns should have little predictive power for future returns, so this trading strategy is not necessarily a good or a bad idea. This will be covered in more detail in Chapter 16.

33. The correlation between these two is high. Therefore, it would make little difference.

All answers should be treated as suspect. They have only been sketched and have not been checked.

CHAPTER 9§

Statistics

This chapter appears in the Survey text only.

THIS chapter attempts to distill the essential concepts of a full course in statistics into thirty-something pages. Thus, it is not an easy chapter, but it is also not as complex and painful as you might imagine.

9·1 Historical and Future Rates of Return

Finance is not philosophy—you have little else but history to guide us.

As an investor, your goal is to find the best possible investment portfolio. Easier said than done. What do you know about how a stock, say, IBM, will perform in the future? Not much. Your prime source of information about how IBM *will* perform is how it *did* perform. If it returned 10% per year over the last 10 years *on average*, maybe it is a good guess that it will return 10% over the next year, too. If it had a risk of plus or minus 20% per year, maybe it is a good guess that it will have a risk of plus or minus 20% over the next year, too. But, is historical performance really representative of future performance?

Don't take history too literally.

Clearly, it makes no sense to assume that future returns will be *exactly* the same as past returns. An investment in a particular six lotto numbers may have paid off big last year, but this does not mean that the exact same investment gamble will work again. More sensibly, you should look at the risk/reward characteristics of the *average* six lotto number investment, from which you would probably conclude that the average six number lotto investment is not a great bet.

For stocks, we use general historical characteristics to indicate future general characteristics.

Similarly, for stocks, it makes more sense to assume that future returns will be *only on average* like past returns in terms of risk and reward. It is not that we believe this to be exactly true, but it is usually our best guess *today*. Of course, we also know that the future will turn out different from the past—some firms will do better, others worse—but we generally have no better information than history. (If you can systematically estimate future risk and reward better than others, you are bound to become rich.)

Sometimes, even general historical characteristics (average) are obviously wrong.

Be warned, though. History is sometimes outright implausible as a predictor of future events. If you had played the lottery for 100 weeks and then won $1 million just by chance, a simple historical average rate of return would be $10,000 per week. Yes, it is a historical average, but it is not the right average *forward-looking*. (Of course, if wyouhad played the lottery 100 million weeks, we would almost surely come to the correct conclusion that playing the lottery is a gamble with a negative expected average rate of return.) Similarly, Microsoft or Wal-Mart are almost surely not going to repeat the spectacular historical stock return performances they experienced over the last 20 years. There are many examples when investors, believing history too much, made spectacularly wrong investment decisions. For example, in 1998-2000, Internet stocks had increased in value by more than 50% per year, and many investors believed that it was almost impossible to lose money on them. Of course, these investors, who believed historical Internet returns were indicative of future Internet returns, lost *all* their money over the following two years.

You have no better choice—so use it, but remain skeptical.

For the most part, the theory of investments—which is the subject of this part of our book—makes it easy on itself. It just *assumes* that you already know a stock's general risk/reward characteristics, and then proceeds to give you advice about what portfolio to choose, given that you already have the correct expectations. The problem of estimating means and variances remains *your* problem. Despite the problems with historical returns, I can only repeat: in many cases, stock's average historical risk-reward behavior is the best guidance you have. You can use this history, even though you should retain a certain skepticism—and perhaps even use common sense to adjust historical averages into more sensible forecasts for the future.

Anecdote: Void Where Prohibited

Persons pretending to forecast the future shall be considered disorderly under subdivision 3, section 901 of the criminal code and liable to a fine of $ 250 and/or six months in prison.

(Section 889, New York State Code of Criminal Procedure.)

9·2 The Data: Twelve Annual Rates of Returns

The goal of this chapter is to explain portfolios and stock returns under uncertainty. This is best done with a concrete example. Table 9.1 contains the actual twelve annual rates of returns from 1991 to 2002 for three possible investments: an S&P 500 mutual fund, IBM stock, and Sony (ADR) shares.

The data example that is used throughout the Investments part of the book.

Table 9.1: Historical Annual Rates of Returns for S&P 500, IBM, and Sony

Year	$\tilde{r}_{\text{S\&P500}}$	$\tilde{r}_{\text{IBM}}$	$\tilde{r}_{\text{Sony}}$	Year	$\tilde{r}_{\text{S\&P500}}$	$\tilde{r}_{\text{IBM}}$	$\tilde{r}_{\text{Sony}}$
1991	+0.2631	−0.2124	−0.1027	1997	+0.3101	+0.3811	+0.3905
1992	+0.0446	−0.4336	−0.0037	1998	+0.2700	+0.7624	−0.2028
1993	+0.0706	+0.1208	+0.4785	1999	+0.1953	+0.1701	+2.9681
1994	−0.0154	+0.3012	+0.1348	2000	−0.1014	−0.2120	−0.5109
1995	+0.3411	+0.2430	+0.1046	2001	−0.1304	+0.4231	−0.3484
1996	+0.2026	+0.6584	+0.0772	2002	−0.2337	−0.3570	−0.0808

Numbers are quoted in percent. You will be working with these returns throughout the rest of the investments part. (Sidenote: Sony is the SNE ADR.) Source: Yahoo!*Finance*.

You should first understand risk and reward, presuming it is now January 1, 2003. Although you are really interested in the returns of 2003 (and beyond), unfortunately all you have are these historical rates of return. You have to make the common assumption that historical returns are good indicators of future returns, because each year was an equally likely and therefore informative outcome drawn from an underlying statistical process. Formally, future returns are **random variables**, because their outcomes are not yet known. Recall from Sections 6·1 and 8·3.E that you can denote a random variable with a tilde over it, to distinguish it from an ordinary non-random variable, e.g.,

Pretend that historical rates of return are representative of future returns. Introduce tilde notation.

$$\tilde{r}_{\text{S\&P500}} \quad , \quad \tilde{r}_{\text{IBM}} \quad , \quad \tilde{r}_{\text{Sony}}$$

However, because you only know historical rates of return, you shall use the historical data series in place of the "tilde-d" future variables.

Conceptually, there is a big difference between average historical realizations and expected future realizations. Just because the long-run historical monthly rate of return average was, say, 10% does not mean that it will be 10% in the future. But, practically, there is often not much difference, because you have no choice but to pretend that the historical return series is representative of the distribution of future returns. To draw the distinction, statisticians often name the unknown expected value by a greek character, and the historical outcome that is used to estimate it by its corresponding English character. For example, μ would be the expected future mean, m would be the historical mean; σ would be the expected future standard deviation, s would be the historical standard deviation. After having drawn this careful conceptual distinction, the statisticians then tell you that they will use the historical mean and historical standard deviation as stand-ins (estimators) for the future mean (the expected value) and standard deviation (the expected standard deviation). This is how it should be done, but it can become very cumbersome when dealing with many statistics for many variables. Because we shall mostly work with historical statistics and then immediately pretend that they are our expectations for the future, let us use a more casual notation: when we claim to compute $\mathcal{E}(\tilde{r})$, the notation of which would suggest an expected return, we really compute only the historical mean (unless otherwise stated). That is, in this finance book, we will keep the distinction rather vague.

Please realize that you are going to make a leap of faith here.

DIG DEEPER

> As with a history, you can think of the tilde as representing not just one month's outcome, but this distribution of historical outcomes. As of *today*, next month's rate of return can be anything. (We do not yet have just one number for it.)

9·3 Univariate Statistics

9·3.A. The Mean

Everyone knows how to compute an average.

If you had invested in a random year, what would you have expected to earn? The reward is measured by the single most important statistic, the expected rate of return (also called **mean** or **average** rate of return). You surely have computed this at one time or another, so let's just state that our means are

$$\mathcal{E}(\tilde{r}_{\text{S\&P500}}) \;=\; 0.101 \;\;,\;\; \mathcal{E}(\tilde{r}_{\text{IBM}}) \;=\; 0.154 \;\;,\;\; \mathcal{E}(\tilde{r}_{\text{Sony}}) \;=\; 0.242$$

Sony was clearly the best investment over these 12 years (mostly due to its spectacular performance in 1999), but IBM and the S&P 500 did pretty well, too.

DIG DEEPER

> Chapter 5 already showed that the average rate of return is not the annualized rate of return. An investment in Sony beginning in 1999 for three years would have had a compound three-year rate of return of $(1+297\%)\cdot(1-51\%)\cdot(1-35\%) \approx 26.5\%$, which is 8.1% annualized. Its average annual rate of return is $[297\%+(-51\%)+(-35\%)]/3 \approx 70.3\%$.
>
> But, what causes the difference? It is the year-to-year volatility! if the rate of return were 8.1% each year without variation, the annualized and average rate of return would be the same. The year-to-year volatility negatively affects the annualized holding rates of return. For a given average rate of return, more volatility means less compound and thus less annualized rate of return.
>
> For purposes of forecasting a single year's return, assuming that each historical outcome was equally likely, you want to work with average rates of returns. For computing long-term holding period performance, you would want to work with compound rates of return.

9·3.B. The Variance and Standard Deviation

A Tale of three investments.

How can you measure portfolio risk? Intuitively, how does the risk of the following investment choices compare?

1. An investment in a bond that yields 10% per year for sure.
2. An investment that yields −15% half the time, and +35% half the time, but only once per year.
3. An investment in the S&P 500.
4. An investment in IBM.
5. An investment in Sony.

You know that the first three investments have a mean rate of return of about +10% per annum. But the mean tells you nothing about the risk.

A naïve attempt at measuring spread fails.

You need a statistic that tells you how variable the outcomes are around the mean. Contestant #1 has no variability, so it is clearly safest. What about your other contestants? For contestant #2, half the time, the investment outcome is 25% below its mean (of +10%); the other half, it is 25% above its mean. Measuring outcomes relative to their means (as we have just done) is so common that it has its own name, **deviation from the mean**. Can you average the deviations from the mean to measure typical variability? Try it. With two years, and the assumption that each year is an equally likely outcome,

$$\text{Bad Variability Measure} \;=\; {}^{1}\!/_{2}\cdot(-25\%) + {}^{1}\!/_{2}\cdot(+25\%) \;=\; 0$$

The root cause of the problem, and better alternatives: variance and standard deviation.

The average deviation is zero, because the minus and plus cancel. Therefore, the simple average of the deviations is *not* a good measure of spread. You need a measure that tells you the typical variability is plus or minus 25%, not plus or minus 0%. Such a better measure must recognize that a negative deviation from the mean is the same as a positive deviation from the mean. The most common solution is to square each deviation from the mean in order to eliminate the "opposite sign problem." The average of these squared deviations is called the **variance**:

$$\mathcal{V}ar \;=\; {}^1\!/_2 \cdot (-25\%)^2 + {}^1\!/_2 \cdot (+25\%)^2 \;=\; {}^1\!/_2 \cdot 0.0625 + {}^1\!/_2 \cdot 0.0625 \;=\; 0.0625$$

The variance and standard deviation measure the expected spread of a random variable.

You can think of the variance as the "average squared deviation from the mean." But the variance of 0.0625 looks nothing like the intuitive spread from the mean, which is plus or minus 25%. However, if you take the square root of the variance, you get the **standard deviation**,

$$Sdv \;=\; \sqrt{\mathcal{V}ar} \;=\; \sqrt{0.0625} \;=\; 25\%$$

which has the intuitively pleasing correct order of magnitude of 25%. Although it is not really correct, it is often convenient to think of the standard deviation as the "average deviation from the mean." (It would be more correct to call it "the square root of the average squared deviation from the mean," but this is unwieldy.)

SIDE NOTE

Aside from its uninterpretable value of 0.0625, there is a second and more important problem interpreting the meaning of a variance. (It did not show up in this example, because rates of return are unitless.) If you are interested in the variability of a variable that has units, like dollars or apples, the units of the variable are usually uninterpretable. For example, if you receive either \$10 or \$20, the deviation from the mean is either $-\$5$ or $+\$5$, and the variance is $(\$5)^2 = \$^2 25$, not \$25—the same way by which multiplying 2 meters by 2 meters becomes 4 square-meters, not 4 meters. Square-meters has a good interpretation (area); dollars-squared does not. The standard deviation takes the square root of the variance, and therefore returns to the same units (dollars) as the original series, $\sqrt{\$^2 25} = \$5$ in the example. Note also that I sometimes use "x%%" to denote $x \cdot (\%^2)$—otherwise, there may be confusion whether $x\%^2$ means $(x\%)^2$ or $x \cdot (\%^2)$. 1%% is $0.01 \cdot 0.01 = 0.0001$, and $\sqrt{1\%\%} = 1\%$.

Standard deviation and variance have the same ordering.

Because the standard deviation is just the square root of the variance, if the variance of one variable is higher than the variance of another variable, so is its standard deviation.

Table 9.2: Deviations From the Mean for S&P 500, IBM, and Sony

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$
1991	+0.1620	−0.3661	−0.3448	1997	+0.2090	+0.2273	+0.1485
1992	−0.0565	−0.5874	−0.2458	1998	+0.1656	+0.6086	−0.4448
1993	−0.0305	−0.0330	+0.2364	1999	+0.0942	+0.0163	+2.7261
1994	−0.1165	+0.1474	−0.1073	2000	−0.2025	−0.3658	−0.7529
1995	+0.2400	+0.0892	−0.1374	2001	−0.2315	+0.2693	−0.5904
1996	+0.1015	+0.5046	−0.1648	2002	−0.3348	−0.5105	−0.3228
				Mean (of Deviations) over all 12 years	0.0	0.0	0.0

Sometimes-important nuisance: For historical data, do not divide the squared deviations by N, but by N-1.

For contestant #3, the S&P 500, you must estimate the variability measures from the historical data series. Recall that to compute the variance, you subtract the mean from each outcome, square the deviations, and then average them. To compute the standard deviation, you then take the square-root of the variance. Table 9.2 does most of the hard work for you, giving you deviations from the mean for the S&P 500, IBM, and Sony. Computing variances from these deviations is now straightforward: square and average. Alas, there is one nuisance complication: because there is a difference between historical realizations (which you have) and true expected future outcomes (which you do not have [we pretended to know this perfectly in the "-15%,+35%"

example]), statisticians divide by $N-1$, not by N. Therefore, the estimated variance (divides by $N-1$) is a little bit higher than the average squared deviation from the mean (divides by N).

SIDE NOTE

> The reason for this $N-1$ adjustment is that the future standard deviation is not actually known, but only estimated, given the historical realizations. This "extra uncertainty" is reflected by the smaller divisor, which inflates the uncertainty estimate.
>
> The best intuition comes from a sample of only one historical data point: What would you believe the variability would be if you only know one realization, say 10%? In this case, you know nothing about variability. If you divided the average squared deviation (0) by N, the variance formula would indicate a zero variability. This is clearly wrong. If anything, you should be especially worried about variability, for you now know nothing about it. Dividing by $N-1 = 0$, i.e., $\mathcal{V}ar = 0/0$, indicates that estimating variability from one sample point makes no sense.
>
> The division by N rather than $N-1$ is not important when there are many historical sample data points, which is usually the case in finance. Thus, most of the time, you could use either method, although you should remain consistent. This book uses the $N-1$ statistical convention, if only because it allows checking computations against the built-in formulas in Excel, OpenOffice, or other statistical packages.

Executing the formulas on historical data and dividing by N-1 yields variances and standard deviations.

To obtain the variance of one investment series, square each deviation from the mean, add these squared terms, and dividing by 11 ($N-1$).

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_{\text{S\&P500}}) &= \frac{(+0.1620)^2 + (-0.0565)^2 + \cdots + (-0.3348)^2}{11} = 0.0362 \\
\mathcal{V}ar(\tilde{r}_{\text{IBM}}) &= \frac{(-0.3661)^2 + (-0.5874)^2 + \cdots + (-0.5105)^2}{11} = 0.1503 \\
\mathcal{V}ar(\tilde{r}_{\text{Sony}}) &= \frac{(-0.3448)^2 + (-0.2458)^2 + \cdots + (-0.3228)^2}{11} = 0.8149 \\
\mathcal{V}ar(\tilde{r}) &= \frac{[\tilde{r}_{t=0} - \mathcal{E}(\tilde{r})]^2 + [\tilde{r}_{t=1} - \mathcal{E}(\tilde{r})]^2 + \cdots + [\tilde{r}_{t=T} - \mathcal{E}(\tilde{r})]^2}{T-1}
\end{aligned}$$

The square roots of the variances are the standard deviations:

$$\begin{aligned}
\mathcal{S}dv(\tilde{r}_{\text{S\&P500}}) &= \sqrt{3.62\%} = 19.0\% \\
\mathcal{S}dv(\tilde{r}_{\text{IBM}}) &= \sqrt{15.03\%} = 38.8\% \\
\mathcal{S}dv(\tilde{r}_{\text{Sony}}) &= \sqrt{81.49\%} = 90.3\%
\end{aligned}$$

Return to your original question. If you were to line up your three potential investment choices—all of which offered about 10% rate of return—it appears that the S&P 500 contestant #3 with its 19% risk is a safer investment than the "-15% or +35%" contestant #2 with its 25% risk. As for the other two investments, IBM with its higher 15%/year average rate of return is also riskier, having a standard deviation of "plus or minus" 38.8%/year. (Calling it "plus or minus" is a common way to express standard deviation.) Finally, Sony was not only the best performer (with its 24.2%/year mean rate of return), but it also was by far the riskiest investment. It had a whopping 90.3%/year standard deviation. (Like the mean, the large standard deviation is primarily caused by one outlier, the +297% rate of return in 1999.)

Preview: Use of risk measures in finance.

You will see that the mean and standard deviation play crucial roles in the study of investments—your ultimate goal will be to determine the portfolio that offers the highest expected reward for the lowest amount of risk. But mean and standard deviation make interesting statistics in themselves. From 1926 to 2002, a period with an inflation rate of about 3% per year, the annual risk and reward of some large asset-class investments were approximately

Asset Class	$\mathcal{E}(\tilde{r})$	$\mathcal{S}dv(\tilde{r})$
Short-Term U.S. Government Treasury Bills	4%	3%
Long-Term U.S. Government Treasury Bonds	5.5%	10%
Long-Term Corporate Bonds	6%	9%
Large Firm Stocks	10%	20%
Small Firm Stocks	15%	30%

Corporate bonds had more credit (i.e., default) risk than Treasury bonds, but were typically shorter-term than long-term government bonds, which explains their lower standard deviation. For the most part, it seems that higher risk and higher reward went hand-in-hand.

Solve Now!

Q 9.1 Use a computer spreadsheet to confirm all numbers computed in this section.

Q 9.2 The annual rates of return on the German DAX index were

1991	+0.1286	1994	−0.0706	1997	+0.4624	2000	−0.0754
1992	−0.0209	1995	+0.0699	1998	+0.1842	2001	−0.1979
1993	+0.4670	1996	+0.2816	1999	+0.3910	2002	−0.4394

Compute the mean, variance, and standard deviation of the DAX.

9·4 Bivariate Statistics: Covariation Measures

9·4.A. Intuitive Covariation

Before we embark on more number-crunching, let us first find some intuitive examples of variables that tend to move together, variables that have nothing to do with one another, and variables that tend to move in opposite directions.

Table 9.3: Covariation Examples

Negative	Zero (or Low)	Positive
Agility versus Weight	IQ versus Gender	Height versus Basketball Scoring
Wealth versus Disease	Wealth versus Tail when flipping coin	Wealth versus Longevity
Age versus Flexibility	Age versus Blood Type	Age versus Being CEO
Snow versus Temperature	Sunspots versus Temperature	Grasshoppers versus Temperature
Your Net Returns versus Broker Commissions Paid	IBM Returns in 1999 versus Exxon Returns in 1986	Returns on S&P500 versus Returns on IBM

Personal statistics (such as weight) apply only to adults. Returns are rates of return on stock investments, net of commissions.

An example of a positive covariation.

Table 9.3 offers some such examples of covariation. For example, it is easier to score in basketball if you are 7 feet tall than if you are 5 feet tall. There is a positive covariation between individuals' heights and their ability to score. It is *not* perfect covariation—there are short individuals who can score a lot (witness John Stockton, the Utah Jazz basketball player, who despite a height of "only" 6-1 would almost surely score more points than the tallest students in my class), and there are many tall people who could not score if their lives depended on it. It is only *on average* that taller players score more. In this example, the reason for the positive covariation is direct causality—it is easier to hit the basket when you are as tall as the basket—but correlation need not come from causality. For example, there is also a positive covariation between shoe size and basketball scoring. It is not because bigger feet make it easier to score, but because taller people have both bigger shoe sizes and higher basketball scores. Never forget: causality induces covariation, but not vice versa.

An example of a zero and a negative covariation.

Zero covariation usually means two variables have nothing to do with one another. For example, there is strong evidence that there is practically no covariation between gender and IQ. Knowing only the gender would not help you a bit in guessing the person's IQ, and vice versa. (Chauvinists guessing wrong, however, tend to have lower IQs.) An example of negative covariation would be agility versus weight. It is usually easier for lighter people to overcome the intrinsic inertia of mass, so they tend to be more agile: therefore, more weight tends to be associated with less agility.

9·4.B. Covariation: Covariance, Correlation, and Beta

A measure of dependence should be positive when two variables tend to move together, negative if they tend to move in opposite directions.

Your goal now is to find measures of covariation that are positive when two variables tend to move together; that are zero when two variables have nothing to do with one another; and that are negative when one variable tends to be lower when the other variable tends to be higher. We will consider three possible measures of covariation: covariance, correlation, and beta. Each has its advantages and disadvantages.

Illustrative Data Series

Work with nine made-up data series.

Start with the nine data series in Table 9.4, the returns on nine assets that I have made up. Let us use asset A as our base asset and consider how assets C through J relate to A. You want to ask such questions as "if data series A were the rate of return on the S&P 500, and data series C were the rate of return on IBM, then how do the two return series covary?" This question is important, because it will help you determine investment opportunities that have lower risk.

Covariation is best to understand with an extreme, albeit absurd example. But you need two measures of covariation, not just one.

How does each series covaries with A? Doing this graphically makes it easier, so Figure 9.1 plots the points. If you look at it, you can see that you shall need more than just one covariation statistic: you need one statistic that tells you how much two variables are related (e.g., whether A has more effect on G or H); and one statistic that tells you how reliable this relationship is (e.g., whether knowing A gives you much confidence to predict G or H). Your first task is to draw lines into each of the eight graphs in Figure 9.1 that best fits the points (do it!)—and then to stare at the points and your lines. How steep are your lines (the relationships) and how reliably do the points cluster around your line? Before reading on, your second task is to write into the eight graphs in Figure 9.1 what your intuition tells you the relation and the reliability are.

Here is what I see.

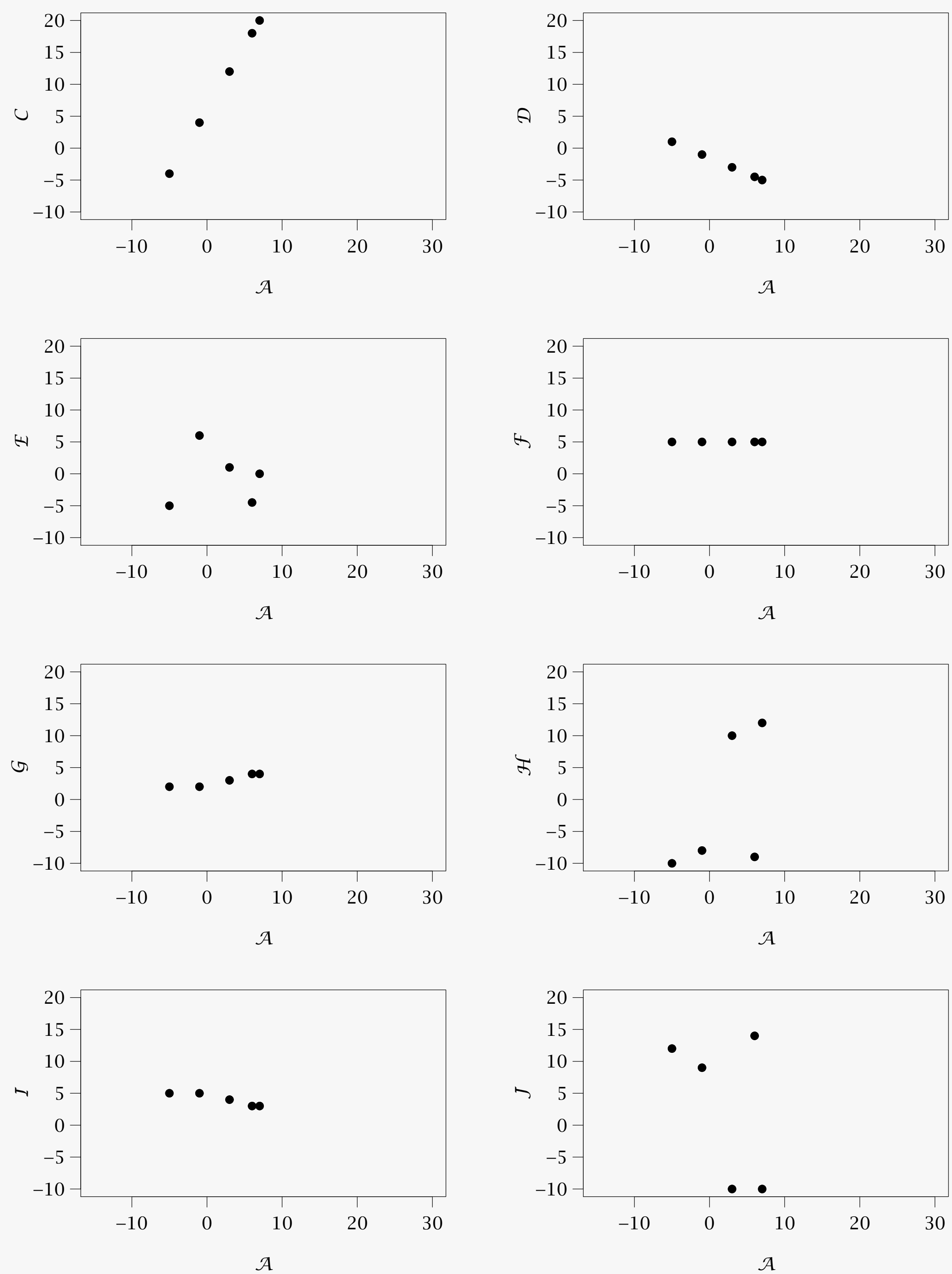
Figure 9.1 seems to show a range of different patterns:

- C and D are strongly related to A. Actually, C is just $2 \cdot (A + 3)$; D is just $-C/2$, so these two relationships are also perfectly reliable.
- E and F have no relationship to A. Though random, there is no pattern connecting A and E. F is 5 regardless of what A is.
- G and H both tend to be higher when A is higher, but in different ways. There is more reliability in the relation between A and G, but even a large increase in A predicts a G that is only a little higher. In contrast, there is less reliability in how A and H covary, but even a small increase in A predicts a much higher H.
- Figures I and J repeat the G/H pattern, but for negative relations.

SIDE NOTE

I cheated in not using my eyeballs to draw lines, but in using the technique of "ordinary least squares" line fitting in Figure 9.3, instead. The lines make it even clearer that when A is high, C, G, and H tend to be high, too; but when A is high, D, I, and J tend to be low. And neither E nor F seem to covary with A. (You will get to compute the slope of this line—the "beta"—later.)

Figure 9.1: C Through J as functions of A



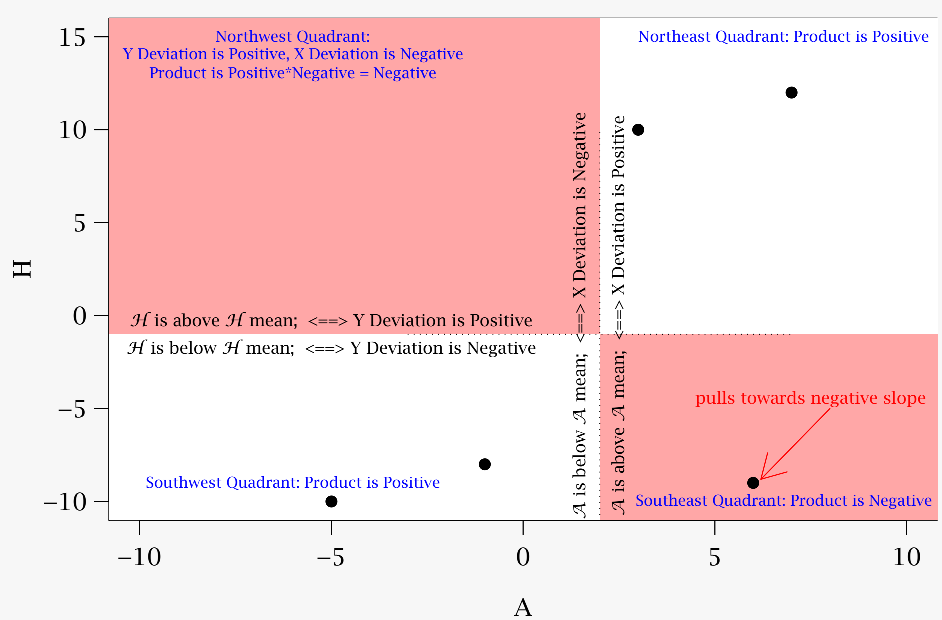
The five observations are marked by circles. *Can you draw a well fitting line? Which series have relations with A? What sign? Which series have reliable relations with A?*

Table 9.4: Illustrative Rates of Return Time Series on Nine Assets

	Portfolio (or Asset or Security)								
Observation	A	C	D	E	F	G	H	I	J
Year 1	−5	−4	+1.0	−10	+5	2	−10	+5	+12
Year 2	+6	+18	−4.5	−9	+5	4	−9	+3	+14
Year 3	+3	+12	−3.0	+2	+5	3	+10	+4	−10
Year 4	−1	+4	−1.0	+12	+5	2	−8	+5	+9
Year 5	+7	+20	−5.0	0	+5	4	+12	+3	−10
Mean	+2	+10	−2.5	−1	+5	+3	−1	+4	+3
Var	25	100	6.25	81	0	1	121	1	144
Sdv	5	10	2.5	9	0	1	11	1	12

Which rate of return series (among portfolios C through J) had low and which had high covariation with the rate of return series of Portfolio A?

Figure 9.2: H as Function of A



Points in the red quadrants pull the overall covariance statistics into a negative direction. Points in the white quadrants pull the overall covariance statistics into a positive direction.

Of course, visual relationships are in the eye of the beholder. You need something more objective that both of us can agree on. Here is how to compute a precise measure. The first step is to determine the means of each X and each Y variable and mark these into the figure—which is done for you in Figure 9.2. The two means divide your data into four quadrants. Now, intuitively, points in the northeast or southwest quadrants (in white) suggest a positive covariation; points that are in the northwest or southeast quadrants (in red) suggest a negative covariation. In other words, the idea of all of the covariation measures is that two series, call them X and Y, are positively correlated

How covariance really works: quadrants above/below means.

- when X tends to be above its mean, Y also tends to be above its mean (upper right quadrant); and
- when X tends to be below its mean, Y also tends to be below its mean (lower left quadrant).

And X and Y are negatively correlated

- when X tends to be above its mean, Y tends to be below its mean (lower right quadrant); and
- when X tends to be below its mean, Y tends to be above its mean (upper left quadrant).

Covariance

Table 9.5: Illustrative Rates of Return Time Series on Nine Assets, De-Meaned

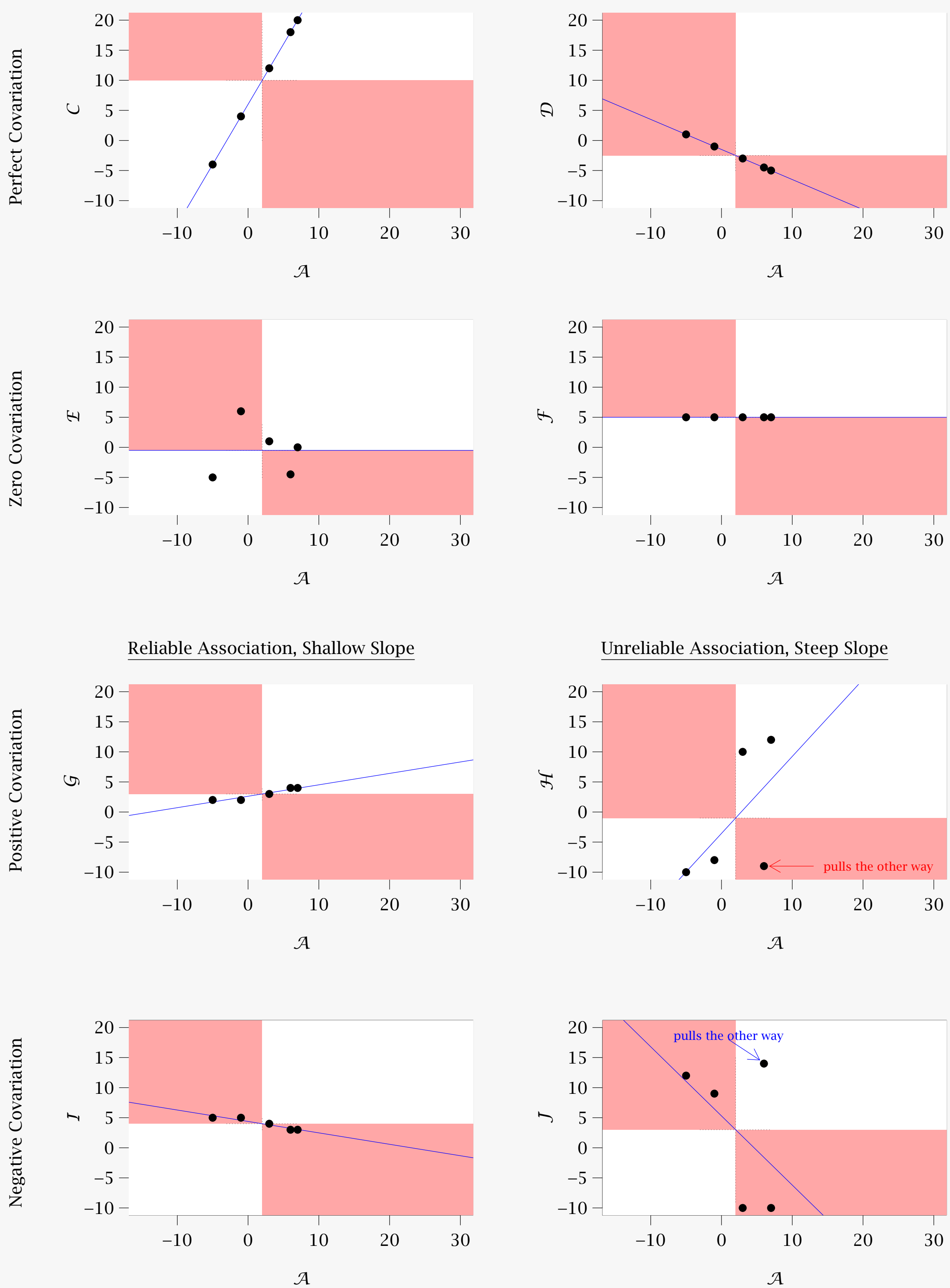
	Rates of Returns on Portfolios (or Assets or Securities)								
Observation	A	C	D	E	F	G	H	I	J
Year 1	−7	−14	+3.5	−9	0	−1	−9	+1	+9
Year 2	+4	+8	−2.0	−8	0	+1	−8	−1	+11
Year 3	+1	+2	−0.5	+3	0	0	+11	0	−13
Year 4	−3	−6	+1.5	+13	0	−1	−7	+1	+6
Year 5	+5	+10	−2.5	+1	0	+1	+13	−1	−13
$\mathcal{E}(\tilde{r})$	0	0	0.0	0	0	0	0	0	0
$\mathcal{V}ar(\tilde{r})$	25	100	6.25	81	0	1	121	1	144
$\mathcal{S}dv(\tilde{r})$	5	10	2.5	9	0	1	11	1	12

It will be easier to work with the series from Table 9.4 if you first subtract the mean from each series.

How can you make a positive number for every point that is either above both the X and Y means, or both below the X and Y means, and a negative number for every point that is above one mean and below the other? Easy! First, you measure each data point in terms of its distance from its mean, so you subtract the mean from each data point, as in Table 9.5. Points in the northeast quadrant are above both means, so both net-of-mean values are positive. Points in the southwest quadrant are below both means, so both net-of-mean values are negative. Points in the other two quadrants have one positive and one negative net-of-mean number. Now, you know that the product of either two positive or two negative numbers is positive, and the product of one positive and one negative number is negative. If you multiply your deviations from the mean, the product has a positive sign if it is in the upper-right and lower-left quadrants (the deviations from the mean are either both positive or both negative), and a negative sign if it is in the upper-left and lower-right quadrants (only one deviation from the mean is positive, the other is negative). A point that has a positive product pulls towards positive covariation, whereas a negative product pulls towards negative covariation.

The main idea is to think about where points lie, within the quadrants in the figure. Multiply the deviations from the mean to get the right sign for each data point relative to each quadrant.

Figure 9.3: B Through I as functions of A, Lines Added



The five observations are marked by circles. The areas north, south, east, and west of the X and Y means are now marked. A cross with arm lengths equal to one standard deviation is also placed on each figure.

Which points push the relationship to be positive, which points push the relationship to be negative?

You want to know the average pull. This is the **covariance**—the average of these products of two variables' deviations from their own means. Try it for A and C (again, following the standard method of dividing by $N-1$ rather than N):

The covariance is the average of the multiplied deviations from their means.

$$\mathcal{C}ov(\mathrm{A},\mathrm{C}) = \frac{(-7)\cdot(-14) + (+4)\cdot(+8) + (+1)\cdot(+2) + (-3)\cdot(-6) + (+5)\cdot(+10)}{4} = 50$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{C}) = \frac{(a_1-\bar{a})\cdot(b_1-\bar{b}) + \cdots + (a_5-\bar{a})\cdot(b_5-\bar{b})}{N-1}$$

Note how in this A versus C relationship, each term in the sum is positive, and therefore pulls the average (the covariance) towards being a positive number. You can see this in the figure, because each and every point lies in the two "positivist" quadrants. Repeat this for the remaining series:

$$\mathcal{C}ov(\mathrm{A},\mathrm{D}) = \frac{(-7)\cdot(+3.5) + (+4)\cdot(-2) + (+1)\cdot(-0.5) + (-3)\cdot(+1.5) + (+5)\cdot(-2.5)}{4} = -12.5$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{E}) = \frac{(-7)\cdot(-9) + (+4)\cdot(-8) + (+1)\cdot(+3) + (-3)\cdot(+13) + (+5)\cdot(+1)}{4} = 0$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{F}) = \frac{(-7)\cdot(0) + (+4)\cdot(0) + (+1)\cdot(0) + (-3)\cdot(0) + (+5)\cdot(0)}{4} = 0$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{G}) = \frac{(-7)\cdot(-1) + (+4)\cdot(+1) + (+1)\cdot(0) + (-3)\cdot(-1) + (+5)\cdot(+1)}{4} = 4.75$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{H}) = \frac{(-7)\cdot(-9) + (+4)\cdot(-8) + (+1)\cdot(+11) + (-3)\cdot(-7) + (+5)\cdot(+13)}{4} = 32$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{I}) = \frac{(-7)\cdot(+1) + (+4)\cdot(-1) + (+1)\cdot(0) + (-3)\cdot(+1) + (+5)\cdot(-1)}{4} = -4.75$$

$$\mathcal{C}ov(\mathrm{A},\mathrm{J}) = \frac{(-7)\cdot(+9) + (+4)\cdot(+11) + (+1)\cdot(-13) + (-3)\cdot(+6) + (+5)\cdot(-13)}{4} = -28.75$$

$$\mathcal{C}ov(X,Y) = \frac{(x_1-\bar{x})\cdot(y_1-\bar{y}) + \cdots + (x_5-\bar{x})\cdot(y_5-\bar{y})}{N-1}$$

Having computed the covariances, look at Figure 9.3. A and D, A and I, and A and J covary negatively on average; A and C, A and G, and A and H covary positively; and A and E, and A and F have zero covariation. Now, take a look at the A versus H figure (also in Figure 9.2) again: there is one lonely point in the lower-right quadrant, marked with an arrow. It tries to pull the line into a negative direction. In the A versus H covariance computation, this point is the $(+4)\cdot(-1)$ term, which is the only negative component in the overall sum. If this one point had not been in the data, the association between A and H would have been more strongly positive than 4.75.

The covariance tells you the sign (whether a relationship is positive or negative), but its magnitude is difficult to interpret—just as you could not really interpret the magnitude of the variance. Indeed, the covariance not only shares the weird squared-units problem with the variance, but the covariance of a variable with itself *is* the variance! This can be seen from the definitions: Both multiply each historical outcome deviation by itself, and then divide by the same number,

Covariance gives the right sign, but not much more. It is often abbreviated as a sigma with two subscripts.

$$\mathcal{C}ov(X,X) = \frac{(x_1-\bar{x})\cdot(x_1-\bar{x}) + \cdots + (x_N-\bar{x})\cdot(x_N-\bar{x})}{N-1}$$

$$= \frac{(x_1-\bar{x})^2 + \cdots + (x_N-\bar{x})^2}{N-1} = \mathcal{V}ar(X)$$

And, just like the variance is needed to compute the standard deviation, the covariance is needed to compute the next two covariation measures (correlation and beta). The covariance statistic is so important and used so often that the Greek letter sigma (σ) with two subscripts has become the standard abbreviation:

Covariance between X and Y: $\mathcal{C}ov(X,Y)$ $\quad \sigma_{X,Y}$

Variance of X: $\mathcal{V}ar(X)$ $\quad \sigma_{X,X} = \sigma_X^2$

These are sigmas with two subscripts. If you use only one subscript, you mean the standard deviation:

Standard Deviation of X: $\mathcal{S}dv(X)$ $\quad \sigma_X$

This is easy to remember if you think of two subscripts as the equivalent of multiplication (squaring), and of one subscript as the equivalent of square-rooting.

Correlation

Correlation is closely related to, but easier to interpret than Covariance.

To better interpret the covariance, you need to somehow normalize it. A first normalization of the covariance gives the **correlation**. It divides the covariance by the standard deviations of both variables. Applying this formula, you can compute

$$\begin{aligned}
\mathit{Correlation}(\mathrm{A},\mathrm{C}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{C})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{C})} = \frac{+50}{10\cdot 5} = +1.00\\
\mathit{Correlation}(\mathrm{A},\mathrm{D}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{D})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{D})} = \frac{-12.5}{5\cdot 2.5} = -1.00\\
\mathit{Correlation}(\mathrm{A},\mathrm{E}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{E})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{E})} = \frac{0}{5\cdot 9} = \pm 0.00\\
\mathit{Correlation}(\mathrm{A},\mathrm{F}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{F})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{F})} = \frac{0}{5\cdot 0} = \text{not defined}\\
\mathit{Correlation}(\mathrm{A},\mathrm{G}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{G})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{G})} = \frac{4.75}{5\cdot 1} = +0.95 \qquad (9.1)\\
\mathit{Correlation}(\mathrm{A},\mathrm{H}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{H})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{H})} = \frac{32}{5\cdot 11} = +0.58\\
\mathit{Correlation}(\mathrm{A},\mathrm{I}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{I})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{I})} = \frac{-4.75}{5\cdot 1} = -0.95\\
\mathit{Correlation}(\mathrm{A},\mathrm{J}) &= \frac{\mathit{Cov}(\mathrm{A},\mathrm{J})}{\mathit{Sdv}(\mathrm{A})\cdot\mathit{Sdv}(\mathrm{J})} = \frac{-28.75}{5\cdot 12} = -0.48\\
\mathit{Correlation}(X,Y) &= \frac{\mathit{Cov}(X,Y)}{\mathit{Sdv}(X)\cdot\mathit{Sdv}(Y)}
\end{aligned}$$

The correlation measures the reliability of the relationship between two variables. A higher *absolute* correlation means more reliability, regardless of the strength of the relationship (slope). The nice thing about the correlation is that it is always between −100% and +100%. Two variables that have a correlation of 100% always perfectly move in the same direction, two variables that have a correlation of -100% always perfectly move in the opposite direction, and two variables that are independent have a correlation of 0%. This makes the correlation very easy to interpret. The correlation is unit-less, regardless of the units of the original variables themselves, and is often abbreviated with the Greek letter rho (ρ). The perfect correlations between A and C or D tell you that all points lie on straight lines. (Verify this visually in Figure 9.3!) The correlations between A and G (95%) and the correlations between A and I (−95%) are almost as strong: the points almost lie on a line. The correlation between A and H, and the correlation between A and J are weaker: the points do not seem to lie on a straight line, and knowing A does not permit you to *perfectly* predict H or J.

SIDE NOTE

If two variable are always acting identically, they have a correlation of 1. Therefore, you can determine the maximum covariance between two variables:

$$1 = \frac{\mathit{Cov}(\tilde{X},\tilde{Y})}{\mathit{Sdv}(\tilde{X})\cdot\mathit{Sdv}(\tilde{Y})} \iff \mathit{Cov}(\tilde{X},\tilde{Y}) = \mathit{Sdv}(\tilde{X})\cdot\mathit{Sdv}(\tilde{Y})$$

It is mathematically impossible for the absolute value of the covariance to exceed the product of the two standard deviations.

Beta

Beta is yet another covariation measure, and has an interesting graphical interpretation.

The correlation cannot tell you that A has more pronounced influence on C than on D: although both correlations are perfect, if A is higher by 1, your prediction of C is higher by 2; but if A is higher by 1, your prediction of D is lower by only −0.5. You need a measure for the slope of the best-fitting line that you would draw through the points. Your second normalization of the covariance does this: it gives you this slope, the **beta**. It divides the covariance by the variance of the X variable (here, A), i.e., instead of one dividing by the standard deviation of Y (as in the correlation), it divides a second time by a standard deviation of X:

$$\begin{aligned}
\beta_{C,A} &= \frac{\mathcal{C}ov(A,C)}{Sdv(A)\cdot Sdv(A)} = \frac{+50}{5\cdot 5} = +2.00\\
\beta_{D,A} &= \frac{\mathcal{C}ov(A,D)}{Sdv(A)\cdot Sdv(A)} = \frac{-12.5}{5\cdot 5} = -0.50\\
\beta_{E,A} &= \frac{\mathcal{C}ov(A,E)}{Sdv(A)\cdot Sdv(A)} = \frac{0}{5\cdot 5} = \pm 0.00\\
\beta_{F,A} &= \frac{\mathcal{C}ov(A,F)}{Sdv(A)\cdot Sdv(A)} = \frac{0}{5\cdot 5} = \pm 0.00\\
\beta_{G,A} &= \frac{\mathcal{C}ov(A,G)}{Sdv(A)\cdot Sdv(A)} = \frac{4.75}{5\cdot 5} = +0.19\\
\beta_{H,A} &= \frac{\mathcal{C}ov(A,H)}{Sdv(A)\cdot Sdv(A)} = \frac{32}{5\cdot 5} = +1.28\\
\beta_{I,A} &= \frac{\mathcal{C}ov(A,I)}{Sdv(A)\cdot Sdv(A)} = \frac{-4.75}{5\cdot 5} = -0.19\\
\beta_{J,A} &= \frac{\mathcal{C}ov(A,J)}{Sdv(A)\cdot Sdv(A)} = \frac{-28.75}{5\cdot 5} = -1.15\\
\beta_{Y,X} &= \frac{\mathcal{C}ov(X,Y)}{Sdv(X)\cdot Sdv(X)} = \frac{\mathcal{C}ov(X,Y)}{\mathcal{V}ar(X)}
\end{aligned}$$

The first subscript on beta denotes the variable on the Y axis, while the second subscript on beta denotes the variable on the X axis. It is the latter that provides the variance (the divisor). Beta got its name from the fact that the most common way to write down the formula for a line is $y = \alpha + \beta \cdot x$, and the best-fitting line slope is exactly what beta is. Unlike correlations, betas are not limited to any range. Beta values of $+1$ and -1 denote the diagonal lines, beta values of 0 and ∞ denote the horizontal and vertical line. Inspection of Figure 9.3 shows that the slope of A versus C is *2 to 1*, while the slope of A versus D is shallower *1 to -2*. This is exactly what beta tells us: $\beta_{C,A}$ is 2.0, while $\beta_{D,A}$ is -0.5. Unlike the correlation, beta cannot tell you whether your line fits perfectly or imperfectly. But, unlike the correlation, beta can tell you how much you should change your prediction of Y if the X values change. And unlike correlation and covariance, the order of the two variables matters in computing beta. For example, $\beta_{G,A} = 0.19$ is not $\beta_{A,G}$:

$$\beta_{A,\,G} = \frac{\mathcal{C}ov(A,G)}{Sdv(G)\cdot Sdv(G)} = \frac{5.00}{1\cdot 1} = 5$$

In a statistical package, beta can be obtained either by computing covariances and variances and then dividing the two; or by running a **Linear Regression**, in which the dependent variable is the Y variable and the independent variable is the X variable.

$$Y = \alpha + \beta\cdot X + \epsilon$$

Both methods yield the same answer.

DIG DEEPER

And, as with all other statistical measures, please keep in mind that you are usually computing a historical beta (line slope), although you usually are really interested in the future beta (line slope)!

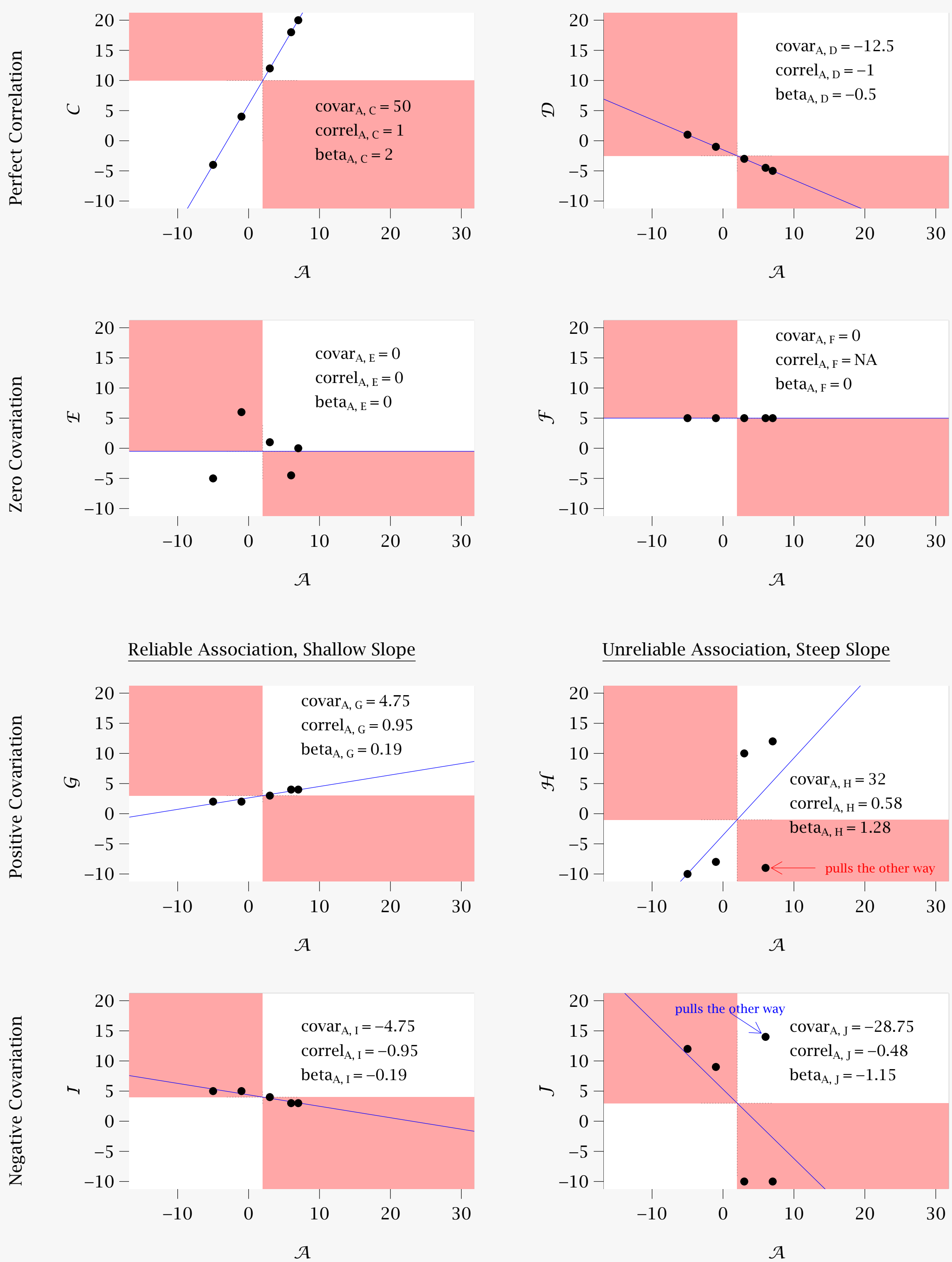
When it comes to stock returns, you really want to know the future slope, although you usually only have the historical slope.

Summary of Covariation Measures

Table 9.6 summarizes the three covariation measures.

Figure 9.4 summarizes everything that you have learned about the covariation of your series. It plots the data points, the quadrants, the best fitting lines, and a text description of the three measures of covariation.

Figure 9.4: C Through J as functions of A with lines and text



The five observations are marked by circles. The areas north, south, east, and west of the X and Y means are now marked. A cross with arm lengths equal to one standard deviation is also placed on each figure.

Table 9.6: Comparison of Covariation Measures

	Units	Magnitude	Order of Variables	computed as	Measures
Covariance	squared	practically meaningless	irrelevant	$\sigma_{X,Y}$	No Intuition
Correlation	unit-less	between −1 and +1	irrelevant	$\sigma_{X,Y} / (\sigma_X \cdot \sigma_Y)$	Reliability
beta (Y,X)	unit-less	meaningful (slope)	important	$\sigma_{X,Y} / \sigma_{X,X}$	Slope
beta (X,Y)	unit-less	meaningful (slope)	important	$\sigma_{X,Y} / \sigma_{Y,Y}$	Slope

All covariation measures share the same sign. If one is positive (negative), so are all others. Recall that $\sigma_{X,X} = \sigma_X^2$, which must be positive.

9·4.C. Computing Covariation Statistics For The Annual Returns Data

Now back to work! It is time to compute the covariance, correlation, and beta for your three investment choices, S&P 500, IBM, and Sony. Return to the deviations from the means in Table 9.2. As you know, to compute the covariances, you add the products of the net-of-mean observations and divide by $(T-1)$—tedious, but not difficult work:

Applying the covariance formula to the historical data.

$$Cov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{IBM}}) = \frac{(+0.162)\cdot(-0.366) + \cdots + (-0.335)\cdot(-0.511)}{11} = 0.0330$$

$$Cov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{Sony}}) = \frac{(+0.162)\cdot(-0.345) + \cdots + (-0.335)\cdot(-0.323)}{11} = 0.0477$$

$$Cov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{Sony}}) = \frac{(-0.366)\cdot(-0.345) + \cdots + (-0.511)\cdot(-0.323)}{11} = 0.0218$$

$$Cov(\tilde{r}_{\text{i}}, \tilde{r}_{j}) = \frac{(\tilde{r}_{\text{i},s=1} - \mathcal{E}(\tilde{r}_{\text{i}}))\cdot(\tilde{r}_{\text{j},s=1} - \mathcal{E}(\tilde{r}_{\text{j}})) + \cdots + (\tilde{r}_{\text{i},s=T} - \mathcal{E}(\tilde{r}_{\text{i}}))\cdot(\tilde{r}_{\text{j},s=T} - \mathcal{E}(\tilde{r}_{\text{j}}))}{T-1} \tag{9.2}$$

All three covariance measures are positive. You know from the discussion on Page 199 that, aside from their signs, the covariances are almost impossible to interpret. Therefore, now compute the correlations, your measure of how well the best-fitting line fits the data. The correlations are the covariances divided by the two standard deviations:

$$Correlation(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{IBM}}) = \frac{3.30\%}{19.0\% \cdot 38.8\%} = 44.7\%$$

$$Correlation(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{Sony}}) = \frac{4.77\%}{19.0\% \cdot 90.3\%} = 27.8\%$$

$$Correlation(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{Sony}}) = \frac{2.18\%}{38.8\% \cdot 90.3\%} = 6.2\%$$

$$Correlation(\tilde{r}_{\text{i}}, \tilde{r}_{\text{j}}) = \frac{Cov(\tilde{r}_{\text{i}}, \tilde{r}_{\text{j}})}{Sdv(\tilde{r}_{\text{i}}) \cdot Sdv(\tilde{r}_{\text{j}})}$$

The S&P 500 has correlated much more with IBM, than the S&P 500 has correlated with Sony (or IBM with Sony). This makes intuitive sense. Both S&P 500 and IBM are U.S. investments, while Sony is a stock that is trading in an entirely different market.

Applying the beta formula to the historical data.

Finally, you might want to compute the beta of $\tilde{r}_{\text{Sony}}$ with respect to the $\tilde{r}_{\text{S\&P500}}$ (i.e., divide the covariance of $\tilde{r}_{\text{Sony}}$ with $\tilde{r}_{\text{S\&P500}}$ by the variance of $\tilde{r}_{\text{S\&P500}}$), and the beta of $\tilde{r}_{\text{IBM}}$ with respect to the $\tilde{r}_{\text{S\&P500}}$. Although you should really write $\beta_{\tilde{r}_{\text{IBM}},\tilde{r}_{\text{S\&P500}}}$, no harm is done if you omit the $\tilde{r}$ for convenience, and just elevate the subscripts when there is no risk of confusion. Thus, you can just write $\beta_{\text{IBM,S\&P 500}}$, instead.

$$
\begin{aligned}
\beta_{\text{IBM,S\&P 500}} &= \frac{3.30\%}{(19.0\%)^2} = 0.91 \; ; \\
\beta_{\text{Sony,S\&P 500}} &= \frac{4.77\%}{(19.0\%)^2} = 1.31 \\
\beta_{\text{i,j}} &= \frac{\mathcal{C}ov(\tilde{r}_{\text{i}}, \tilde{r}_{\text{j}})}{\mathcal{S}dv(\tilde{r}_{\text{i}}) \cdot \mathcal{S}dv(\tilde{r}_{\text{j}})}
\end{aligned}
\qquad (9.3)
$$

Beta is the slope of the best-fitting line when the rate of return on S&P 500 is on the X-axis and the rate of return on IBM (or Sony) is on the Y-axis. Note that although Sony was correlated less with the S&P 500 than IBM was correlated with the S&P 500, it is Sony that has the steeper slope. Correlation and beta do measure different things. The next chapters will elaborate more on the importance of beta in finance.

The Marquis De Sade would not have been happy. Ok, neither would Mary Poppins.

You have now computed all the statistics that this book will use: means, variances, standard deviations, covariances, correlations, and betas. Only modestly painful, I hope. The next chapter will use no new statistics, but it will show how they work in the context of portfolios.

Solve Now!

Q 9.3 What is the correlation of a random variable with itself?

Q 9.4 What is the slope (beta) of a random variable with itself?

Q 9.5 Return to the historical rates of return on the DAX from Question 9.2 on Page 193. Compute the covariances, correlations and betas for the DAX with respect to each of the other three investment securities.

Q 9.6 Very advanced Question: Compute the annual rates of return on a portfolio of 1/3 IBM and 2/3 Sony. Then compute the beta of this portfolio with respect to the S&P 500. How does this compare to the beta of IBM with respect to the S&P 500, and the beta of Sony with respect to the S&P 500?

9·5 Summary

The chapter covered the following major points:

- Finance often uses statistics based on historical rates of return as standings to predict statistics for future rates of return. This is a leap of faith—often, but not always correct.
- Tildes denote random variables—a distribution of possible future outcomes. In practice, the distribution is often given by historical data.
- The expected rate of return is a measure of the reward. It is often forecast from the historical mean.
- The standard deviation—and its intermediate input, the variance—are measures of the risk. The standard deviation is (practically) the square-root of the average squared deviation from the mean.
- Covariation measures how two variables move together. Causality induces covariation, but not vice versa. Two variables can covary, even if neither variable would be the cause of the other.
- Like variance, the covariance is difficult to interpret. Thus, covariance is often only an intermediate number on the way to more intuitive statistics.

- The correlation is the covariance divided by the standard deviation of both variables. It measures how reliable a relationship between two variables is. The order of variables does not matter. The correlation is always between −1 and +1.
- The beta is the covariance divided the standard deviation of the variable on the X axis squared (which is the variance). It measures how steep a relationship between two variables is. The order of variables matters: $\beta_{A,B}$ is different from $\beta_{B,A}$.

Deeply Dug Appendix

A More Statistical Theory

a. Historical and Future Statistics

Physical processes often have known properties. Stock returns do not.

The theory usually assumes that although you do not know the *outcome* of a random variable, you do know the statistics (such as mean and standard deviation) for the outcomes. That is, you can estimate or judge a random variable's unknown mean, standard deviation, covariance (beta), etc. Alas, while this is easy for the throw of a coin or a die, where you know the physical properties of what determines the random outcome, this is not so easy for stock returns. For example, what is the standard deviation of next month's rate of return on PepsiCo?

Use historical statistics as estimators of future statistics.

You just do not have a better alternative than to assume that PepsiCo's returns are manifestations of the same statistical process over time. If you want to know the standard deviation of PepsiCo's next month's rate of return, you typically must assume that each historical monthly rate of return—at least over the last couple of years—was a draw from the same distribution. Therefore, you can use the historical rates of return, assuming each one was an equally likely outcome, to estimate the future standard deviation. Analogously, the mechanics of the computation for obtaining the estimated future standard deviation are exactly the same as those you used to obtain an actual historical standard deviation.

This works well for standard deviations and covariation statistics, but not for means.

But, using historical statistics and then arguing that they are representative of the future is a bit of a leap. Empirical observation has taught us that doing so works well for standard deviations and covariation measures: that is, the historical statistics obtained from monthly rates of returns over the last 3 to 5 years appear to be fairly decent predictors of future betas and standard deviations. Unfortunately, the historical mean rates of return are fairly unreliable predictors of the future rates of returns.

Q 9.7 When predicting the outcome of a die, why do you not use historical statistics on die throws as predictors of future die throws?

Q 9.8 Are historical financial securities' mean rates of return good predictors of future mean rates of return?

Q 9.9 Are historical financial securities' standard deviations and correlations of rates of return good predictors of their future equivalents?

b. Improving Future Estimates From Historical Estimates

Extreme outcomes have more of two components: higher expected outcome and a higher error term (which will not repeat).

The principal remaining problem in the reliability of historical estimates of covariances for prediction is what statisticians call "regression to the mean." That is, the most extreme historical estimates are likely caused not only by the underlying true estimates, but even more so by chance. For example, if all securities had a true standard deviation of 30% per annum, over a particular year some might show a standard deviation of 40%, while others might show a standard deviation of 20%. Those with the high 40% historical standard deviations are most likely to have lower than their historical standard deviations (dropping back to 30%). Those with the low 20% historical standard deviations are most likely to have higher than their historical standard deviations (increasing back to 30%). This can also manifest itself in market beta estimates. Predicting future betas by running a regression with historical rate of return data is too naïve. The reason is that a stock that happened to have a really high return on one day will show too high a beta if the overall stock market happened to have gone up this day and too low a beta if the overall stock market happened to have gone down this day. Such extreme observations tend not to repeat under the same market conditions in the future.

Shrinkage just reduces the estimate, hoping to adjust for extremes' errors.

Statisticians handle such problems with a technique called "shrinkage." The historical estimates are reduced ("shrunk") towards a more common mean. Naturally, the exact amount by which historical estimates should be shrunk and what number they should be shrunk towards is a very complex technical problem—and doing it well can make millions of dollars. This book is definitely not able to cover this subject appropriately. Still, reading this book, you might wonder if there is something both quick-and-dirty and reasonable that you can do to obtain better estimates of future mean returns, better estimates of future standard deviations, and better estimates of future betas.

Advice: take the average of the market historical statistic and your individual stock historical statistic. It probably predicts better.

The answer is yes. Here is a two minute non-formal heuristic estimation job: To predict a portfolio statistic, average the historical statistic on your particular portfolio with the historical statistic on the overall stock market. There are better and more sophisticated methods, but this averaging is likely to predict better than the historical statistic of the particular portfolio by itself. (With more time and statistical expertise, you could use other information, such as beta, the industry historical rate of return, or the average P/E ratio of the portfolio, to produce even better guesstimates of future portfolio behavior.) For example, the market beta for the overall market is "1.0," so my prescription is to average the estimated beta and 1.0. Commercial vendors of market beta estimates do something similar, too. *Bloomberg* computes the weighted average of the estimated market beta and the number one, with weights of 0.67 and 0.33, respectively, *Value Line* reverses these two weights. *Ibbotson Associates*, however, does something more sophisticated, shrinking beta not towards one, but towards a "peer group" market beta.

An example of shrinking.

↦ *Table 10.7 on Page 234*

Let us apply some shrinking to the statistics in Table 10.7. If you were asked to guesstimate an expected annual rate of return for Wal-Mart over the next year, you would not quote Wal-Marts historical 31.5% as your estimate of Wal-Mart's future rate of return. Instead, you could quote an average of 31.5% and 6.3% (the historical rate of return on the market from 1997 to 2002), or about 20% per annum. (This assumes that you are not permitted to use more sophisticated models, such as the CAPM.) You would also guesstimate Wal-Mart's risk to be the average of 31.1% and 18.7%, or about 25% per year. Finally, you would guesstimate Wal-Mart's market beta to be about 0.95. The specific market index to which you shrink matters little (the Dow-Jones 30 or the S&P500)—but it does matter that you do shrink somehow! An even better target to shrink towards would be the industry average statistics. (Some researchers go as far as to estimate only industry betas, and forego computing the individual firm beta altogether! This is shrinking to a very large degree.) However, good shrinking targets are beyond the scope of this book. Would you like to bet that the historical statistics are better guesstimates than the shrunk statistics? (If so, feel free to invest your money into Wal-Mart, and deceive yourself that you will likely earn a mean return 31.5%! Good luck!)

What works reasonably well, what does not.

Here is a summary of some recommendations. Based on regressions using five years of historical monthly data, to predict one-year-ahead statistics, you can use reasonable shrinkage methods for large stocks (e.g., members of the S&P500) as follows:

Mean	Nothing works too well (i.e.,predicting the future from the past).
Market-Model Alpha	Nothing works too well.
Market-Model Beta	Average the historical beta with the number "1." For example, if the regression coefficient (covariance/variance) is 4, use a beta of 2.5.
Standard Deviation	Average the historical standard deviation of the stock and the historical standard deviation of the S&P500. Then increase by 30%, because, historically, for unknown reasons, volatility has been increasing.

Recall that the market model is the linear regression in which the x variable is the rate of return on the S&P 500, and the y variable is the rate of return on the stock in which you are interested.

c. Other Measures of Spread

There are measures of risk other than the variance and standard deviation, but they are obscure enough to deserve your ignorance (at least until an advanced investments course). One such measure is the mean absolute deviation (MAD) from the mean. For the example of a rate of return of either +25% or −25%,

$$\text{MAD} \;=\; {}^{1}/_{2} \cdot |(-25\%)| + {}^{1}/_{2} \cdot |(+25\%)| \;=\; {}^{1}/_{2} \cdot 25\% + {}^{1}/_{2} \cdot 25\% \;=\; 25\%$$

In this case, the outcome happens to be the same as the standard deviation, but this is not generally the case. The MAD gives less weight than the standard deviation to observations far from the mean. For example, if you had three returns, −50%, −50% and +100%, the mean would be 0%, the standard deviation 70.7%, and the MAD 66.7%.

Another measure of risk is the semivariance ($\mathcal{SV}$), which relies only on observations below zero or below the mean. That is, all positive returns (or deviations from the mean) are simply ignored. For the example of +25% or −25%,

$$\mathcal{SV} \;=\; {}^{1}/_{2} \cdot (-25\%)^2 + {}^{1}/_{2} \cdot (0) \;=\; {}^{1}/_{2} \cdot 0.0625 \;=\; 0.03125$$

The idea is that investors fear only realizations that are negative (or below the mean).

Finally, note that the correlation has another nice interpretation: the correlation squared is the R^2 in a bivariate OLS regression with a constant.

d. Translating Mean and Variance Statistics Into Probabilities

Translating standard deviations into more intuitive risk assessments

Although you now know enough to compute a measure of risk, you have not bothered to explore how likely outcomes are. For example, if a portfolio's expected rate of return is 12.6% per year, and its standard deviation is 22% per year, what is the probability that you will lose money (earn below 0%)? What is the probability that you will earn 15% or more? 20% or more?

Stock returns often assume a normal (bell-shaped) distribution.

It turns out that if the underlying distribution looks like a bell curve—and many common portfolio return distributions have this shape—there is an easy procedure to translate mean and standard deviation into the probability that the return will end being less than x. In fact, this probability correspondence is the only advantage that bell shaped distributions provide! Everything else works regardless of the actual shape of the distribution.

An example: Z-score and probability.

For concreteness sake, assume you want to determine the probability that the rate of return on this portfolio is less than +5%:

Step 1: Subtract the mean from 5%. In the example, with the expected rate of return of 12.6%, the result is 5% − 12.6% = −7.6%.

Step 2: Divide this number by the standard deviation. In ths example, this is −7.6% divided by 22%, which comes to −0.35. This number is called the **Score** or **Z-score**.

Step 3: Look up the probability for this Score in the **Cumulative Normal Distribution Table** in Appendix 1·3. For the score of −0.35, this probability is about 0.36.

↦ *Appendix Section 1·3 on Page 978*

In sum, you have determined that if returns are drawn from a distribution with a mean of 12.6% and a standard deviation of 22%, then the probability of observing a single rate of return of +5% or less is about 36%. It also follows that the probability that a return is greater than +5% must be 100% − 36% = 64%.

SIDE NOTE

In the real world, this works well enough—but not perfectly. Do not get fooled by theoretical pseudo-accuracy. Anything between 30% and 40% is a reasonable prediction here.

How well does the Z-score fit in the history?

Now recall portfolio P in Table 10.1. P had a mean of 12.6% and a standard deviation of 22%. You have just computed that about one third of the 12 annual portfolio returns should be below +5%. 1991, 1993, 1994, 1995, 1996, 1997, 1998, and 1999 performed better than +5%; 1992, 2000, 2001, and 2002 performed worse. As predicted by applying the normal distribution table, about one third of the annual returns were 5% or less.

How likely is it that you will lose money?

A common question is “what is the probability that the return will be negative?” Use the same technique,

Step 1: Subtracting the mean of 0% yields 0.0% − 12.6% = −12.6%.

Step 2: Dividing −12.6% by the standard deviation of 22% yields the score of −0.57.

Step 3: For this score of −0.57, the probability is about 28%.

In words, the probability that the rate of return will be negative is around 25% to 30%. And, therefore, the probability that the return will be positive is around 70% to 75%. The table shows that 4 out of the 12 annual rates of return are negative. This is most likely **sampling error**: with only 12 annual rates of return, it was impossible for the distribution of data to accurately follow a bell shape.

DIG DEEPER

Many portfolio returns have what is called “fat tails.” This means that the probability of extreme outcomes—especially extreme negative outcomes—is often higher than suggested by the normal distribution table. For example, if the mean return were 30% (e.g. for a multi-year return) and the standard deviation were 10%, the score for a value of 0 is −3. The table therefore suggests that the probability of drawing a negative return should be 0.135%, or about once in a thousand periods. Long experience with financial data suggests that this is often much too overconfident for the real world. In some contexts, the true probability of even the most negative possible outcome (−100%) may be as high as 1%, even if the Z-score suggests 0.0001%!

e. Correlation and Causation

Correlation does not imply Causation.

A warning: covariation is related to, but not the same as **Causation**. If one variable “causes” another, then the two variables will be correlated. But the opposite does not hold. For example, snow and depression are positively correlated, but neither causes the other. Instead, there is another variable (winter) that has an influence on both snow and depression.

Solve Now!

Q 9.10 If the mean is 20 and the standard deviation is 15, what is the probability that the value will turn out to be less than 0?

Q 9.11 If the mean is 10 and the standard deviation is 20, what is the probability that the value will turn out to be positive?

Q 9.12 If the mean is 50 and the standard deviation is 20, what is the probability that the value will turn out to be greater than 80?

Q 9.13 If the mean is 50 and the standard deviation is 20, what is the probability that the value will turn out to be greater than 30?

[No keyterm list for statistics-g.]

Anecdote: Long Term Capital Management

Long-Term Capital Management (**LTCM**), a prominent hedge fund run by top finance professors and Wall Street traders, collapsed in 1999 in what their quantitative models believed to be a “10 sigma” (i.e., a 10 standard deviation) event. According to the normal distribution probability table, such an event that has a score of −10 should occur with a probability of less than 0.0001%, or 1 in 1,000,000 periods of trading.
You can conclude that either their models were wildly over-optimistic, or their assumption of a normal distribution was incorrect, or the 1 in 1,000,000 actually occurred. Chances are that it was a little bit of all three.
(In essence, LTCM’s model believed that it would be exceedingly unlikely that all their bets would go sour at the same time. Of course, they did all go sour together, so the LTCM principals lost most of their wealth.)

End of Chapter Problems

13 “Solve Now” Answers

1. Do it!
2. The mean was 9.837%. The variance was 7.7%. standard deviation was 27.74%.

3.
$$Correlation(\tilde{x},\tilde{x}) = \frac{\mathcal{C}ov(\tilde{X},\tilde{X})}{Sdv(\tilde{X})\cdot Sdv(\tilde{X})} = \frac{\mathcal{V}ar(\tilde{X})}{\mathcal{V}ar(\tilde{X})} = 1$$

4.
$$\beta_{\tilde{x},\tilde{x}} = \frac{\mathcal{C}ov(\tilde{X},\tilde{X})}{\mathcal{V}ar(\tilde{X})} = \frac{\mathcal{V}ar(\tilde{X})}{\mathcal{V}ar(\tilde{X})} = 1$$

5.
$$\begin{aligned} \mathcal{C}ov(\tilde{r}_{\text{DAX}},\tilde{r}_{\text{S\&P500}}) &= 0.0394 \\ \mathcal{C}ov(\tilde{r}_{\text{DAX}},\tilde{r}_{\text{IBM}}) &= 0.0464 \\ \mathcal{C}ov(\tilde{r}_{\text{DAX}},\tilde{r}_{\text{Sony}}) &= 0.1264 \end{aligned}$$

You have already computed the standard deviation of S&P 500, IBM, and Sony as 19.0%, 38.8%, and 90.3%; and for the DAX, as 27.74%. Therefore,

$$\begin{aligned} Correlation(\tilde{r}_{\text{DAX}},\tilde{r}_{\text{S\&P500}}) &= 74.6\% \\ Correlation(\tilde{r}_{\text{DAX}},\tilde{r}_{\text{IBM}}) &= 43.1\% \\ Correlation(\tilde{r}_{\text{DAX}},\tilde{r}_{\text{Sony}}) &= 50.5\% \end{aligned}$$

The beta of the DAX with respect to the S&P 500 is

$$\begin{aligned} \beta_{\tilde{r}_{\text{DAX}},\tilde{r}_{\text{S\&P500}}} &= \frac{0.0394}{0.0362} = 1.088 \\ \beta_{\tilde{r}_{\text{DAX}},\tilde{r}_{\text{IBM}}} &= \frac{0.0464}{0.388^2} = 0.308 \\ \beta_{\tilde{r}_{\text{DAX}},\tilde{r}_{\text{Sony}}} &= \frac{0.1264}{0.903^2} = 0.155 \end{aligned}$$

6. This is to confirm the digging-deeper on Page 215.

7. Because you know the true distribution of future die throws. The historical values are measured with errors.
8. No.
9. Yes, reasonably so.

10. The score is $(0 - 20)/15 = -1.3$. Therefore, the probability is 9.68%, i.e., roughly 10%.
11. The score is $(0 - 10)/20 = -0.5$. Therefore, the probability is around 30% that the value will be negative, or 70% that it will be positive.
12. The score is $(80 - 50)/20 = +1.5$. Therefore, the probability is around 93% that the value will be below 80, or 7% that it will be above 80.
13. The score is $(30 - 50)/20 = -1.0$. Therefore, the probability is around 16% that the value will be 30, and 84% that it will be above 30.

All answers should be treated as suspect. They have only been sketched and have not been checked.

CHAPTER 10§

Statistics of Portfolios

This chapter appears in the Survey text only.

THE previous chapter explained how to measure risk and reward for an investment—standard deviation and expected rate of return. This chapter explains how to measure these statistics in the context of portfolios. It may be the most tedious chapter in the book. But it is also important: these formulas will be used in subsequent chapters, where you will want to find the risks and rewards of many portfolios. You cannot understand investments without having read this chapter—without understanding the rules for working with statistics in a portfolio context. Your ultimate goal in reading this chapter is to ingest (understand how to use) the methods described in Table 10.5.

Admittedly, some of this chapter "overdoes" it—it tries to explain where the algebra comes from, even though in the end you only need to know the rules. Consider the extra pages to be "useful reference."

Table 10.1: Historical Rates of Returns and Statistics for S&P 500, IBM, Sony, and a portfolio P

Historical Annual Rates of Returns

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$
1991	+0.263	−0.212	−0.103	−0.1758	1997	+0.310	+0.381	+0.391	+0.3842
1992	+0.045	−0.434	−0.004	−0.2903	1998	+0.270	+0.762	−0.203	+0.4407
1993	+0.071	+0.121	+0.479	+0.2400	1999	+0.195	+0.170	+2.968	+1.1028
1994	−0.015	+0.301	+0.135	+0.2457	2000	−0.101	−0.212	−0.511	−0.3116
1995	+0.341	+0.243	+0.105	+0.1969	2001	−0.130	+0.423	−0.348	+0.1659
1996	+0.203	+0.658	+0.077	+0.4647	2002	−0.234	−0.357	−0.081	−0.2647
					Mean over all 12 years	+0.101	+0.154	0.242	+0.183

Quoted as Deviations from the Mean

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{P}$
1991	+0.1620	−0.3661	−0.3448	−0.3590	1997	+0.2090	+0.2273	+0.1485	+0.2010
1992	−0.0565	−0.5874	−0.2458	−0.4735	1998	+0.1656	+0.6086	−0.4448	+0.2575
1993	−0.0305	−0.0330	+0.2364	+0.0568	1999	+0.0942	+0.0163	+2.7261	+0.9196
1994	−0.1165	+0.1474	−0.1073	+0.0625	2000	−0.2025	−0.3658	−0.7529	−0.4948
1995	+0.2400	+0.0892	−0.1374	+0.0137	2001	−0.2315	+0.2693	−0.5904	−0.0173
1996	+0.1015	+0.5046	−0.1648	+0.2815	2002	−0.3348	−0.5105	−0.3228	−0.4479
					Mean (of Deviations) over all 12 years	0.0	0.0	0.0	0.0

Statistics: The covariances (and variances) computed in the previous chapter:

$$\mathcal{V}ar(\tilde{r}_{S\&P500}) = 0.0362\ ,\quad \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{IBM}) = 0.0330\ ,\quad \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) = 0.0477$$

$$\mathcal{C}ov(\tilde{r}_{IBM}, \tilde{r}_{S\&P500}) = 0.0330\ ,\quad \mathcal{V}ar(\tilde{r}_{IBM}) = 0.1503\ ,\quad \mathcal{C}ov(\tilde{r}_{IBM}, \tilde{r}_{Sony}) = 0.0218$$

$$\mathcal{C}ov(\tilde{r}_{Sony}, \tilde{r}_{S\&P500}) = 0.0477\ ,\quad \mathcal{C}ov(\tilde{r}_{Sony}, \tilde{r}_{IBM}) = 0.0218\ ,\quad \mathcal{V}ar(\tilde{r}_{Sony}) = 0.8149$$

The new portfolio P covariances, computed from the twelve historical returns above, are

$$\mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{P}) = 0.0379\ ,\quad \mathcal{C}ov(\tilde{r}_{IBM}, \tilde{r}_{P}) = 0.1075$$

$$\mathcal{C}ov(\tilde{r}_{Sony}, \tilde{r}_{P}) = 0.2862\ ,\quad \mathcal{C}ov(\tilde{r}_{P}, \tilde{r}_{P}) = 0.1671$$

The standard deviations are therefore

$$\mathcal{S}dv(\tilde{r}_{S\&P500}) = 19.0\%\ ,\quad \mathcal{S}dv(\tilde{r}_{IBM}) = 38.8\%\ ,\quad \mathcal{S}dv(\tilde{r}_{Sony}) = 90.3\%\ ,\quad \mathcal{S}dv(\tilde{r}_{P}) = 40.9\%\ .$$

The return of portfolio P is $\tilde{r}_P = 66.7\% \cdot \tilde{r}_{IBM} + 33.3\% \cdot \tilde{r}_{Sony}$. Note: to keep investment weights constant at 66.7% and 33.3%, you must rebalance the portfolio every year.

10·1 Two Investment Securities

Table 10.1 summarizes what you did in the previous chapter, plus it adds statistics for a portfolio P.

Our goal in this section is to explore the properties of a portfolio P that invests twice as much into IBM as it invests into Sony. Table 10.1 begins with all the information you computed in the previous chapter: historical rates of return, means, deviations from the means, variances, covariances, and standard deviations. The only novelty is that the performance of a portfolio P in each year is now also in the table—as if it were a stock that you could have purchased. The table also repeats the calculations for the covariances and variances for P. Please check the calculations—do not go on until you have convinced yourself both that you can compute the basic statistics for P yourself, and that the table contains no mistakes.

Solve Now!

Q 10.1 Compute the $\tilde{r}_P$ related statistics from the twelve historical rates of returns in Table 10.1.

Q 10.2 Is there an error in Table 10.1? If so, can you find it?

10·1.A. Expected Rates of Returns

The expected rate of return is the investment-weighted average.

You knew this one even before you ever opened my book. If you expect 20% in your first stock and 30% in your second stock, and you invest half in each, you expect to earn 25%. In our case, in which portfolio P is defined by

$$P \equiv (2/3 \text{ IBM}, 1/3\text{Sony})$$

the expected rate of return on P of 18.3%. You could directly work this out from the 12 observations in the time-series in the first panel of Table 10.1, so you should confirm now that it is also the investment-weighted average of the expected rates of return on our portfolio,

$$\begin{aligned} \mathcal{E}(\tilde{r}_P) &= \mathcal{E}(w_{\text{IBM}} \cdot \tilde{r}_{\text{IBM}} + w_{\text{Sony}} \cdot \tilde{r}_{\text{Sony}}) \\ &= w_{\text{IBM}} \cdot \mathcal{E}(\tilde{r}_{\text{IBM}}) + w_{\text{Sony}} \cdot \mathcal{E}(\tilde{r}_{\text{Sony}}) \\ &= 66.7\% \cdot 15.4\% + 33.3\% \cdot 24.2\% = 18.3\% \end{aligned}$$

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_P \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with expected rates of return of a portfolio by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned} \mathcal{E}(\tilde{r}_P) &= \mathcal{E}(w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2) \\ &= w_1 \cdot \mathcal{E}(\tilde{r}_1) + w_2 \cdot \mathcal{E}(\tilde{r}_2) \end{aligned}$$

10·1.B. Covariance

Compute the covariance the slow way: from the twelve historical returns.

A more interesting question is about the covariance of our portfolio P (2/3 in IBM, 1/3 in Sony) with the returns on some other portfolio, say, the S&P 500. In Table 10.1, the covariance worked out from the twelve historical returns is

$$\mathcal{C}ov(\ \tilde{r}_{\text{S\&P500}}\ ,\ \tilde{r}_{\text{P}}\ ) = \frac{(-0.3590)\cdot(+0.1620)+\cdots+(-0.4479)\cdot(-0.3348)}{11} = 0.0379$$

(This computation works with the deviations from the means.) But how does our P portfolio's covariance with the S&P 500 (0.0379) relate to the covariances of its two portfolio components with the S&P 500 (0.0330 for IBM, and 0.0477 for Sony)?

The covariance of the rate of return with something else is the investment-weighted average.

In the previous section, you learned that the expected rate of return on our portfolio is the investment-weighted average of the expected rates of the portfolio constituents. Can you do the same for covariance—i.e., is the portfolio covariance equal to the weighted sum of its portfolio constituents' covariances? Yes!

$$\begin{array}{ccccc} 66.7\%\cdot 0.0330 & + & 33.3\%\cdot 0.0477 & = & 0.0379 \\ w_{\text{IBM}}\cdot\mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ ) & + & w_{\text{Sony}}\cdot\mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ ) & = & \mathcal{C}ov(\ \tilde{r}_{\text{P}}\ ,\ \tilde{r}_{\text{S\&P500}}\ ) \end{array} \tag{10.1}$$

If you want to find out the covariance of our portfolio with the S&P 500 (or any other security), all you need to do is compute the weighted average of its constituents. You do not need to recompute the covariance from scratch in the tedious multi-step manner if you want to experiment with different portfolio weights!

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_{\text{P}} \equiv w_1\cdot r_1 + w_2\cdot r_2$.

You can work with the covariances of your portfolio P with any other portfolio X by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned} Cov(\ \tilde{r}_{\text{P}}\ ,\ \tilde{r}_{\text{X}}\ ) &= Cov(\ w_1\cdot\tilde{r}_1 + w_2\cdot\tilde{r}_2\ ,\ \tilde{r}_{\text{X}}\ ) \\ &= w_1\cdot Cov(\ \tilde{r}_1\ ,\ \tilde{r}_{\text{X}}\ ) + w_2\cdot Cov(\ \tilde{r}_2\ ,\ \tilde{r}_{\text{X}}\ ) \end{aligned}$$

If you like our σ notation, you can write this as

$$\sigma_{\text{P,X}} = w_1\cdot\sigma_{1,\text{X}} + w_2\cdot\sigma_{2,\text{X}}$$

As usual, we just omit the $\tilde{r}$ in the sigma subscripts, so that we avoid double subscripts. (Incidentally, this covariance law is only interesting, because we need it to work out the laws for variance and beta.)

10·1.C. Beta

The covariance of the rate of return with something else is the investment-weighted average.

Our next question is: what is the beta of our portfolio P with respect to another security—here again the S&P 500? That is, how can you express the portfolio beta for P in terms of the betas of its two constituents (2/3 in IBM, 1/3 in Sony). Recall from the previous chapter how beta is defined: you divide the covariance between $\tilde{X}$ and $\tilde{Y}$ by the variance of the $\tilde{X}$ variable,

$$\begin{aligned}
\beta_{\text{IBM,S\&P 500}} &= \frac{\mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} = \frac{0.0330}{0.0362} \approx 0.91 \\
\beta_{\text{Sony,S\&P 500}} &= \frac{\mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} = \frac{0.0477}{0.0362} \approx 1.32 \\
\beta_{\text{P,S\&P 500}} &= \frac{\mathcal{C}ov(\ \tilde{r}_{\text{P}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} = \frac{0.0379}{0.0362} \approx 1.05 \\
\beta_{Y,X} &= \frac{\mathcal{C}ov(\ \tilde{r}_X\ ,\ \tilde{r}_Y\ )}{\mathcal{V}ar(\tilde{r}_X)}
\end{aligned} \tag{10.2}$$

Note that the second subscript on beta is the variance denominator, and that we again omit the $\tilde{r}$ in the beta subscripts so as to avoid double subscripts.

The beta of the rate of return with something else is the investment-weighted average.

In the previous subsections, you learned that both the expected rate of return and the covariance of our portfolio with another portfolio are the investment-weighted statistics of the portfolio constituents, respectively. Can you do the same for beta—i.e., is the portfolio beta equal to the weighted sum of its portfolio betas? Yes!

$$\begin{matrix}
66.7\% \cdot 0.91 & + & 33.3\% \cdot 1.32 & = & 1.05 \\
w_{\text{IBM}} \cdot \beta_{\text{IBM,S\&P 500}} & + & w_{\text{Sony}} \cdot \beta_{\text{Sony,S\&P 500}} & = & \beta_{\text{P,S\&P 500}}
\end{matrix}$$

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_{\text{P}} \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with betas of a portfolio by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned}
\beta_{\text{P,X}} &= \beta_{(w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2\ ,\ \tilde{r}_{\text{X}})} \\
&= w_1 \cdot \beta_{1,\text{X}} + w_2 \cdot \beta_{2,\text{X}}
\end{aligned}$$

SIDE NOTE

Two points. First, because you know the covariance law, you can easily show this algebraically,

$$\begin{aligned}
\beta_{\tilde{r}_{\text{P}},\text{S\&P 500}} &= \frac{\mathcal{C}ov(\ \tilde{r}_{\text{P}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= \frac{\mathcal{C}ov(\ w_{\text{IBM}} \cdot \tilde{r}_{\text{IBM}} + w_{\text{Sony}} \cdot \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= \frac{w_{\text{IBM}} \cdot \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ ) + w_{\text{Sony}} \cdot \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= w_{\text{IBM}} \cdot \frac{\mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} + w_{\text{Sony}} \cdot \frac{\mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{S\&P500}}\ )}{\mathcal{V}ar(\tilde{r}_{\text{S\&P500}})} \\
&= w_{\text{IBM}} \cdot \beta_{\text{IBM}\ ,\ \text{S\&P 500}} + w_{\text{Sony}} \cdot \beta_{\text{Sony}\ ,\ \text{S\&P 500}}
\end{aligned}$$

Second, $\beta_{X,P} \neq w_1 \cdot \beta_{X,1} + w_2 \cdot \beta_{X,2}$. Weighted averaging works only for the first beta subscript, not the second.

10·1.D. Variance

The variance of the rate of return is not the investment-weighted average.

Our next question is: how does the variance of the rate of return of our portfolio relate to the covariances of its constituents? This time, our trick does not work. The variance of a portfolio is *not* the investment-weighted average of the portfolio constituents:

$$\begin{array}{ccccccc} 0.1671 & \neq & 66.7\% \cdot 0.1503 & + & 33.3\% \cdot 0.8149 & = & 0.3719 \\ \mathcal{V}ar(\tilde{r}_P) & \neq & w_{\text{IBM}} \cdot \mathcal{V}ar(\tilde{r}_{\text{IBM}}) & + & w_{\text{Sony}} \cdot \mathcal{V}ar(\tilde{r}_{\text{Sony}}) & & \end{array}$$

(Incidentally, the variance of the portfolio is much lower than just the average of its constituents; you will explore this in great detail in later chapters—after you learn how to work with variances.)

Eliminate the first $\tilde{r}_P$ from the variance...ahem, the covariance with itself.

To find out how the variance can be decomposed, you now have to cleverly use our covariance law. You already know that the variance of a random variable is the covariance with itself—and you have even computed it from the twelve historical returns,

$$\mathcal{V}ar(\tilde{r}_P) = \mathcal{C}ov(\ \tilde{r}_P\ ,\ \tilde{r}_P\ ) = 0.1671$$

Now, drop in the definition of our portfolio, but just once,

$$\mathcal{V}ar(\tilde{r}_P) = \mathcal{C}ov(\ 66.7\% \cdot \tilde{r}_{\text{IBM}} + 33.3\% \cdot \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_P\ ) = 0.1671$$

and use our covariance law to pull out the weights and to create a sum,

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_P) &= \mathcal{C}ov(\ 66.7\% \cdot \tilde{r}_{\text{IBM}} + 33.3\% \cdot \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_P\ ) \\ &= 66.7\% \cdot \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_P\ ) + 33.3\% \cdot \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_P\ ) \end{aligned}$$

Table 10.1 has these covariances, so you can check that we have not committed a mistake yet,

$$\begin{array}{ccccccc} \mathcal{V}ar(\tilde{r}_P) & = & 66.7\% \cdot 0.1075 & + & 33.3\% \cdot 0.2862 & = & 0.1671 \\ & & w_{\text{IBM}} \cdot \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_P\ ) & + & w_{\text{Sony}} \cdot \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_P\ ) & & \end{array}$$

You are still getting the same number through algebra that you obtained by direct computation of the variance (from the twelve historical rates of return in Table 10.1).

Eliminate the second $\tilde{r}_P$ from each term.

Now you must handle each of these two covariance terms by themselves: use the covariance law to pull out the weights and distribute the sum, and substitute in the covariance inputs from Table 10.1,

$$\begin{aligned} \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_P\ ) &= \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ 66.7\% \cdot \tilde{r}_{\text{IBM}} + 33.3\% \cdot \tilde{r}_{\text{Sony}}\ ) \\ &= 66.7\% \cdot \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{IBM}}\ ) + 33.3\% \cdot \mathcal{C}ov(\ \tilde{r}_{\text{IBM}}\ ,\ \tilde{r}_{\text{Sony}}\ ) \\ &= 66.7\% \cdot 0.1503 + 33.3\% \cdot 0.0218 = 0.1075 \\ \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_P\ ) &= \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ 66.7\% \cdot \tilde{r}_{\text{IBM}} + 33.3\% \cdot \tilde{r}_{\text{Sony}}\ ) \\ &= 66.7\% \cdot \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{IBM}}\ ) + 33.3\% \cdot \mathcal{C}ov(\ \tilde{r}_{\text{Sony}}\ ,\ \tilde{r}_{\text{Sony}}\ ) \\ &= 66.7\% \cdot 0.0218 + 33.3\% \cdot 0.8149 = 0.2862 \end{aligned}$$

(Again, you can check that the algebra is right, because using the information from Table 10.1 gives you the same answers for $\mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{P}})$ and $\mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{P}})$.) Now put it all together,

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_{\text{P}}) &= 0.1671 \\
&= 66.7\% \cdot 0.1075 \;+\; 33.3\% \cdot 0.2862 \\
&= w_{\text{IBM}} \cdot [\mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{S\&P500}})] \;+\; w_{\text{Sony}} \cdot [\mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{S\&P500}})] \\
&= 66.7\% \cdot \overbrace{[66.7\% \cdot 0.1503 \;+\; 33.3\% \cdot 0.0218]}^{0.1075} \\
&\quad + 33.3\% \cdot \underbrace{[66.7\% \cdot 0.0218 \;+\; 33.3\% \cdot 0.8149]}_{0.2862} \\
&= w_{\text{IBM}} \cdot [\, w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{S\&P500}}) \;+\; w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{Sony}}) \,] \\
&\quad + w_{\text{Sony}} \cdot [\, w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{S\&P500}}) \;+\; w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{Sony}}) \,]
\end{aligned}$$

Tedious substitutions—but no higher math required. Take the last expression, multiply out (remembering that the covariance of anything with itself is the variance), realize that the two terms that contain both Sony and IBM are the same, rearrange the terms, and you get

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_{\text{P}}) &= (66.7\%)^2 \cdot 0.1503 \;+\; (33.3\%)^2 \cdot 0.8149 \;+\; 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218 \\
&= 0.1671, \\
\mathcal{V}ar(\tilde{r}_{\text{P}}) &= w_{\text{IBM}}^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{IBM}}) \;+\; w_{\text{Sony}}^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{Sony}}) \;+\; 2 \cdot w_{\text{Sony}} \cdot w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{IBM}})
\end{aligned} \tag{10.3}$$

You are done: this is how the variance of a portfolio is expressed in terms of the variances and covariances of its constituent securities. It is the sum of the variances, each multiplied by its weight squared, plus two times each weight times the pairwise covariance. And you know this is right, because the answer is still the same 0.1671 that you computed directly from the historical rates of return for portfolio P in Table 10.2!

IMPORTANT: Say your portfolio P consists of an investment w_1 in security 1 and an investment w_2 in security 2. Therefore, its rate of return is $r_{\text{P}} \equiv w_1 \cdot r_1 + w_2 \cdot r_2$.

You can work with variances as follows:

$$\mathcal{V}ar(\tilde{r}_{\text{P}}) = w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) \;+\; w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) \;+\; 2 \cdot w_1 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) \tag{10.4}$$

This is not the investment-weighted average of its constituents.

If you liked our σ notation, you can write this briefer as

$$\begin{aligned}
\sigma_P^2 = \sigma_{P,P} &= w_X \cdot \sigma_{X,X} \;+\; w_Y \cdot \sigma_{Y,Y} \;+\; 2 \cdot w_X \cdot w_Y \cdot \sigma_{X,Y} \\
&= w_X \cdot \sigma_X^2 \;+\; w_Y \cdot \sigma_Y^2 \;+\; 2 \cdot w_X \cdot w_Y \cdot \sigma_{X,Y}
\end{aligned}$$

Standard Deviation

To compute the standard deviation, you always have to first compute the variance, and then take the square-root. You *cannot* take any shortcuts here—like computing the average standard deviation of the portfolio's constituents.

Solve Now!

Q 10.3 Assume portfolio S consists of 25% IBM and 75% Sony. Compute the covariance of the rate of return of S with the rates of return on IBM, Sony, and S&P 500. (If you feel shaky, do it with both a historical time-series in a table, and with the formulas.)

Q 10.4 Continue with portfolio S. Compute the beta of portfolio S with respect to the S&P 500, denoted $\beta_{S,S\&P\ 500}$. (If you feel shaky, do it with a historical time-series in a table, then directly from the covariances, and finally with the beta-combination formula.)

Q 10.5 Continue with portfolio S. Compute its variance. (If you feel shaky, do it with both a historical time-series in a table, and the formulas.)

10·2 Three and More Investment Securities

We now work with portfolios of more than two securities, often expressed in summation notation.

Of course, you rarely want a portfolio with just two investments—usually, your portfolio will have more than two investments. Let me remind you of how summation notation works. As in Chapter 8, we later use i as a counter to enumerate each and every possible investment, from N choices (e.g., stocks). w_i are the investment weights that define the portfolio. The return of any portfolio Q defined by these weights in each and every time period is

$$r_Q = \sum_{i=1}^{N} w_i \cdot r_i$$

but because you do not yet know the returns,

$$\tilde{r}_Q = \sum_{i=1}^{N} w_i \cdot \tilde{r}_i$$

10·2.A. Expected Returns, Covariance, Beta

Expected returns, covariances, and betas can be averaged, using the investment-weighted proportions.

For the three statistics from the previous section that we could average—the expected return, the covariance, and beta—the generalization from two securities to any number (N) of securities is easy: you can just take a weighted sum of all the components, not just of the first two components. This is what our next important box expresses.

IMPORTANT: (For a portfolio Q that consists of w_1 investment in security 1, w_2 investment in security 2, all the way up to w_N investment in security N, and which therefore has a rate of return of $r_Q \equiv w_1 \cdot r_1 + w_2 \cdot r_2 + \cdots + w_N \cdot r_N = \sum_{i=1}^{N} w_i r_i$):

You can work with expected rates of return, covariances, and betas by taking the investment-weighted average of its constituents, as follows:

$$\begin{aligned}
\mathcal{E}(\tilde{r}_Q) &= w_1 \cdot \mathcal{E}(\tilde{r}_1) + w_2 \cdot \mathcal{E}(\tilde{r}_2) + \cdots + w_N \cdot \mathcal{E}(\tilde{r}_N) = \sum_{i=1}^{N} w_i \cdot \mathcal{E}(\tilde{r}_i) \\
\sigma_{P,X} &= w_1 \cdot \sigma_{1,X} + w_2 \cdot \sigma_{2,X} + \cdots + w_N \cdot \sigma_{N,X} = \sum_{i=1}^{N} w_i \cdot \sigma_{i,X} \\
\beta_{P,X} &= w_1 \cdot \beta_{1,X} + w_2 \cdot \beta_{2,X} + \cdots + w_N \cdot \beta_{N,X} = \sum_{i=1}^{N} w_i \cdot \beta_{i,X}
\end{aligned} \tag{10.5}$$

Checking the formulas for a particular portfolio, that we shall use later, too.

Let's use these formulas on a new portfolio Q that has 70% invested in the S&P 500, 20% invested in IBM, and 10% invested in Sony. You want to know the expected rate of return of our portfolio ($\mathcal{E}(\tilde{r}_Q)$), its covariance with S&P 500 ($\mathcal{C}ov(\tilde{r}_Q, \tilde{r}_{S\&P500}) \equiv \sigma_{Q,S\&P\ 500}$), and its beta with respect to S&P 500 ($\beta_{Q,S\&P\ 500}$). Corresponding to the formulas' numbering schemes, security 1 is S&P 500, security 2 is IBM, and security 3 is Sony. Table 10.1 provided all the necessary expected returns and covariances; betas were in Formula 10.2; and your portfolio investment weights are given.

$$
\begin{array}{rcccccccc}
\tilde{r}_{Q} & = & 70\% \cdot \tilde{r}_{S\&P500} & + & 20\% \cdot \tilde{r}_{IBM} & + & 10\% \cdot \tilde{r}_{Sony} & & \\
\mathcal{E}(\tilde{r}_{Q}) & = & 70\% \cdot 10.1\% & + & 20\% \cdot 15.4\% & + & 10\% \cdot 24.2\% & = & 12.57\% \\
 & = & w_{S\&P\,500} \cdot \mathcal{E}(\tilde{r}_{S\&P500}) & + & w_{IBM} \cdot \mathcal{E}(\tilde{r}_{IBM}) & + & w_{Sony} \cdot \mathcal{E}(\tilde{r}_{Sony}) & & \\
 & = & \sum_{i=1}^{3} w_i \cdot \mathcal{E}(\tilde{r}_i) & = & \sum_{S\&P\,500,IBM,Sony} w_i \cdot \mathcal{E}(\tilde{r}_i) & ; & & & \\
\\
\sigma_{Q,S\&P\,500} & = & 70\% \cdot 0.03622 & + & 20\% \cdot 0.03298 & + & 10\% \cdot 0.04772 & = & 0.03672 \\
 & = & w_{S\&P\,500} \cdot \sigma_{S\&P\,500,S\&P\,500} & + & w_{IBM} \cdot \sigma_{IBM,S\&P\,500} & + & w_{Sony} \cdot \sigma_{Sony,S\&P\,500} & & \\
 & = & \sum_{i=1}^{3} w_i \cdot \sigma_{i,S\&P\,500} & = & \sum_{S\&P\,500,IBM,Sony} w_i \cdot \sigma_{i,S\&P\,500} & ; & & & \\
\\
\beta_{Q,S\&P\,500} & = & 70\% \cdot 1.000 & + & 20\% \cdot 0.910 & + & 10\% \cdot 1.317 & = & 1.0138 \\
 & = & w_{S\&P\,500} \cdot \beta_{S\&P\,500,S\&P\,500} & + & w_{IBM} \cdot \beta_{IBM,S\&P\,500} & + & w_{Sony} \cdot \beta_{Sony,S\&P\,500} & & \\
 & = & \sum_{i=1}^{3} w_i \cdot \beta_{i,S\&P\,500} & = & \sum_{S\&P\,500,IBM,Sony} w_i \cdot \beta_{i,S\&P\,500} & & & &
\end{array}
\tag{10.7}
$$

Table 10.2 gives the historical rates of return of this portfolio next to its constituents. Please confirm from the twelve historical returns for Q and S&P 500 that the above three statistics are correct.

Table 10.2: Historical Annual Rates of Returns for portfolio Q

Historical Annual Rates of Returns

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{Q}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{Q}$
1991	+0.263	−0.212	−0.103	+0.131	1997	+0.310	+0.381	+0.391	+0.332
1992	+0.045	−0.434	−0.004	−0.056	1998	+0.270	+0.762	−0.203	+0.319
1993	+0.071	+0.121	+0.479	+0.121	1999	+0.195	+0.170	+2.968	+0.468
1994	−0.015	+0.301	+0.135	+0.063	2000	−0.101	−0.212	−0.511	−0.165
1995	+0.341	+0.243	+0.105	+0.298	2001	−0.130	+0.423	−0.348	−0.042
1996	+0.203	+0.658	+0.077	+0.281	2002	−0.234	−0.357	−0.081	−0.243
					Mean over all 12 years	+0.101	+0.154	0.242	+0.1257

Quoted as Deviations from the Mean

Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{Q}$	Year	$\tilde{r}_{S\&P500}$	$\tilde{r}_{IBM}$	$\tilde{r}_{Sony}$	$\tilde{r}_{Q}$
1991	+0.1620	−0.3661	−0.3448	−0.006	1997	+0.2090	+0.2273	+0.1485	+0.207
1992	−0.0565	−0.5874	−0.2458	−0.182	1998	+0.1656	+0.6086	−0.4448	+0.193
1993	−0.0305	−0.0330	+0.2364	+0.004	1999	+0.0942	+0.0163	+2.7261	+0.342
1994	−0.1165	+0.1474	−0.1073	+0.063	2000	−0.2025	−0.3658	−0.7529	−0.290
1995	+0.2400	+0.0892	−0.1374	+0.172	2001	−0.2315	+0.2693	−0.5904	−0.167
1996	+0.1015	+0.5046	−0.1648	+0.156	2002	−0.3348	−0.5105	−0.3228	−0.369
					Mean (of Deviations) over all 12 years	0.0	0.0	0.0	0.0

The return of portfolio Q is $\tilde{r}_{Q} = 70\% \cdot \tilde{r}_{S\&P500} + 20\% \cdot \tilde{r}_{IBM} + 10\% \cdot \tilde{r}_{Sony}$.

DIG DEEPER

Expectations, covariances, and betas are so-called linear function, because

$$f(a+b) = f(a) + f(b) \tag{10.8}$$

For our expectations, covariances, and beta, a would be defined as $w_1 \cdot \tilde{r}_1$, and b as $w_2 \cdot \tilde{r}_2$. For example, $\mathcal{E}(w_1 \cdot \tilde{r}_1 + w_2 \cdot \tilde{r}_2) = \mathcal{E}(w_1 \cdot \tilde{r}_1) + \mathcal{E}(w_2 \cdot \tilde{r}_2)$.

It is not difficult to prove that if a function is linear, then it works for more than two securities in a portfolio. Just replace b with $c+d$

$$f(a+(c+d)) = f(a) + f(c+d)$$

and apply formula 10.8 again on the $c+d$ term to get

$$f(a+(c+d)) = f(a) + f((c+d)) = f(a) + f(c) + f(d)$$

Solve Now!

Q 10.6 Confirm that the computations for the expected rate of return and the covariance in Formula 10.6 are correct by directly computing these statistics from the historical timeseries in Table 10.2.

Q 10.7 Consider a portfolio T that consists of 25% S&P 500, 35% IBM, and 40% Sony. What is its expected rate of return, its covariance with the S&P 500, and its beta with respect to the S&P 500? (If you feel shaky, compute this from both the twelve historical rates of return on T, and from the formulas.)

10·2.B. Variance

Panic Warning!

Here is where it gets more complicated. What is the variance of portfolio Q, consisting of 70% in S&P 500, 20% in IBM, and 10% in Sony? Actually, it will get more tedious, but not more complex. All you need to do is to apply the covariance law twice.

I made it easy on us: portfolio Q is a combination of S&P 500 and P.

Recall the portfolio Q that invested 66.7% in IBM and 33.3% in Sony. If you invest 70% in S&P 500 and 30% in P, you end up with portfolio Q, because the remaining 30% in P are appropriately split ($w_{\text{IBM}} = 30\% \cdot 66.7\% = 20\%$ and $w_{\text{Sony}} = 30\% \cdot 33.3\% = 10\%$):

$$\begin{aligned}
\tilde{r}_{\text{Q}} &= 70\% \cdot \tilde{r}_{\text{S\&P500}} + \quad (20\% \cdot \tilde{r}_{\text{IBM}} + 10\% \cdot \tilde{r}_{\text{Sony}}) \\
&= 70\% \cdot \tilde{r}_{\text{S\&P500}} + 30\% \cdot (66.7\% \cdot \tilde{r}_{\text{IBM}} + 33.3\% \cdot \tilde{r}_{\text{Sony}}) \\
&= 70\% \cdot \tilde{r}_{\text{S\&P500}} + \quad 30\% \cdot \tilde{r}_{\text{P}}
\end{aligned}$$

You already know how to work with two securities—and you know the other inputs, too.

With only two securities (S&P 500 and P) now, you can use our variance formula 10.4:

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_{\text{Q}}) &= \mathcal{V}ar(w_{\text{S\&P 500}} \cdot \tilde{r}_{\text{S\&P500}} + w_{\text{P}} \cdot \tilde{r}_{\text{P}}) \\
&= (w_{\text{S\&P 500}})^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{S\&P500}}) + (w_{\text{P}})^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{P}}) + 2 \cdot w_{\text{S\&P 500}} \cdot w_{\text{P}} \cdot \mathcal{C}ov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{P}}) \quad (10.9) \\
&= (70\%)^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{S\&P500}}) + (30\%)^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{P}}) + 2 \cdot 30\% \cdot 70\% \cdot \mathcal{C}ov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{P}})
\end{aligned}$$

Actually, you already know all three remaining unknowns: 0.0362 was the variance of the S&P 500, given in Table 10.1. You had worked out $\mathcal{V}ar(\tilde{r}_{\text{P}})$ in Formula 10.3,

$$\mathcal{V}ar(\tilde{r}_{\text{P}}) = (66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218 = 0.1671$$

$$\mathcal{V}ar(\tilde{r}_{\text{P}}) = w_{\text{IBM}}^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{IBM}}) + w_{\text{Sony}}^2 \cdot \mathcal{V}ar(\tilde{r}_{\text{Sony}}) + 2 \cdot w_{\text{Sony}} \cdot w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{IBM}})$$

and $\mathcal{C}ov(\tilde{r}_{\text{P}}, \tilde{r}_{\text{S\&P500}})$ in Formula 10.1,

$$\mathcal{C}ov(\tilde{r}_{\text{P}}, \tilde{r}_{\text{S\&P500}}) = 66.7\% \cdot 0.0330 + 33.3\% \cdot 0.0477 = 0.0379$$

$$w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{S\&P500}}) + w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{S\&P500}})$$

Now just substitute these terms into Formula 10.9 to get

$$
\begin{aligned}
\mathcal{V}ar(\tilde{r}_{Q}) &= (70\%)^2 \cdot \mathcal{V}ar(\tilde{r}_{IBM}) + (30\%)^2 \cdot \mathcal{V}ar(\tilde{r}_{P}) + 2 \cdot 70\% \cdot 30\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{P}) \\
&= (70\%)^2 \cdot 0.0362 \\
&\quad + (30\%)^2 \cdot \underbrace{[(66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218]}_{\mathcal{V}ar(\tilde{r}_{P})} \\
&\quad + 2 \cdot 70\% \cdot 30\% \cdot \underbrace{[66.7\% \cdot 0.0330 + 33.3\% \cdot 0.0477]}_{\mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{P})} \\
&= (70\%)^2 \cdot 0.0362 + (30\%)^2 \cdot 0.1671 + 2 \cdot 70\% \cdot 30\% \cdot 0.0379 \\
&= 0.0487 \; .
\end{aligned}
\tag{10.10}
$$

Please confirm this from the twelve annual rates of return for portfolio Q in Table 10.2.

Although we are done—we have our answer for the variance of our portfolio Q—let's do some more algebra "just for fun." Take the middle form from the previous formula,

A lengthy detailed step-by-step rewrite of the answer shows us what the answer really consists of: three variance terms and three pairwise covariance terms.

$$
\begin{aligned}
\mathcal{V}ar(\tilde{r}_{Q}) &= (70\%)^2 \cdot 0.0362 \\
&\quad + (30\%)^2 \cdot [(66.7\%)^2 \cdot 0.1503 + (33.3\%)^2 \cdot 0.8149 + 2 \cdot 66.7\% \cdot 33.3\% \cdot 0.0218] \\
&\quad + 2 \cdot 70\% \cdot 30\% \cdot [66.7\% \cdot 0.0330 + 33.3\% \cdot 0.0477] \\
&= 0.04869 \; .
\end{aligned}
$$

Now, multiply the (30%) and $(30\%)^2$ terms into the parentheses, and pull the 30% and 30%% into the adjacent weights,

$$
\begin{aligned}
&= (70\%)^2 \cdot 0.0362 \\
&\quad + [(30\% \cdot 66.7\%)^2 \cdot 0.1503 + (30\% \cdot 33.3\%)^2 \cdot 0.8149 + 2 \cdot (30\% \cdot 66.7\%) \cdot (30\% \cdot 33.3\%) \cdot 0.0218] \\
&\quad + 2 \cdot 70\% \cdot [(30\% \cdot 66.7\%) \cdot 0.0330 + (30\% \cdot 33.3\%) \cdot 0.0477] \\
&= 0.04869 \; .
\end{aligned}
$$

Execute the 30% multiplication, multiply in the 2 · 70%, eliminate some parentheses, and reorder terms,

$$
\begin{aligned}
&= (70\%)^2 \cdot 0.0362 + (20\%)^2 \cdot 0.1503 + (10\%)^2 \cdot 0.8149 \\
&\quad + 2 \cdot (20\%) \cdot (10\%) \cdot 0.0218 + 2 \cdot (70\%) \cdot (20\%) \cdot 0.0330 + 2 \cdot (70\%) \cdot (10\%) \cdot 0.0477 \\
&= 0.04869 \; .
\end{aligned}
$$

Now stare at this formula, and recall the form of the variance formula in 10.4. What does this formula consist of? Well, it is just

↦ *Variance Formula 10.4 on Page 217*

$$
\begin{aligned}
\mathcal{V}ar(\tilde{r}_{Q}) = w_{S\&P\,500}^2 \cdot \mathcal{V}ar(\tilde{r}_{S\&P500}) &+ w_{IBM}^2 \cdot \mathcal{V}ar(\tilde{r}_{IBM}) + w_{Sony}^2 \cdot \mathcal{V}ar(\tilde{r}_{Sony}) \\
&+ 2 \cdot w_{IBM} \cdot w_{Sony} \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) \\
&+ 2 \cdot w_{S\&P\,500} \cdot w_{IBM} \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{IBM}) \\
&+ 2 \cdot w_{S\&P\,500} \cdot w_{Sony} \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony})
\end{aligned}
$$

This generalizes, too!

IMPORTANT: To obtain the variance of a portfolio that invests $w_1, w_2, \ldots, w_N$ into N securities, do the following:

1. For each security, square its weights and multiply it by the variance.
2. For each pair of different securities, multiply two times the first weight times the second weight times the securities' covariance.
3. Add up all these terms.

For example, for three securities, the formula is

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_Q) = w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) \quad + \quad w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) \quad + \quad w_3^2 \cdot \mathcal{V}ar(\tilde{r}_3) \\ + \; 2 \cdot w_1 \cdot w_2 \cdot \mathcal{C}ov(\,\tilde{r}_1 \;,\; \tilde{r}_2\,) \\ + \; 2 \cdot w_1 \cdot w_3 \cdot \mathcal{C}ov(\,\tilde{r}_1 \;,\; \tilde{r}_3\,) \\ + \; 2 \cdot w_2 \cdot w_3 \cdot \mathcal{C}ov(\,\tilde{r}_2 \;,\; \tilde{r}_3\,) \end{aligned}$$

For portfolios with many securities, there are relatively more covariance than variance terms.

If you do not believe me, feel free to repeat this exercise for four securities. (The principle remains the same, but it becomes a lot more messy.) Incidentally, the number of covariance terms increases more rapidly than the number of variance terms. With four securities, you will have four variance terms, and six pairwise covariance terms. With ten securities, you will have ten variance terms, and forty-five pairwise covariance terms. For one-hundred stocks, there are one-hundred variance terms and 2,475 covariance terms. With more and more securities in the portfolio, there are fewer and fewer "own return variance" terms, and more and more "return covariance terms." Thus, on first glance, it seems that the overall portfolio variance could be driven more by the many covariance terms than the few variance terms. This will play a major role in the next chapters.

Solve Now!

Q 10.8 Continue with our portfolio T that consists of 25% S&P 500, 35% IBM, and 40% Sony. What is its standard deviation of return? Compute this both from the twelve historical rates of return on T, and from the formulas.

10·2.C. Advanced Nerd Section: Variance with N Securities and Double Summations

The Formal Definitions.

Formula 10.5 used $\sum$ notation to write sums more compactly. The point of this section is to show how to write the variance formula with two summations signs—a lot more compactly, and perhaps easier to remember. We shall first write it down—don't panic—and then explain and use it. The variance of the rate of return of a portfolio is

$$\mathcal{V}ar(\tilde{r}_P) = \mathcal{V}ar\left(\sum_{i=1}^{N} w_i \cdot \tilde{r}_i\right) = \sum_{j=1}^{N} \left\{ \sum_{k=1}^{N} [w_j \cdot w_k \cdot \mathcal{C}ov(\,\tilde{r}_j \;,\; \tilde{r}_k\,)] \right\} \tag{10.11}$$

What does this formula mean? Write out the terms in this formula to eliminate the summation signs. Concentrate on one step at a time. Start with the innermost parentheses: j is still unknown, so leave j untouched. Just write out the sum for k, which consists of N terms,

$$\sum_{k=1}^{N} [w_j \cdot w_k \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_k)]$$

$$= \left\{ \underbrace{w_j \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_1)}_{\text{our } k = 1 \text{ term}} + \underbrace{w_j \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_2)}_{\text{our } k = 2 \text{ term}} + \cdots + \underbrace{w_j \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_N)}_{\text{our } k = N \text{ term}} \right\}$$

and plug it back into Formula 10.11,

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_P) &= \sum_{j=1}^{N} \left\{ \sum_{k=1}^{N} \left[w_j \cdot w_k \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_k) \right] \right\} \\ &= \sum_{j=1}^{N} \left\{ w_j \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_1) + w_j \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_2) + \cdots + w_j \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_N) \right\} \end{aligned}$$

So far, so good. Now this formula tells you that you have N summation terms, each of which is itself N summation terms, so you have a total of N^2 summation terms in the variance. Do what you just did again—write out the sum for j,

$$\mathcal{V}ar(\tilde{r}_P) = \sum_{j=1}^{N} \left\{ w_j \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_1) + w_j \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_2) + \cdots + w_j \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_N) \right\} =$$

$$\begin{array}{llllllll}
\text{our j = 1 term} \rightarrow & & \underline{w_1 \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_1)} & + & w_1 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) & + \cdots + & \boxed{w_1 \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_N)} \\
\text{our j = 2 term} \rightarrow & + & w_2 \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_1) & + & \underline{w_2 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_2)} & + \cdots + & w_2 \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_N) \\
\text{our j = 3 term} \rightarrow & + & w_3 \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_3, \tilde{r}_1) & + & w_3 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_3, \tilde{r}_2) & + \cdots + & w_3 \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_3, \tilde{r}_N) \\
 & + & \cdots & + & \cdots & + \cdots + & \cdots \\
\text{our j = } N \text{ term} \rightarrow & + & \boxed{w_N \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_N, \tilde{r}_1)} & + & w_N \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_N, \tilde{r}_2) & + \cdots + & \underline{w_N \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_N, \tilde{r}_N)}
\end{array} \tag{10.12}$$

This has everything written out, and no longer needs scary summation notation—here it is the length which makes the formula appear intimidating. Would you rather memorize the long form in Formula 10.12 or the short double summation sign Formula 10.11? They both mean exactly the same thing—the double summation is merely abbreviated notation. I find it easier to remember the short form—if need be, I can always expand it into the long form. Before you forget about the long form, though, is this formula really the same as the step-by-step procedure on Page 222? Look at the terms on the diagonal in Formula 10.12, which are underlined. These are covariances of variables with themselves—which are just the variances multiplied by weights squared. Now look at the off-diagonal terms—each term appears twice, because both multiplication and covariances don't care about order. The big-mess Formula 10.12 can also be expressed as

The double summation formula is just an easy-to-remember abbreviation, and is precise. It says exactly what you already knew: variance and pairwise covariance terms.

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_P) &= [\text{sum up N diagonal variance terms}] \\ &+ \left[\text{sum up remaining } N^2 - N \text{ covariance terms}\right] \\ &= \left[\text{for each security i, sum up each } w_i^2 \text{ times the i-th variance}\right] \\ &+ \left[\text{for each possible pair i and j, sum up twice } w_i \text{ times } w_j \text{ times the i versus j covariance}\right] \end{aligned} \tag{10.13}$$

which is exactly what was stated on Page 222: to compute an overall portfolio variance, sum up all the constituent variances ($\mathcal{V}ar(\tilde{r}_i)$), each multiplied by its squared weight (w_i^2); and then add each pairwise covariance ($\mathcal{C}ov(\tilde{r}_i, \tilde{r}_j)$), multiplied by two times its two weights ($2 \cdot w_i \cdot w_j$).

IMPORTANT: The mean and variance formulas for portfolios deserve memorizing if you want to concentrate in investments:

$$\mathcal{E}(\tilde{r}_P) = \sum_{i=1}^{N} w_i \cdot \mathcal{E}(\tilde{r}_i)$$

$$\mathcal{V}ar(\tilde{r}_P) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \cdot w_j \cdot \mathcal{C}ov(\tilde{r}_i, \tilde{r}_j)$$

If you prefer sigma notation, this is even shorter: $\sigma_{P,P} = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \cdot w_j \cdot \sigma_{i,j}$.

Application of the formula makes it clear that it is the same thing.

Before you forget about double summations, let us just confirm that the formula gives us the same variance for our portfolio Q:

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_Q) &= \sum_{i \in \{\text{S\&P 500, IBM, Sony}\}} \left\{ \sum_{j \in \{\text{S\&P 500, IBM, Sony}\}} [w_i \cdot w_j \cdot \mathcal{C}ov(\tilde{r}_i, \tilde{r}_j)] \right\} \\
&= \Big[w_{\text{S\&P 500}} \cdot w_{\text{S\&P 500}} \cdot \mathcal{C}ov(\tilde{r}_{\text{S\&P 500}}, \tilde{r}_{\text{S\&P 500}}) + w_{\text{IBM}} \cdot w_{\text{S\&P 500}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{S\&P 500}}) \\
&\qquad + w_{\text{Sony}} \cdot w_{\text{S\&P 500}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{S\&P 500}}) \Big] \\
&+ \Big[w_{\text{S\&P 500}} \cdot w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{S\&P 500}}, \tilde{r}_{\text{IBM}}) + w_{\text{IBM}} \cdot w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{IBM}}) \\
&\qquad + w_{\text{Sony}} \cdot w_{\text{IBM}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{IBM}}) \Big] \\
&+ \Big[w_{\text{S\&P 500}} \cdot w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{S\&P 500}}, \tilde{r}_{\text{Sony}}) + w_{\text{IBM}} \cdot w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{Sony}}) \\
&\qquad + w_{\text{Sony}} \cdot w_{\text{Sony}} \cdot \mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{Sony}}) \Big] \\
&= [70\% \cdot 70\% \cdot 0.0362 + 20\% \cdot 70\% \cdot 0.0330 + 10\% \cdot 70\% \cdot 0.0218] \\
&+ [70\% \cdot 20\% \cdot 0.0330 + 20\% \cdot 20\% \cdot 0.1503 + 10\% \cdot 20\% \cdot 0.0218] \\
&+ [70\% \cdot 10\% \cdot 0.0477 + 20\% \cdot 10\% \cdot 0.0218 + 10\% \cdot 10\% \cdot 0.8149] \\
&= 0.04869
\end{aligned} \tag{10.14}$$

and you have the answer you already knew!

10·2.D. Another Variance Example: PepsiCo, CocaCola, and Cadbury

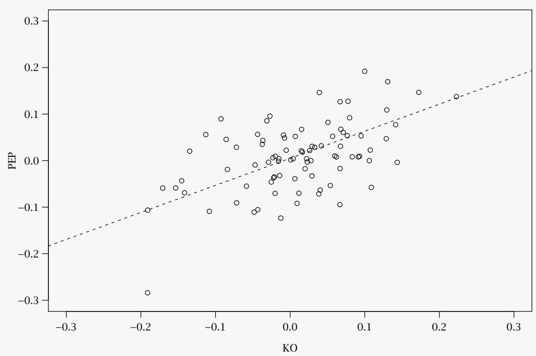
The table provides only statistics for the rates of return, not the series themselves.

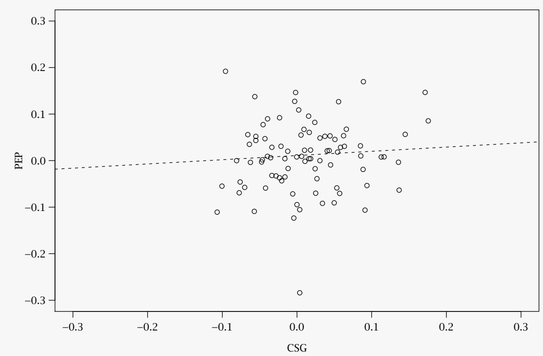
Let us now use portfolio formulas on a second example with monthly data, based on the historical means, standard deviations, and correlations of Coca Cola, PepsiCo, and Cadbury Schweppes, over the 1995-2002 period. This example is deliberately reminiscent of the example from the previous section, but we are now given only the correlations, not the detailed historical return series themselves. Table 10.3 shows that the average return of each of these three stocks was about 1% per month, or 10%–12% per year. The monthly means were significantly lower than the monthly standard deviations. Over the sample period, Cadbury Schweppes had higher performance and lower risk than either PepsiCo or Coca Cola. Note also how high the 53% correlation between PepsiCo and Coca Cola is, especially relative to the 10.8% and 9.9% Cadbury Schweppes correlations. Figure 10.1 plots the data points. Coca Cola stock seems to behave more like PepsiCo stock than like Cadbury Schweppes stock.

Table 10.3: PepsiCo, Coca Cola, and Cadbury Schweppes Monthly Return Statistics, from September 1995 to August 2002.

	Investment	Means	Standard Deviations	Correlations PEP	KO	CSG
1	PEP	0.83%	7.47%	100.0%		
2	KO	0.90%	8.35%	53.2%	100.0%	
3	CSG	1.19%	6.29%	10.8%	9.9%	100.0%

Figure 10.1: 1,765 Daily Stock Returns of PepsiCo versus Coca Cola, and PepsiCo versus Cadbury Schweppes from August 1995 to August 2002





As always, assume that historical means, standard deviations, and correlations are indicative of future means, standard deviations, and correlations. Now determine the risk and reward of a portfolio B defined by

A real-world example that computes the risk and reward of a portfolio of three stocks.

$$B \equiv (20\% \text{ in PEP, } 30\% \text{ in KO, and } 50\% \text{ in CSG})$$

The unknown rate of return on this portfolio is

$$\tilde{r}_B = 20\% \cdot \tilde{r}_{PEP} + 30\% \cdot \tilde{r}_{KO} + 50\% \cdot \tilde{r}_{CSG}$$

The reward is easy. It is

$$\mathcal{E}(\tilde{r}_B) = 20\% \cdot 0.83\% + 30\% \cdot 0.90\% + 50\% \cdot 1.19\% = 1.03\%$$
$$\mathcal{E}(\tilde{r}_B) = 20\% \cdot \mathcal{E}(\tilde{r}_{PEP}) + 30\% \cdot \mathcal{E}(\tilde{r}_{KO}) + 50\% \cdot \mathcal{E}(\tilde{r}_{CSG})$$

The portfolio risk is more difficult. As inputs, you need covariances, not standard deviations or correlations. The covariances of variables with themselves (i.e., the variances) of the three stocks can be computed from the standard deviations, by squaring:

$$\sigma_{\tilde{r}_{PEP},\tilde{r}_{PEP}} = (0.0747)^2 = 0.005585$$
$$\sigma_{\tilde{r}_{KO},\tilde{r}_{KO}} = (0.0835)^2 = 0.006967$$
$$\sigma_{\tilde{r}_{CSG},\tilde{r}_{CSG}} = (0.0629)^2 = 0.003956$$

The covariances have to be computed from the correlations. Recall Formula 9.1,

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \cdot \sigma_2} \iff \sigma_{1,2} = \rho_{1,2} \cdot \sigma_1 \cdot \sigma_2$$

With this formula, you can compute the covariances,

$$\begin{aligned}
\sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{KO}}} &= \rho_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{KO}}} \cdot \sigma_{\tilde{r}_{\text{PEP}}} \cdot \sigma_{\tilde{r}_{\text{KO}}} = 0.532 \cdot 0.0747 \cdot 0.0835 = 0.003318\\
\sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{CSG}}} &= \rho_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{CSG}}} \cdot \sigma_{\tilde{r}_{\text{PEP}}} \cdot \sigma_{\tilde{r}_{\text{CSG}}} = 0.108 \cdot 0.0747 \cdot 0.0629 = 0.000507\\
\sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{CSG}}} &= \rho_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{CSG}}} \cdot \sigma_{\tilde{r}_{\text{KO}}} \cdot \sigma_{\tilde{r}_{\text{CSG}}} = 0.099 \cdot 0.0835 \cdot 0.0629 = 0.000522
\end{aligned}$$

You now have all inputs that you need to compute the portfolio return variance:

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_B) &= \mathcal{V}ar\left(20\% \cdot \tilde{r}_{\text{PEP}} + 30\% \cdot \tilde{r}_{\text{KO}} + 50\% \cdot \tilde{r}_{\text{CSG}}\right)\\
&= (20\%)^2 \cdot \sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{PEP}}} + (30\%)^2 \cdot \sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{KO}}} + (50\%)^2 \cdot \sigma_{\tilde{r}_{\text{CSG}},\tilde{r}_{\text{CSG}}}\\
&\quad + 2 \cdot 20\% \cdot 30\% \cdot \sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{KO}}} + 2 \cdot 30\% \cdot 50\% \cdot \sigma_{\tilde{r}_{\text{PEP}},\tilde{r}_{\text{CSG}}} + 2 \cdot 20\% \cdot 50\% \cdot \sigma_{\tilde{r}_{\text{KO}},\tilde{r}_{\text{CSG}}}\\
&= (20\%)^2 \cdot 0.005585 + (30\%)^2 \cdot 0.006967 + (50\%)^2 \cdot 0.003956\\
&\quad + 2 \cdot 20\% \cdot 30\% \cdot 0.003318 + 2 \cdot 30\% \cdot 50\% \cdot 0.000507 + 2 \cdot 20\% \cdot 50\% \cdot 0.000522\\
&= 0.002496
\end{aligned}$$

Therefore, the risk of portfolio B is

$$\mathcal{S}dv(\tilde{r}_B) = \sqrt{\mathcal{V}ar(\tilde{r}_B)} = \sqrt{0.002496} = 5.00\%$$

Note that this standard deviation is lower than the standard deviation of each of the three stocks by themselves (Table 10.3). This is caused by "diversification," which is explored in great detail in the next chapter.

Solve Now!

Q 10.9 Compute the standard deviation (risk) of portfolio B from Table 10.2 to 4 digits. Is it the same as what we computed in the text?

Q 10.10 Compute the expected rate of return (reward) and standard deviation (risk) for another portfolio, called BF, that invests 10.7% in the S&P 500, 64.5% in IBM and 24.8% in Sony. Compute this both from the formulas and from a historical rate of return series of this portfolio.

Q 10.11 Compute the risk and reward for portfolio CF:

$$w_{\text{S\&P 500}} = 0.5, \quad w_{\text{IBM}} = 0.5, \quad w_{\text{Sony}} = 0$$

Q 10.12 Compute the risk and reward for portfolio DF:

$$w_{\text{S\&P 500}} = 0.614, \quad w_{\text{IBM}} = 0.288, \quad w_{\text{Sony}} = 0.98$$

Q 10.13 What is the risk and reward of a portfolio EF that invests 10% in PEP, 10% in KO, and 80% in CSG?

Q 10.14 What is the slope (beta) of the lines in Figure 10.1?

10·3 Historical Statistics For Some Asset-Class Index Portfolios

This section describes the 1997-2002 investment performance of some Vanguard funds.

Enough with statistical torture! Let's look at the historical performances of some realistic investment portfolios. Table 10.4 describes historical rates of return from portfolios managed by **Vanguard** over the January 1997 to October 2002 period. (**Vanguard** is a prominent low-cost provider of **index funds**.) Each Vanguard fund purchases a large number of securities, often simply everything that qualifies in an asset category (e.g., all suitable bonds, all European stocks, all real estate investment trusts, etc.), and without much attempt at picking winners within each class. Naturally, although you really are interested in forward-looking statistics, standing here today, all you have are historical statistics. So, look at the properties of historical rates of return of these portfolios.

SIDE NOTE

Experience shows that historical means are not good predictors of future means, but historical standard deviations and betas are good predictors of their future equivalents. See the local Nerd Appendix a.

Bonds offered only low rates of return.

The first statistic that this chapter described was the mean. The **historical mean** (also called **sample mean**) of the monthly rates of return describes how you would have fared on average. The second column in Table 10.4 shows that over the sample period, the short-term government bond fund earned a rate of return of about 60 basis points (*per year*). The intermediate government bond fund earned 100 basis points; municipal bonds earned 80 basis points; corporate junk bonds earned 180 basis points; corporate convertible bonds earned about 2.4%; and so on.

Equities offered higher rates of return.

Continuing on to pure equity (stock) investments, you can see that the 500 large stocks in Vanguard's S&P 500 fund earned about 4.9% per year. The tax-managed version of the same investment strategy minimizes trading (to minimize capital gains). It did even better than the unmanaged version, earning a rate of return of 6.0% per year. **Value firms** are large and unexciting companies, and **growth firms** are small, fast-growing, exciting companies. Yet, following a long-standing historical trend, value firms earned higher rates of return than growth firms. (During this particular sample period, small growth firms did best, though.) Neither could outperform the S&P 500 over the sample period. Among industries, **health care firms** earned the highest rates of return, and **utilities** firms earned the lowest rates of return. International investors fared especially poorly in this sample period: Japanese stocks in particular lost 7.4% *per annum* over the sample period.

SIDE NOTE

The Vanguard S&P500 index portfolio outperformed the S&P500 index itself (second to last line in the table). This is presumably [a] because the fund also received dividends, which are not counted in the S&P index itself; and [b] because the fund could lend out securities to short-sellers and thereby earn some extra return. Incidentally, to reduce transaction costs, index funds usually do not hold the 500 stocks in the exact right proportion, which causes further **tracking error**—a deviation of the performance of the index fund's rate of return from the index percent change.

Equities were riskier than bonds.

The second set of statistics that this chapter described included the standard deviation. The third column in Table 10.4 shows that short-term and medium-term government bonds and municipal bonds were exceptionally safe. Their monthly rates of return varied only a little over the sample period. A large cluster of investment strategies had risks of about 15% to 25% per year, including the overall S&P 500 stock market index investment strategy. The riskiest stock market investment strategy in the sample period would have been U.S. Gold and Metals, whereas the safest would have been **Real Estate Investment Trust** (**REITs**) and **Utilities** firms.

An important statistic, used later again, is the covariation of investments with the stock market.

The third set of statistics that this chapter described was covariation, which included beta. The last column in the Table 10.4 shows the beta of the rate of return of each investment portfolio with the rate of return on the S&P 500 index,

$$\beta_{\text{B, S\&P 500}} = \frac{\mathcal{C}ov(\,\tilde{r}_{\text{B}}\,,\,\tilde{r}_{\text{S\&P 500}}\,)}{\mathcal{V}ar(\tilde{r}_{\text{S\&P 500}})}$$

These "market-betas" tell us how much a particular investment portfolio's rate of return covaried with the rate of return on the S&P 500. A beta of 1 tells us that the rate of return of a portfolio tended to covary one-to-one with the rate of return in the U.S. stock market. A beta of 0 tells us

Table 10.4: Asset Class Portfolios, Based on Monthly Rates of Returns, Annualized, January 1997 to October 2002.

	Annualized Returns				Monthly Returns					Ann.Market-Model	
Asset	Mean	StdDev	%Neg	%Pos	Worst	Q2	Median	Q3	Best	α_i	$\beta_{i,S\&P}$
govbonds: short-term	0.6%	2.0%	43%	51%	−1.5%	−0.3%	0.1%	0.4%	1.2%	0.7%	−0.0
govbonds: intermediate	1.0%	4.4%	44%	53%	−3.2%	−0.7%	0.2%	0.8%	2.9%	1.2%	-0.0
bond: long-term munis	0.8%	4.5%	43%	54%	−3.1%	−1.1%	0.4%	0.9%	2.7%	1.0%	-0.0
bond: high-yield (junk) bonds	1.8%	16.2%	46%	54%	−15.0%	−2.3%	0.4%	2.9%	9.9%	−1.5%	0.7
bond: corporate convertibles	2.4%	15.4%	50%	49%	−12.8%	−2.7%	−0.2%	3.2%	10.6%	−0.5%	0.6
u.s. s&p500	4.9%	18.5%	46%	54%	−14.5%	−3.0%	0.7%	4.7%	9.5%	0.1%	1.0
u.s. tax managed	6.0%	21.3%	46%	54%	−17.6%	−3.4%	0.4%	5.2%	9.6%	0.6%	1.1
u.s. value firms	1.6%	18.1%	46%	53%	−16.1%	−3.2%	0.3%	3.8%	8.6%	−2.7%	0.9
u.s. growth firm	−2.1%	25.7%	49%	51%	−21.7%	−5.2%	0.8%	5.6%	15.2%	−8.2%	1.3
u.s. small growth firms	7.7%	25.5%	46%	54%	−19.3%	−4.6%	0.3%	6.3%	20.1%	3.1%	1.0
u.s. small-cap firms	2.7%	22.6%	49%	51%	−19.3%	−4.3%	0.3%	5.2%	16.5%	−1.3%	0.8
u.s. energy firms	5.2%	23.7%	53%	47%	−18.0%	−4.1%	−0.5%	2.8%	19.0%	2.3%	0.6
u.s. gold and metals	3.6%	35.1%	51%	47%	−19.9%	−6.1%	−0.8%	6.0%	39.3%	1.1%	0.5
u.s. health care firms	13.6%	14.0%	36%	64%	−10.5%	−1.0%	1.8%	3.3%	10.3%	11.3%	0.5
u.s. real estate inv. trusts	−0.4%	12.6%	50%	49%	−9.2%	−2.2%	−0.2%	2.0%	9.6%	−1.2%	0.2
u.s. utilities firms	−2.3%	13.9%	57%	43%	−11.9%	−2.5%	−0.7%	2.4%	7.7%	−4.2%	0.4
intl emerging market firms	−4.4%	28.0%	50%	50%	−26.1%	−6.2%	−0.4%	4.5%	15.4%	−9.7%	1.1
intl european firms	1.9%	17.6%	41%	56%	−13.1%	−3.4%	0.3%	3.2%	9.7%	−1.7%	0.7
intl pacific firms	−7.4%	21.3%	59%	40%	−12.1%	−5.3%	−2.0%	4.0%	17.6%	−10.6%	0.7
intl growth firms	−1.6%	17.5%	46%	53%	−13.2%	−3.2%	0.5%	3.1%	9.6%	−5.1%	0.7
intl value firms	−4.2%	18.7%	57%	43%	−15.0%	−3.1%	−0.7%	2.9%	13.0%	−7.5%	0.7
s&p 500 index	4.8%	18.5%	46%	54%	−14.6%	−3.1%	0.5%	5.0%	9.7%	0.0%	1.0
dow jones 30 index	6.3%	18.7%	43%	57%	−15.1%	−3.6%	0.9%	4.0%	10.6%	1.8%	0.9

that the rate of return of a portfolio tended to be unrelated to what happened to the U.S. stock market.

SIDE NOTE

The linear regression by which the beta measure can be obtained is so common that it is called the **market model**, and this particular beta is called the **market beta**. It can be obtained by running the time-series regression

$$\tilde{r}_{B} = \alpha + \beta \cdot \tilde{r}_{S\&P\ 500}$$

Again, we use a historical beta as estimate for the future beta.

Equities covaried more with the s&p than bonds.

The last column shows that government bonds had practically no covariation with the S&P 500. Corporate bonds, energy stocks, precious metals, health care stocks, real estate investment trusts (REITs) and utilities had very mild covariation, indicated by betas around 0.5. The next-most correlated segment are international stocks, having betas of around 0.7. But many other portfolios varied about 1-to-1 with the overall stock market. Note that U.S. growth firms swung even more than 1-to-1 with changes in the stock market: The beta of 1.3 tells us that a 10% increase in the stock market tended to be associated with a 13% increase in the growth firm portfolio. This is typical.

Can you read and understand tables of historical performances now?

For your curiosity, there are two more tables with the same statistics: Table a describes the historical performance of non-U.S. stock markets; and Table b describes the historical performance of the 30 stocks that constitute the Dow-Jones 30 Index. At this point, you can read and interpret the table as well as I can, so enjoy!

SIDE NOTE

Table 10.4 also shows some other statistics, such as the percent of all months that earned a positive rate of return. (Naturally, one minus this percent are the months in which the portfolio had a negative return.) The Table further shows the single worst month, the single best month, the median month (half of all return months were better, half were worse). Finally, although this is beyond what we have covered so far, the alpha (α) in the Table is sometimes interpreted as a risk-adjusted reward measure—the higher the better.

Some other random observations: The table shows no systematic relationship between risk and rate of return over the sample period. However, it is the case that the least risky and least covarying investment strategies (government bonds) provided a very modest, but positive average rate of return. With hindsight, it would have been terrific to invest in U.S. health care stocks: they had the most spectacular return, plus a very modest risk. Naturally, with hindsight, you could have selected the right six numbers for the lottery. So, which numbers can you trust to be indicative of their future equivalent? First and foremost, covariation measures. They tend to be very stable. Next, standard deviations are reasonably stable. Historical means, however, are very untrustworthy as predictors of the future: it is not especially likely that health care firms will continue to outperform other stocks, and that Japanese firms will continue to underperform other stocks.

Solve Now!

All questions refer to Table 10.4.

Q 10.15 Which investment class portfolio would have done best over the sample period? Do you believe this will continue?

Q 10.16 Which investment class portfolio would have done worst over the sample period? Do you believe this will continue?

Q 10.17 Assuming you had held only one asset class, which investment class portfolio was safest during the sample period? Do you believe this will continue?

Q 10.18 Assuming you had held only one asset class, which investment class portfolio was riskiest during the sample period? Do you believe this will continue?

Q 10.19 Which asset class portfolio had the lowest covariation with the S&P 500 index? Do you believe this will continue?

Q 10.20 Which asset class portfolio had the highest covariation with the S&P 500 index? Do you believe this will continue?

10·4 Summary

The chapter covered the following major points:

- The formulas in this chapter decompose the statistics of a portfolio return in terms of the statistics of its constituent securities' portfolio returns.

  The formulas are merely alternative computations. You can instead write out the time-series of the portfolio's rates of return and compute the portfolio statistics directly from this distribution.

  You shall use these formulas later, because you want to consider portfolios when you vary the weights. The formulas express the overall portfolio statistics in terms of investment weights, which will make it easier to choose the best portfolio.
- For three statistics, you can take investment-weighted averages:
  1. The portfolio expected rate of return is the investment-weighted average of its components' expected rates of return.
  2. The portfolio covariance with anything else is the investment-weighted average of its components' covariance with this anything else.
  3. The portfolio beta with respect to anything else is the investment-weighted average of its components' beta with respect to this anything else.
- The portfolio variance can not be computed as the investment-weighted average of its components' variances. Instead, it is computed as follows:
  1. For each security, square its weights and multiply it by the variance.
  2. For each pair of different securities, multiply two times the first weight times the second weight times the securities' covariance.
  3. Add up all these terms.

  There are other ways to compute the variance. In particular, you can instead compute the historical portfolio rate of return for each time period, and then compute the variance from this univariate time-series. Or you can use the double summation Formula 10.11.

↦ *Table 10.4 on Page 228*

- For a sense of order-of-magnitude, Table 10.4 provides recent return statistics for some common asset-class portfolios. The appendix gives equivalent statistics for the Dow-Jones 30 stocks and for foreign stock markets.

Appendix

Table 10.5: Summary of Portfolio Algebra in the Context of the Chapter Example

Statistic	Notation	Three Input Securities: S&P 500	IBM	Sony	Investment-Weighted Average	Formula	Portfolio of 70% IBM, 20% IBM, and 10% Sony
Expected Return	$\mathcal{E}(\tilde{r}_i)$	19.0%	38.8%	90.3%	Yes	$\sum_{i=1}^{N} w_i \cdot \mathcal{E}(\tilde{r}_i)$	$70\% \cdot 19.0\% + 20\% \cdot 38.8\% + 10\% \cdot 90.3\% = 40.9\%$
Covariance, e.g. with x=S&P 500	$\sigma_{i,\text{S\&P 500}}$	0.0362	0.0330	0.0477	Yes	$\sum_{i=1}^{N} w_i \cdot \mathcal{C}ov(\tilde{r}_i, \tilde{r}_x)$	$70\% \cdot 0.0362 + 20\% \cdot 0.0330 + 10\% \cdot 0.0477 = 0.03672$
Beta, e.g. with x= S&P 500	$\beta_{i,\text{S\&P 500}}$	1.00	0.91	1.32	Yes	$\sum_{i=1}^{N} w_i \cdot \beta_{(\tilde{r}_i, \tilde{r}_x)}$	$70\% \cdot 1.00 + 20\% \cdot 0.91 + 10\% \cdot 1.32 = 1.01$
Variance	$\sigma_{i,i} = \sigma_i^2$	0.036	0.150	0.815	**No**	$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i \cdot w_j \cdot \sigma_{i,j}$	Requires three variance terms and the three mutual covariance terms. The latter are not provided in this table. For method, see Page 222, and Formulas 10.10 and 10.14. It is 0.0487 here.
Standard Deviation	σ_i	19%	39%	90%	**No**		Squareroot of variance (which is $\sqrt{0.0487} = 41\%$ here).

The goal of this chapter was to explain these portfolio rules.
Know what they mean and how to use them!

A More Historical Statistics

a. Country Fund Rates of Return

Table 10.6: AMEX Country Funds, Based on Monthly Rates of Returns, Annualized, January 1997 to October 2002.

	Annualized Returns				Monthly Returns					Ann.Market-Model	
Asset	Mean	StdDev	%Neg	%Pos	Worst	Q2	Median	Q3	Best	α_i	$\beta_{i,S\&P}$
australia index fund amex	−0.4%	23.1%	47%	49%	−17.5%	−4.2%	0.0%	3.4%	14.5%	−4.1%	0.8
canada index fund amex	5.0%	23.5%	40%	57%	−22.4%	−4.0%	0.7%	5.5%	11.2%	−0.2%	1.1
sweden index fund amex	1.2%	30.3%	46%	53%	−21.2%	−4.9%	0.6%	5.3%	21.7%	−4.4%	1.2
germany index fund amex	3.9%	28.6%	46%	53%	−24.0%	−4.6%	0.9%	5.1%	24.4%	−1.2%	1.1
hong kong index fund amex	−5.7%	36.0%	56%	41%	−28.1%	−7.4%	−1.3%	5.7%	38.4%	−11.7%	1.2
japan index fund amex	−8.0%	24.2%	59%	41%	−16.5%	−5.9%	−1.2%	3.7%	21.1%	−11.4%	0.7
belgium index fund amex	5.9%	26.6%	44%	56%	−19.0%	−2.7%	0.4%	4.0%	37.9%	2.3%	0.8
netherlands index fund amex	0.2%	22.2%	46%	53%	−17.3%	−3.3%	0.3%	3.6%	14.4%	−3.9%	0.9
austria index fund amex	1.2%	21.0%	47%	49%	−19.7%	−4.2%	0.0%	4.4%	11.9%	−1.0%	0.5
spain index fund amex	6.5%	25.0%	44%	53%	−22.6%	−4.1%	0.3%	4.8%	15.5%	2.0%	0.9
france index fund amex	6.2%	23.5%	43%	57%	−15.0%	−2.5%	0.8%	3.9%	16.8%	2.0%	0.9
singapore index fund amex	−7.5%	37.7%	49%	46%	−27.0%	−6.5%	0.0%	3.6%	40.3%	−13.8%	1.3
uk index fund amex	0.2%	15.4%	43%	54%	−11.9%	−3.2%	0.6%	3.1%	8.6%	−3.1%	0.7
mexico index fund amex	11.3%	36.1%	44%	53%	−35.3%	−7.0%	3.0%	9.0%	20.6%	4.7%	1.4
s&p 500 index	4.8%	18.5%	46%	54%	−14.6%	−3.1%	0.5%	5.0%	9.7%	0.0%	1.0

b. Dow-Jones Constituents

Table 10.7: Dow Jones Constituents, Based on Monthly Rates of Returns, Annualized, January 1997 to October 2002.

	Annualized Returns				Monthly Returns					Ann.Market-Model	
Asset	Mean	StdDev	%Neg	%Pos	Worst	Q2	Median	Q3	Best	α_i	$\beta_{i,S\&P}$
alcoa	15.1%	41.3%	49%	51%	−23.9%	−6.2%	0.4%	8.2%	51.1%	8.8%	1.3
american express	17.3%	31.4%	36%	63%	−29.3%	−3.0%	1.7%	8.4%	16.9%	11.3%	1.2
boeing	−2.9%	34.6%	46%	53%	−34.6%	−7.1%	0.6%	6.6%	20.2%	−6.0%	0.7
citigroup	24.9%	37.4%	43%	57%	−34.0%	−5.8%	2.4%	9.0%	25.8%	17.6%	1.5
caterpillar	9.6%	34.0%	46%	54%	−17.4%	−5.9%	0.8%	6.7%	40.8%	5.7%	0.8
du pont	4.4%	29.0%	50%	50%	−17.0%	−6.1%	−0.2%	6.5%	21.7%	0.6%	0.8
disney	0.3%	32.2%	46%	54%	−26.8%	−6.4%	0.4%	5.0%	24.2%	−4.0%	0.9
eastman kodak	−5.5%	34.6%	53%	47%	−34.4%	−6.6%	−0.5%	5.5%	24.2%	−8.4%	0.6
general electric	12.8%	28.4%	50%	49%	−17.7%	−4.1%	−0.2%	5.9%	19.2%	7.6%	1.1
general motors	5.1%	36.7%	50%	49%	−24.1%	−5.0%	−0.5%	5.8%	25.4%	0.1%	1.0
home depot	22.8%	35.1%	41%	59%	−20.6%	−4.1%	2.8%	8.8%	30.2%	16.8%	1.2
honeywell	6.0%	44.5%	44%	56%	−38.4%	−4.1%	0.7%	6.5%	51.0%	0.1%	1.2
hewlett packard	4.0%	48.2%	51%	49%	−32.0%	−9.1%	−1.4%	8.7%	35.4%	−3.1%	1.5
ibm	20.2%	39.6%	50%	50%	−22.6%	−5.8%	−0.6%	7.8%	35.4%	13.2%	1.5
intel	15.8%	53.1%	46%	54%	−44.5%	−9.6%	0.8%	11.4%	33.9%	7.5%	1.7
international paper	6.0%	36.0%	47%	53%	−22.3%	−6.3%	0.5%	5.3%	27.0%	1.6%	0.9
johnson and johnson	19.5%	26.4%	43%	57%	−16.0%	−2.7%	1.5%	6.2%	17.4%	17.0%	0.5
jp morgan	5.9%	42.5%	46%	54%	−30.6%	−6.3%	0.7%	8.2%	32.9%	−1.9%	1.6
coca-cola	3.7%	30.7%	49%	51%	−19.1%	−4.7%	0.1%	6.7%	22.3%	0.9%	0.6
mcdonalds	0.6%	28.8%	44%	56%	−25.7%	−5.7%	1.0%	5.9%	17.7%	−3.1%	0.8
3m	12.8%	26.0%	44%	56%	−15.8%	−3.9%	1.1%	6.0%	25.8%	10.4%	0.5
philip morris	6.9%	33.0%	41%	57%	−23.7%	−4.6%	0.8%	5.3%	24.4%	5.1%	0.4
merck	12.1%	32.0%	49%	51%	−21.7%	−5.6%	0.1%	6.5%	22.8%	9.3%	0.6
microsoft	28.4%	50.0%	47%	51%	−34.4%	−8.7%	0.1%	11.9%	40.8%	20.0%	1.7
proctor and gamble	13.7%	28.4%	41%	59%	−35.4%	−1.8%	1.4%	5.3%	24.7%	12.6%	0.2
sbc communications	8.1%	34.4%	50%	50%	−18.8%	−7.0%	−0.2%	5.7%	29.3%	4.5%	0.8
att	−5.7%	40.3%	54%	46%	−23.8%	−8.5%	−2.6%	6.1%	39.1%	−10.3%	1.0
united technologies	18.1%	34.1%	40%	60%	−32.0%	−3.4%	1.9%	7.0%	24.6%	12.6%	1.1
wal-mart	31.5%	31.1%	33%	66%	−20.8%	−3.2%	3.0%	8.2%	26.4%	27.2%	0.9
exxon	7.1%	18.3%	50%	50%	−10.2%	−3.5%	−0.1%	4.7%	17.7%	4.9%	0.5
dow jones 30 index	6.3%	18.7%	43%	57%	−15.1%	−3.6%	0.9%	4.0%	10.6%	1.8%	0.9
s&p 500 index	4.8%	18.5%	46%	54%	−14.6%	−3.1%	0.5%	5.0%	9.7%	0.0%	1.0

[No keyterm list for statspfios-g.]

End of Chapter Problems

20 "Solve Now" Answers

1.

$$\mathcal{E}(\tilde{r}_P) = \frac{(-0.176) + (-0.290) + \cdots + (-0.265)}{12} = 0.183$$

$$\mathcal{V}ar(\tilde{r}_P) = \frac{(-0.359)^2 + (-0.474)^2 + \cdots + (-0.448)^2}{11} = 0.167$$

$$\mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_P) = \frac{(+0.1620)\cdot(-0.359) + (-0.0565)\cdot(-0.474) + \cdots + (-0.3348)\cdot(-0.448)}{11} = 0.0379$$

$$\mathcal{C}ov(\tilde{r}_{IBM}, \tilde{r}_P) = \frac{(-0.3661)\cdot(-0.359)\cdot + (-0.5874)\cdot(-0.474) + \cdots + (-0.5105)\cdot(-0.448)}{11} = 0.1075$$

$$\mathcal{C}ov(\tilde{r}_{Sony}, \tilde{r}_P) = \frac{(-0.3448)\cdot(-0.359)\cdot + (-0.2458)\cdot(-0.474) + \cdots + (-0.3228)\cdot(-0.448)}{11} = 0.2862$$

$$\mathcal{S}dv(\tilde{r}_P) = \sqrt{\mathcal{V}ar(\tilde{r}_P)} = \sqrt{16.71\%} = 40.9\%$$

2. There is no error.

3.

$$\mathcal{C}ov(\tilde{r}_{IBM}, \tilde{r}_S) = 0.05397, \quad \mathcal{C}ov(\tilde{r}_{Sony}, \tilde{r}_S) = 0.6166, \quad \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_S) = 0.04403$$

4.

$$\beta_{S,S\&P\,500} = \mathcal{C}ov(\tilde{r}_S, \tilde{r}_{S\&P500})/\mathcal{V}ar(\tilde{r}_{S\&P500}) = 0.04403/0.03622 \approx 1.22$$

$$\beta_{S,S\&P\,500} = w_{IBM}\cdot\beta_{IBM,S\&P\,500} + w_{Sony}\cdot\beta_{Sony,S\&P\,500}$$

$$= 25\%\cdot 0.910 + 75\%\cdot 1.317 \approx 1.22$$

5.

$$\mathcal{V}ar(\tilde{r}_P) = \mathcal{C}ov(\tilde{r}_S, \tilde{r}_S) = \mathcal{V}ar(\tilde{r}_P) = (25\%)^2\cdot 0.15035 + (75\%)^2\cdot 0.81489$$

$$+2\cdot 25\%\cdot 75\%\cdot 0.02184 = 0.4760$$

6. Do it!

$$\mathcal{E}(\tilde{r}_Q) = \frac{0.0131 + \cdots + (-0.243)}{12} \approx 12.57\%$$

$$\mathcal{C}ov(\tilde{r}_Q, \tilde{r}_{S\&P500}) = \frac{(-0.006\cdot 0.1620) + \cdots + (-0.369\cdot -0.3348)}{12} \approx ??.$$

7.

$$\begin{aligned}
\mathcal{E}(\tilde{r}_S) &= 25\% \cdot 10.1\% + 35\% \cdot 15.4\% + 40\% \cdot 24.2 \approx 17.59\% \\
\mathcal{C}ov(\tilde{r}_S, \text{S\&P 500}) &= 25\% \cdot 0.03622 + 35\% \cdot 0.03298 + 40\% \cdot 0.04772 \approx 0.0397 \\
\beta_{S,\text{S\&P 500}} &= 25\% \cdot 1.00 + 35\% \cdot 0.9104 + 40\% \cdot 1.3172 \approx 1.096
\end{aligned}$$

8. For the historical computation, compute the returns and their deviations:

Year	1991	1992	1993	1994	1995	1996
Historical Return	−0.04965	−0.14210	+0.25130	+0.15549	+0.21218	+0.31198
Deviation from Mean	−0.22556	−0.31801	+0.07539	−0.02043	+0.03626	+0.13607

Year	1997	1998	1999	2000	2001	2002
Historical Return	+0.36708	+0.25242	+1.29560	−0.30389	−0.02390	−0.21557
Deviation from Mean	+0.19117	+0.07651	+1.11969	−0.47980	−0.19981	−0.39148

because the portfolio mean is 17.59%. The variance is therefore

$$\mathcal{V}ar(\tilde{r}_S) = \frac{0.0509 + 0.1011 + \cdots + 0.1533}{11} \approx 0.1725$$

The alternative calculation is

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_S) &= 0.25^2 \cdot 0.03622 + 0.35^2 \cdot 0.15035 + 0.40^2 \cdot 0.81849 \\
&\quad +2 \cdot 0.25 \cdot 0.35 \cdot 0.03298 \\
&\quad +2 \cdot 0.25 \cdot 0.40 \cdot 0.04772 \\
&\quad +2 \cdot 0.35 \cdot 0.45 \cdot 0.02184 \\
&= 0.1516 + 0.02220 \\
&\approx 0.1738
\end{aligned}$$

and the difference is rounding error (of which half is in the Sony variance term).

9. The variance is 0.0487, the standard deviation is 0.2207. (Divide by $N-1$, not by N.)
10. $\mathcal{E}(\tilde{r}_{BF}) = 17.00\%$, $\mathcal{S}dv(\tilde{r}_{BF}) = 35.65\%$
11. $\mathcal{E}(\tilde{r}_{CF}) = 12.74\%$, $\mathcal{S}dv(\tilde{r}_{CF}) = 25.12\%$.
12. $\mathcal{E}(\tilde{r}_{DF}) = 13.00\%$, $\mathcal{S}dv(\tilde{r}_{DF}) = 22.90\%$.
13.

$$\mathcal{E}(\tilde{r}_{EF}) = 10\% \cdot 0.83\% + 10\% \cdot 0.90\% + 80\% \cdot 1.19\% = 1.125\%$$

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_{EF}) &= 10\% \cdot (7.47\%)^2 + 10\% \cdot (8.35\%)^2 + 80\% \cdot (6.29\%)^2 \\
&\quad +2\cdot 10\%\cdot 10\%\cdot 53.2\% + 2\cdot 10\%\cdot 80\%\cdot 10.8\% + 2\cdot 10\%\cdot 80\%\cdot 9.9\% \\
&= 0.0004858 \\
\rightarrow \mathcal{S}dv(\tilde{r}_{EF}) &= 6.97\%
\end{aligned}$$

14. We want the slope of a line where KO is the X variable. Therefore, the slope is

$$\begin{aligned}
\beta_{PEP,KO} &= \frac{\mathcal{C}ov(\tilde{r}_{PEP}, \tilde{r}_{KO})}{\mathcal{V}ar(\tilde{r}_{KO})} = \frac{0.003318}{0.006967} \approx 0.48 \\
\beta_{PEP,CSG} &= \frac{\mathcal{C}ov(\tilde{r}_{PEP}, \tilde{r}_{CSG})}{\mathcal{V}ar(\tilde{r}_{CSG})} = \frac{0.0005}{0.004} \approx 0.13
\end{aligned}$$

15. Health care firms. Unlikely: You know that historical expected rates of return are not reliable predictors of future expected rates of return.

16. Pacific (Japanese) Firms. Unlikely: You know that historical expected rates of return are not reliable predictors of future expected rates of return.
17. Government Bonds, short term. Likely: Historical standard deviations tend to be good predictors of future standard deviations.
18. Gold and Metals, then Emerging Stock Market Investments. (These are stock markets from developing countries.) Likely: Historical standard deviations tend to be good predictors of future standard deviations.
19. Look at the final column, $\beta_{i,S\&P\ 500}$. Government bonds of all kinds had almost no correlation with the S&P 500. Among more risky securities, REITs were almost uncorrelated, too. Likely: historical covariations tend to be good predictors of future covariations.
20. Growth firms. These contain many technology firms. Likely: historical covariations tend to be good predictors of future covariations.

All answers should be treated as suspect. They have only been sketched and have not been checked.

CHAPTER 11[§]

The Principle of Diversification

Eggs and Baskets

This chapter appears in the Survey text only.

HAVING the statistical artillery now in place to describe risk (i.e., the standard deviation), you are ready to abandon the previously maintained assumption of investor risk-neutrality. Henceforth, you will no longer be assumed to be indifferent among investments with the same expected rates of return. Instead, you can now prefer the less risky investment *if* two investment options have the same expected rate of return.

11·1 What Should You Care About?

What we have assumed about investor preferences?

For the remainder of this book, we are assuming that you care only about the risk and reward of our portfolios, and at one particular point in the future. (You may however reinvest your portfolio at this point to earn more returns.) You care about no other characteristics of your portfolio, or whether a bigger portfolio at this point in time might cause a lower portfolio at the next point in time. What does this mean, and how reasonable are these assumptions?

What non-financial preferences have you assumed away?

First, you are assuming that you do not care about anything other than financial returns. Instead, you could care about whether your portfolio companies invest ethically, e.g., whether a firm in your portfolio produces cigarettes or cancer cures. In real life, few investors care about what their portfolio firms are actually doing. Even if *you* care, you are too small to be able to influence companies one way or the other—and other investors stand ready to purchase any security you may spurn. Aside, if you purchase an ordinary mutual fund, you will hold all sorts of companies—companies whose behavior you may or may not like.

What have you assumed away by looking at the portfolio problem in isolation?

Second, you are assuming that external influences do not matter—you consider your portfolio's outcome by itself without regard for anything else. For example, this means that you do not seek out portfolios that offer higher rates of return if you were to lose your job. *This would not be a bad idea*—you probably should prefer a portfolio with a lower mean (given the same standard deviation), just as long as the better outcomes occur in recessions when you are more likely to lose your job. You should definitely consider such investment strategies—but unfortunately few investors do so in the real world. (If anything, the empirical evidence suggests that many investors seem to do the exact opposite of what they should do if they wanted to ensure themselves against employment risk.) Fortunately, our tools would still work with some modifications if you define your portfolio return to include your labor income.

What return preferences have you assumed away?

Third, you are assuming that risk and reward is all you care about. But it is conceivable that you might care about other portfolio characteristics. For example, the following two portfolios both have a mean return of 20% and a standard deviation of 20%.

Pfio "Symmetric"	with 50% probability, a return of 0%	with 50% probability, a return of +40%
Pfio "Skewed"	with 33% probability, a return of –8%	with 67% probability, a return of +34%

Are you really indifferent between the two? They are not the same. The symmetric portfolio cannot lose money, while the skewed portfolio can. On the other hand, the skewed portfolio has the better return more frequently. By focusing only on mean and standard deviation, you have assumed away any preference between these two portfolios. Few investors in the real world actively invest with an eye towards portfolio return skewness, so ignoring it is acceptable.

What multi-period return processes have you assumed away?

Fourth, you are assuming that you want to maximize your portfolio value at one specific point in time. This could be problematic if, for example, in the symmetric portfolio case it were true that a return of 0% was always later followed by a return of 100%, while a return of +40% was always followed by an unavoidable return of –100% (nuclear war!), then you might not care about the return at the end of the first measurement period. This is so unrealistic that you can ignore this issue for most practical purposes.

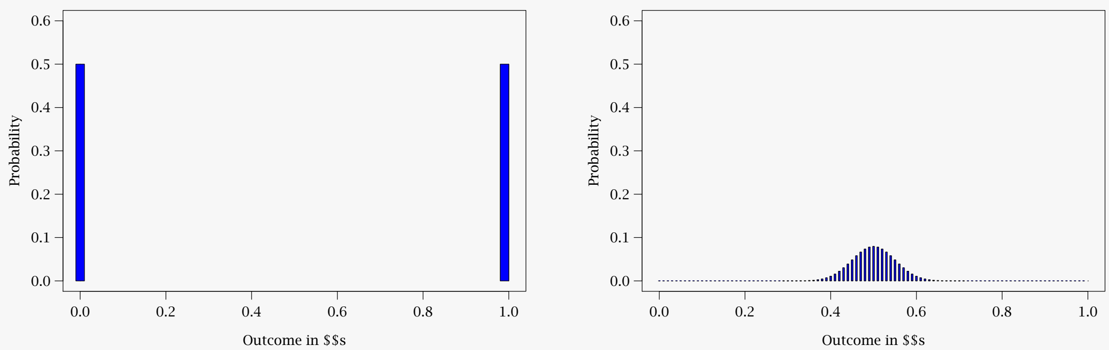
IMPORTANT: The remainder of this book assumes that you care only about the risk (standard deviation) and reward (expected return) of your portfolio.

11·2 Diversification: The Informal Way

Let us assume that you dislike wealth risk. An important method to reduce this risk—and the cornerstone of the area of investments—is **diversification**, which means investing not only in one but in many different assets. We shall expound on it in great detail, but it can be explained with a simple example. Compare two bets. The first bet depends on the outcome of a single coin toss. If heads, the bet pays off $1 (if tails, the bet pays off nothing). The second bet depends on the outcome of 100 coin tosses. Each coin toss, if heads, pays off $1/100$ of a dollar, zero otherwise. The expected outcome of either bet is 50 cents. But, as Figure 11.1 shows, the standard deviation of the payoff—the risk—of the latter bet is much lower.

Don't put all your eggs into one basket (on one bet), if you dislike variance!

Figure 11.1: Payoffs Under Two Bets With Equal Means.



The left figure is the distribution of payoff where you bet $1 on heads for 1 coin throw. The right figure is the distribution of payoff where you bet $0.01 on heads for each of 100 coin throws.

We are assuming that investors dislike risk. Putting money into many different bets, rather than one big bet, accomplishes this goal. But, is this a good strategy on the roulette table, too? Should you bet your entire money on red (one big bet only), or should you bet it one dollar at a time on red? From a purely financial perspective, the answer is that the single bet is better: if you bet one dollar at a time, you are indeed likely to have a lower variance of payoffs around your expected rate of return. Unfortunately, in roulette, your expected rate of return is *negative*. The casino would be perfectly happy to have you pay up your (negative) expected rate of return *without any risk* for each roll of the ball. Indeed, if your strategy is to gamble until you either are bankrupt or have doubled your money, you are more likely not to go bankrupt if you make fewer but bigger bets.

What about gambling?

Anecdote: Ancient Wisdom: How Cato (the Elder) diversified.

Plutarch, a famous Historian who lived in the first century AD, notes Cato's famous statement that, if people wished to obtain money for shipping business [from him], they should form a large association and when the association had fifty members and as many ships, he would take one share in the company.

11·3 Diversification: The Formal Way

11·3.A. Uncorrelated Securities

The base case: independent security returns.

↦ Unknown 8·3 on Page 176

Recall from Section 8·3 that a portfolio's return, r_P, is

$$r_P \equiv \sum_{i=1}^{N} w_i \cdot r_i$$

where P is the overall portfolio, w_i is the investment weight (proportion) in asset i, and i is a counter that enumerates all assets from 1 to N. If you do not yet know the return outcomes, then your returns are random variables, so

$$\tilde{r}_P \equiv \sum_{i=1}^{N} w_i \cdot \tilde{r}_i$$

It is now time to put the laws of expectations and standard deviations (from Section 10·2) to good use. To illustrate how diversification works, make the following admittedly unrealistic assumptions:

1. All securities offer the same expected rate of return (mean):

   $$\mathcal{E}(\tilde{r}_i) = 5\% \quad \text{for all i} \;\; ;$$

2. All securities have the same risk (standard deviation of return):

   $$\mathcal{S}dv(\tilde{r}_i) \equiv \sigma(\tilde{r}_i) \equiv \sigma_i = 40\% \quad \text{for all i}$$

   which is roughly the annual rate of return standard deviation for a typical U.S. stock;
3. All securities have rates of return that are independent from one another. Independence implies that security returns have zero covariation with one another, so $\mathcal{C}ov(\tilde{r}_i, \tilde{r}_j) = 0$ for any two securities i and j—just as long as i is not j:

   $$\mathcal{C}ov(\tilde{r}_i, \tilde{r}_j) = \sigma_{i,j} = 0 \qquad \text{for all different i and j}$$

   (This last assumption is the most unrealistic of the three.)

The portfolio definition.

What are the risk and return characteristics of the portfolio, $\tilde{r}_P$, if it contains N securities? For an equal-weighted portfolio with N securities, each investment weight is $1/N$, so the portfolio rate of return is

$$\tilde{r}_P = \sum_{i=1}^{N} w_i \cdot \tilde{r}_i = \sum_{i=1}^{N} (1/N) \cdot \tilde{r}_i = 1/N \cdot \tilde{r}_1 + 1/N \cdot \tilde{r}_2 + \cdots + 1/N \cdot \tilde{r}_N$$

The Expected Rate of Return.

Let's start with one security. In this case, $r_P \equiv r_1$, so

$$\mathcal{E}(\tilde{r}_P) = \mathcal{E}(\tilde{r}_1) = 5\% \;\; , \quad \mathcal{S}dv(\tilde{r}_P) = \mathcal{S}dv(\tilde{r}_1) = \sqrt{\mathcal{V}ar(\tilde{r}_1)} = 40\%$$

Two securities now. The portfolio consists of a 50-50 investment in securities 1 and 2:

$$\tilde{r}_P = 1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2$$

In this portfolio, it is easy to see that the average expected rate of return on a portfolio is the average of the expected rates of return on its components:

$$\begin{aligned}\mathcal{E}(\tilde{r}_P) &= \mathcal{E}(1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2) \\ &= 1/2 \cdot \mathcal{E}(\tilde{r}_1) + 1/2 \cdot \mathcal{E}(\tilde{r}_2) \\ &= 1/2 \cdot 5\% + 1/2 \cdot 5\% = 5\%\end{aligned}$$

More generally, it is not a big surprise that the rate of return is 5%, no matter how many securities enter the portfolio:

$$\begin{aligned}\mathcal{E}(\tilde{r}_P) &= \sum_{i=1}^{N} \mathcal{E}(1/N \cdot \tilde{r}_i) = \sum_{i=1}^{N} 1/N \cdot \mathcal{E}(\tilde{r}_i) \\ &= \sum_{i=1}^{N} 1/N \cdot 5\% = 5\%\end{aligned}$$

The base case: Variance and Standard Deviation are lower for 2 securities.

It is when you turn to the portfolio risk characteristics that it becomes interesting. The variance and standard deviation of the rate of return on the portfolio P are more interesting. Begin with two securities:

$$\begin{aligned}\mathcal{V}ar(\tilde{r}_P) &= \mathcal{V}ar(1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2) \\ &= (1/2)^2 \cdot \mathcal{V}ar(\tilde{r}_1) + (1/2)^2 \cdot \mathcal{V}ar(\tilde{r}_2) + 2 \cdot (1/2) \cdot (1/2) \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) \\ &= 1/4 \cdot 0.16 + 1/4 \cdot 0.16 + 2 \cdot 1/2 \cdot 1/2 \cdot 0 \\ &= 1/4 \cdot 0.16 + 1/4 \cdot 0.16 \\ &= 1/2 \cdot 0.16 \\ \Longleftrightarrow \; \mathcal{S}dv(\tilde{r}_P) &= \sqrt{\mathcal{V}ar(\tilde{r}_P)} = \sqrt{1/2 \cdot 0.16} = 70.7\% \cdot 40\% = 28.3\%\end{aligned} \tag{11.1}$$

You could drop out the covariance term, because we have assumed security returns to be independent. Pay close attention to the final line: the portfolio of one security had a risk of 40%. The portfolio of two securities has the lower risk of 28.3%. This is important—diversification at work!

SIDE NOTE

If you find it easier to understand the formula if you see some data, below are two sample series that are consistent with our assumptions: each has 5% mean and 40% standard deviation, and they have zero mutual covariance. The final column is the rate of return on the portfolio P that invests half in each security, thus appropriately rebalanced each year, of course . You can confirm that the standard deviation of this portfolio is indeed the same 28.284% that you have just computed via the formula.

Year	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_P$	Year	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_P$
1980	25.420	-11.476	6.972	1985	21.268	-35.560	-7.146
1981	39.376	62.064	50.720	1986	-45.524	30.620	-7.452
1982	41.464	-19.356	11.054	1987	42.976	59.616	51.296
1983	30.500	-41.256	-5.378	1988	-10.324	30.360	10.018
1984	-67.552	15.772	-25.890	1989	-27.604	-40.784	-34.194
				$\mathcal{E}$	5.000	5.000	5.000
				$\mathcal{S}dv$	40.000	40.000	28.284

Variance and Standard Deviation for N securities.

Now, compute the standard deviation for an arbitrary number of securities in the portfolio. Recall the variance formula,

$$\mathcal{V}ar(\tilde{r}_P) = \mathcal{V}ar\left(\sum_{i=1}^{N} c_i \cdot \tilde{r}_i\right) = \sum_{j=1}^{N} \left\{\sum_{k=1}^{N} [c_j \cdot c_k \cdot \mathcal{C}ov(\tilde{r}_j, \tilde{r}_k)]\right\}$$

$$
\begin{array}{lcccccc}
 & w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) & + & w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) & + \cdots + & w_N^2 \cdot \mathcal{V}ar(\tilde{r}_N) \\
+ & 2 \cdot w_1 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) & + & 2 \cdot w_1 \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_3) & + \cdots + & 2 \cdot w_1 \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_N) \\
+ & 2 \cdot w_2 \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_1) & + & 2 \cdot w_2 \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_3) & + \cdots + & 2 \cdot w_2 \cdot w_N \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_N) \\
+ & \vdots & + & \ddots & + \ddots + & \vdots \\
+ & 2 \cdot w_N \cdot w_1 \cdot \mathcal{C}ov(\tilde{r}_N, \tilde{r}_1) & + & 2 \cdot w_N \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_N, \tilde{r}_3) & + \cdots + & 2 \cdot w_N \cdot w_{N-1} \cdot \mathcal{C}ov(\tilde{r}_N, \tilde{r}_{N-1})
\end{array}
\tag{11.2}
$$

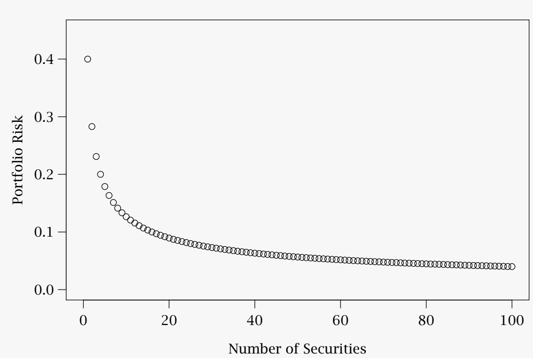
In our example, all the covariance terms are zero, all the weights are $1/N$, and all individual security variances are the same, so this is

$$
\begin{array}{rcl}
\mathcal{V}ar(\tilde{r}_P) & = & (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_1) + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_2) + \cdots + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_N) \\
 & = & (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) + \cdots + (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) \\
 & = & N \cdot (1/N)^2 \cdot \mathcal{V}ar(\tilde{r}_i) = (1/N) \cdot \mathcal{V}ar(\tilde{r}_i) \\
\Longleftrightarrow \quad \mathcal{S}dv(\tilde{r}_P) & = & \sqrt{1/N \cdot \mathcal{V}ar(\tilde{r}_i)} = \sqrt{1/N} \cdot \mathcal{S}dv(\tilde{r}_i) = \sqrt{1/N} \cdot 40\%
\end{array}
$$

Reflect on the effectiveness of diversification: a lot for the first few additions, then less and less.

This formula states that for 1 security, the risk of the portfolio is 40%; for 2 securities, it is 28.3% (as also computed in Formula 11.1); for 4 securities, it is $\sqrt{1/4} \cdot 40\% = 20\%$; for 16 securities, it is 10%; for 100 securities, it is 4%, and for 10,000 securities, is 0.4%. In a portfolio of infinitely many securities, the risk of the portfolio gradually disappears. In other words, you would practically be certain to earn the expected rate of return (here 5%). Now take a look at the risk decline in Figure 11.2 to see how more securities help to reduce risk. The square root function on N declines steeply for the first few securities, but then progressively less so for subsequent securities. Going from 1 to 4 securities reduces the risk by 50%. The next 5 securities (going from 4 to 9 securities) only reduce the risk by 17% (from $\sqrt{1/4} = 50\%$ to $\sqrt{1/9} = 33\%$). To drop the risk from 50% to 25% requires 12 extra securities; to drop the risk from 50% to 10% requires 96 extra securities. In other words, *if* security returns are independent, diversification works really well in the beginning, but less and less as more securities are added. It is important to have, say, a dozen independent securities in the portfolio, which drops the portfolio risk by two-thirds; additional diversification through purchasing more securities is nice, but it is not as important, in relative terms, as these first dozen securities.

Figure 11.2: Diversification if security returns were independent.



11·3.B. Correlated Securities

When does diversification fail? Recall that the maximum possible correlation of 1 implies that

Diversification fails when security returns are perfectly positively correlated.

↦ Max Correlation: 9·4.B on Page 200

$$Correlation(\tilde{X},\tilde{Y}) = +1 \iff Cov(\tilde{X},\tilde{Y}) = Sdv(\tilde{X}) \cdot Sdv(\tilde{Y})$$

Therefore, the covariance of two perfectly correlated securities' rates of returns (in our example) is

$$\begin{aligned} Cov(\tilde{r}_i,\tilde{r}_j) &= 40\% \cdot 40\% = 0.16 \\ Cov(\tilde{r}_i,\tilde{r}_j) &= Sdv(\tilde{r}_i) \cdot Sdv(\tilde{r}_j) \end{aligned}$$

The variance of a portfolio of two such stocks is

$$\begin{aligned} Var(\tilde{r}_P) &= Var(1/2 \cdot \tilde{r}_1 + 1/2 \cdot \tilde{r}_2) \\ &= 1/2^2 \cdot Var(\tilde{r}_1) + 1/2^2 \cdot Var(\tilde{r}_2) + 2 \cdot 1/2 \cdot 1/2 \cdot Cov(\tilde{r}_1,\tilde{r}_2) \\ &= 1/4 \cdot 0.16 + 1/4 \cdot 0.16 + 2 \cdot 1/2 \cdot 1/2 \cdot 0.16 \\ &= 1/2 \cdot 0.16 + 1/2 \cdot 0.16 = 0.16 \\ Sdv(\tilde{r}_P) &= \sqrt{Var(\tilde{r}_P)} = 40\% \end{aligned}$$

In other words, when two securities are perfectly correlated, diversification does not reduce portfolio risk. It should come as no surprise that you cannot reduce the risk of a portfolio of one PepsiCo share by buying another PepsiCo share. (The returns of PepsiCo shares are perfectly correlated.) In general, the smaller or even more negative the covariance term, the better diversification works.

IMPORTANT:

- Diversification fails when underlying securities are perfectly positively correlated.
- Diversification reduces portfolio risk better when its underlying securities are less correlated.
- Diversification works perfectly (reducing portfolio risk to zero) when underlying securities are perfectly negatively correlated.

Question 11.3 at the end of this section asks you to prove the last point.

11·3.C. Measures of Contribution Diversification: Covariance, Correlation, or Beta?

Covariance, Correlation, and Beta are multiple possible measures of covariation.

It seems that diversification works better when the covariation between investment securities is smaller. The correct measure of overall risk remains, of course, the standard deviation of the portfolio's rate of return. But, the question you are now interested in is "What is the best measure of the contribution of an individual security to the risk of a portfolio?" You need a measure of the risk contribution of just one security inside the portfolio to the overall portfolio risk.

Candidates to measure how a new security helps portfolio diversification.

Let's assume that you already hold a portfolio Y, and you are now considering adding "a little bit" of security i. How much does this new security help or hurt your portfolio through diversification? Because you are adding fairly little of this security, it is a reasonable approximation to assume that the rest of the portfolio remains as it was (even though the new security really becomes part of portfolio Y and thereby changes Y). The three candidates to measure how correlated the new investment opportunity i is with the rest of your portfolio Y are

$$\text{The Covariance:} \quad \mathcal{C}ov(\tilde{r}_i, \tilde{r}_Y) \quad \text{(Uninterpretable)}$$

$$\text{The Correlation:} \quad \frac{\mathcal{C}ov(\tilde{r}_i, \tilde{r}_Y)}{\mathcal{S}dv(\tilde{r}_i) \cdot \mathcal{S}dv(\tilde{r}_Y)} \quad \text{(Interpretable)}$$

$$\text{The Portfolio Beta:} \quad \frac{\mathcal{C}ov(\tilde{r}_i, \tilde{r}_Y)}{\mathcal{V}ar(\tilde{r}_Y)} \quad \text{(Interpretable)}$$

↦ *Covariance: Formula 11.1 on Page 243*

Although all three candidates share the same sign, each measure has its own unique advantage. The covariance is used directly in the portfolio formula, but its value is difficult to interpret. The correlation is easiest to interpret, because it lies between −1 and +1. However, its real problem as a measure of risk for a new security (which you want to add to your portfolio) is that it ignores a security's relative variability.

Debunking correlation: it is not a good risk measure.

This last fact merits an explanation. Consider two securities that both have perfect correlation with your portfolio. In our example, there are only two equally likely possible outcomes:

	Your Portfolio	Security A	Security B
Outcome 1	+24%	+12%	+200%
Outcome 2	–12%	–6%	–100%
$\mathcal{E}(\tilde{r})$	+6%	+3%	50%
$\mathcal{S}dv(\tilde{r})$	+18%	+9%	150%

Now assume you had \$75 in your portfolio, but you are adding \$25 of either A or B. Therefore, your new combined portfolio rate of return would be

	Your Portfolio Y Plus Security A	Your Portfolio Plus Security B
Case 1	$75\% \cdot (+24\%) + 25\% \cdot (+12\%) = 21.0\%$	$75\% \cdot (+24\%) + 25\% \cdot (+200\%) = 68\%$
Case 2	$75\% \cdot (-12\%) + 25\% \cdot (-6\%) = -10.5\%$	$75\% \cdot (-12\%) + 25\% \cdot (-100\%) = -34\%$
$\mathcal{E}(\tilde{r}_P)$	5.25%	51.00%
$\mathcal{S}dv(\tilde{r}_P)$	15.75%	153.00%

Adding stock B causes your portfolio risk to go up more. It adds more portfolio risk. Yet, the correlation between your portfolio and either security A or security B was the same. The correlation would not have told you that B is the riskier add-on.

But the beta of a stock with your portfolio still works as a good risk measure!

In contrast to correlation, our third candidate for measuring risk contribution tells you the right thing. The beta of security A with respect to your portfolio Y is 0.5 (which you can compute either with the formula $\mathcal{C}ov(\tilde{r}_A, \tilde{r}_Y)/\mathcal{V}ar(\tilde{r}_Y)$, or by recognizing that A is always one-half of Y); the beta of security B with respect to your portfolio Y is 8.33. Therefore, beta tells you that adding security B would increase your overall portfolio risk more than adding security A. Unlike correlation, beta takes into account the *scale* of investments.

A concrete example that shows that correlation is not a great diversification measure.

You can also find this scale problem within the context of our earlier three-investments scenario with the annual returns of the S&P 500, IBM, and Sony. On Page 204, you found that, compared to IBM, Sony had a higher beta with the market, but a lower correlation. Does a portfolio I consisting of one-half S&P 500 and one-half IBM have more or less risk than a portfolio S consisting of one-half S&P 500 and one-half Sony?

$$\mathcal{V}ar(\tilde{r}_I) = (1/2)^2 \cdot 3.62\% + (1/2)^2 \cdot 15.03\% + 2 \cdot 1/2 \cdot 1/2 \cdot 3.30\% = 5.075\%$$
$$\mathcal{V}ar(\tilde{r}_S) = (1/2)^2 \cdot 3.62\% + (1/2)^2 \cdot 81.49\% + 2 \cdot 1/2 \cdot 1/2 \cdot 4.77\% = 21.874\%$$

Even though Sony has lower correlation with the S&P 500 than IBM, Sony's higher variance negates this advantage: the S portfolio is riskier than the I portfolio. The covariances reflect this accurately: the covariance of the S&P 500 with Sony is higher than the covariance of the S&P 500 with IBM. Beta preserves this ordering, too, because both covariances are divided by the same variance (the variance of $\tilde{r}_{\text{S\&P500}}$). It is only the correlation that would have misleadingly indicated that Sony would have been the better diversifier. This, then, is our main insight:

IMPORTANT: The beta of any security i with respect to our portfolio (called $\beta_{i,Y}$) is a measure of the security's risk contribution to the portfolio, because it properly takes scale into account. The lower the beta of security i with respect to portfolio Y, the better security i works at diversifying portfolio Y's overall risk.

The other two measures.

The covariance would have worked equally well, but it is more difficult to intuitively interpret. Correlation is not suitable as a quantitative measure, because it fails to recognize investment scales.

Beta has a nice slope interpretation, too.

The remainder of this section just provides some additional intuition: Beta has a nice slope interpretation. If you graph the rate of return on your overall portfolio Y (e.g., the S&P 500) on the x axis, and the rate of return on the new security i (e.g., Sony) on the y axis, then beta represents the slope of a line that helps you predict how the rate of return of security i will turn out *if* you know how the rate of return on your portfolio turned out.

Beta can be visually judged by graphing the rate of return on a security against the rate of return on the overall portfolio. It is the slope of the line.

Figure 11.3 shows three securities with very different betas with respect to your portfolio. In graph (c), security POS has a positive beta with respect to your portfolio. You would therefore expect security POS to not help you much in your quest to diversify: when your portfolio Y does better, so does U. In the figure, U has a beta with respect to your portfolio Y of 3. This indicates that if Y were to ear an additional 5% rate of return, the rate of return on the security POS would be expected to change by an additional 15%. More importantly, if your portfolio Y hits hard times, asset POS would be hit even harder! You would be better off purchasing only a very, very small amount of such a security; otherwise, you could quickly end up with a portfolio that looks more like POS than like Y. In contrast, in graph (a), you see that if you purchased security NEG would expect this NEG to help you considerably in diversification: when your Y does worse, NEG does better! With a beta of −3, the security NEG tends to go up by an additional 15% when the rest of your portfolio Y goes down by an additional 5%. Therefore, NEG provides excellent "insurance" against downturns in Y. Finally, in graph (b), the security ZR has a zero beta, which is the case when ZR 's rates of return are independent of P's rates of return. You already know that securities with no correlation can help you quite nicely in helping diversify your portfolio risk.

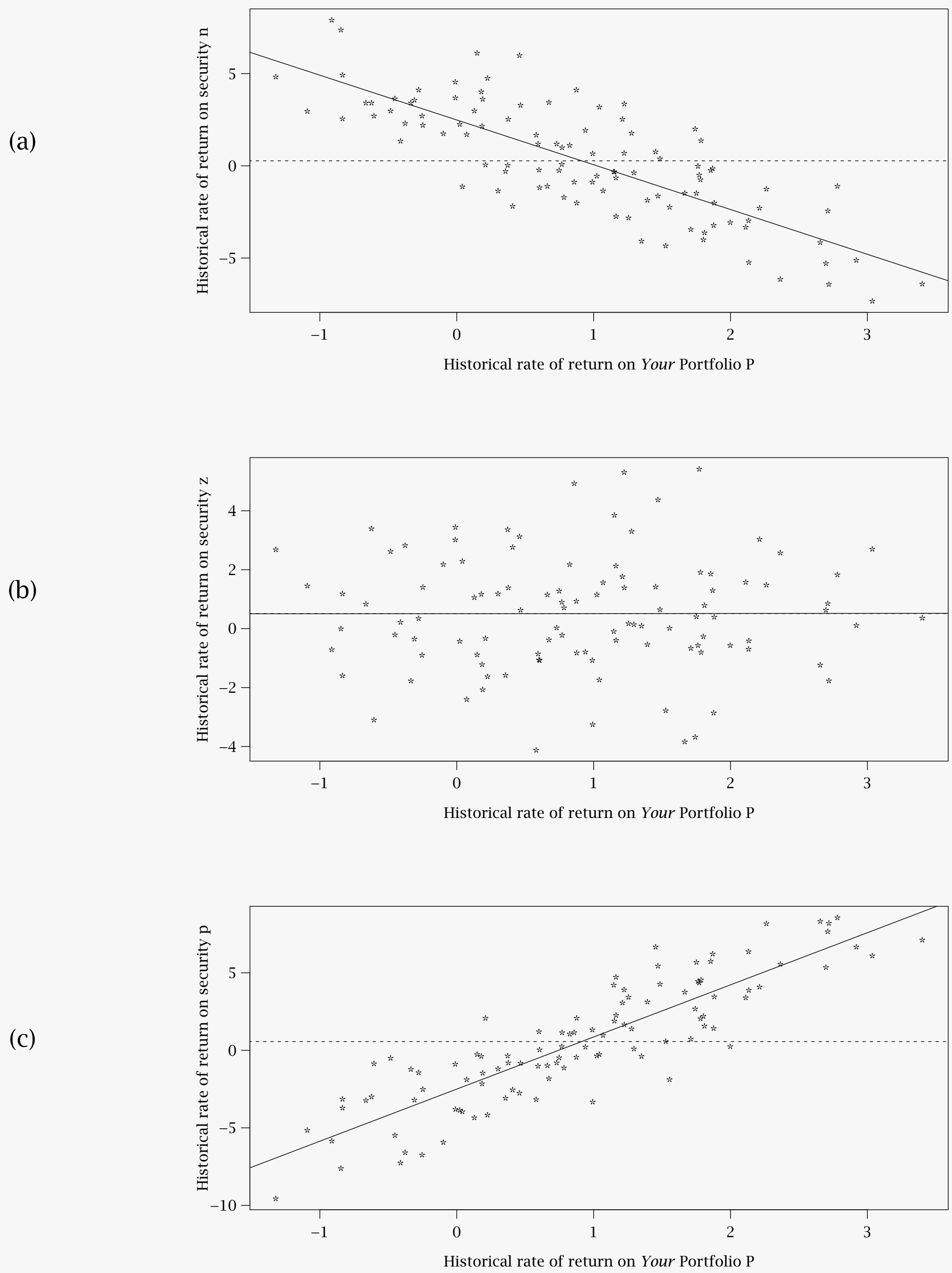
The intercept in Figure 11.3 is sometimes called the "alpha," and it can measure how much expected rate of return the security is likely to offer, holding its extra risk constant. For example, if stocks i and j have lines as follows

$$\mathcal{E}(\tilde{r}_i) \approx +15\% + 1.5 \cdot \mathcal{E}(\tilde{r}_P)$$
$$\mathcal{E}(\tilde{r}_j) \approx -10\% + 1.5 \cdot \mathcal{E}(\tilde{r}_P)$$
$$\mathcal{E}(\tilde{r}_k) \approx \alpha_k + \beta_{k,P} \cdot \mathcal{E}(\tilde{r}_P)$$

the α of stock i is 15%, while the α of stock j is −10%. It appears that stock i offers a holder of portfolio P a lot of positive return compared to j, holding its exposure to portfolio P constant. Naturally, you would like stocks with high alphas and low betas—but such opportunities are difficult to find, especially if you already hold a reasonably good (well-diversified) portfolio.

SIDE NOTE

Figure 11.3: Three Different Beta Measures of Risk for Security i.



The historical betas of these three securities, n, z, and p, with respect to your portfolio P are based on returns whose true betas with respect to your portfolio P are −3, 0, and +3 in the three graphs, respectively. Historical data is indicative, but not perfect in telling you the true beta.

Solve Now!

Q 11.1 Assume that every security has a mean of 12%, and a standard deviation of 30%. Further, assume that each security has no covariation with any security. What are the risk and reward of a portfolio of N stocks that invests equal amount in each security?

Q 11.2 Assume that every security has a mean of 12%, and a standard deviation of 30%. Further, assume that each security has 0.0025 covariation with any security. What are the risk and reward of a portfolio of N stocks that invests equal amount in each security? If you find this difficult, solve this for 2 stocks. Can you guess what the risk is for a portfolio of infinitely many such stocks?

Q 11.3 Compute the variance of a two-stock portfolio if the two securities are not perfectly positively, but perfectly negatively, correlated.

Q 11.4 You own a $1,000 portfolio P, whose expected rate of return has a mean of 10% and a standard deviation of 20%. You are considering buying a security Q that has a mean of 15% and a standard deviation of 50%. The correlation between the rates of return on P and Q is 20%.

(a) What is the covariance between the rate of return of P and Q?
(b) What is the beta between the rate of return on P and Q?
(c) Consider purchasing $1,000 in Q. What would the portfolio risk be?

A new security, named N has appeared. It has a mean of 150% and a standard deviation of 500%, and the same 20% correlation with P. (Sidenote: Such a security could be created by a fund that borrows in order to purchase more than 100% in Q.)

(d) What is the covariance between the rate of return of P and N?
(e) What is the beta between the rate of return on P and N?
(f) Consider purchasing $1,000 in N. What would the portfolio risk be?
(g) Q and N have equal correlation with portfolio P. Does it follow that they would both be equal risk-contributors, if added to the portfolio?

11·4 Does Diversification Work in the Real World?

You now understand the theory. But can you make it work in the real world?

11·4.A. Diversification Among The Dow-Jones 30 Stocks

Diversification depends on the specific portfolio.

To see whether portfolio diversification works in the real world, you should look at some specific securities. Clearly, the degree to which diversification works must depend on the weights of the specific securities you look at within the context of your portfolio.

Anecdote: Value-At-Risk (VAR)

The latest in risk measurement techniques among banks and other financial institutions—and a great step forward if executed correctly—is **VaR** (**Value-at-Risk**). It often replaces older risk-scoring systems, in which (for example) all commercial loans received one score, government loans another, etc. The goal of Value at Risk is to compute the risk (standard deviation) when all investment (loans) are evaluated in a portfolio framework. Value at Risk can come to very different conclusions than these older systems, especially if payoffs to loans are very negatively or very positively correlated.

The risk of investing in a Dow-Jones 30 Stock.

↦ DJ30 Section 8·3.C on Page 179

You can get an intuitive feel for the effectiveness of diversification in the U.S. stock markets if you look at the 30 stocks constituting the Dow-Jones 30. The Dow-Jones company has chosen 30 stocks in its Dow-Jones 30 Index that avoid industry and firm-type concentration. However, the Dow-Jones contains only very large firms, both in market capitalization and sales.

The average stock in the Dow-Jones has a risk of about 30% to 40%; the overall portfolio has a risk just about half.

Table 11.1 shows the risks (standard deviations of the rate of returns) for the 30 Dow-Jones stocks, measured either from January 1997 to October 2002 or from January 1994 to October 2002. Their risks ranged from about 18% to 50% (annualized); the typical stock's risk averaged about 30% from 1994 to 2002, and 35% from 1997 to 2002. Just investing in one randomly chosen stock is fairly risky. However, when you compute the risk of the index itself, the index's risk was only 16.5% from 1994 to 20002 and 18.7% from 1997 to 2002. Naturally, this is still pretty risky. But the Dow-Jones index risk is only about half the risk of its average stock.

Inferring correlations.

Are the returns of these 30 stocks positively correlated? Note that 18% is only about half of the 35% risk that would obtain if all the securities' returns were perfectly positively correlated. Diversification works: the correlation among the Dow-Jones 30 stock returns is not close to +1. However, the correlation among the Dow-30 stocks is also not zero. Let us do some back-of-the-envelope calculations. If the Dow-Jones 30 Index were an equal-weighted portfolio (it is not!) of uncorrelated securities (it is not!), you would have expected it to have a risk of about $\sqrt{1/30} \cdot 35\% \approx 6.4\%$ per year. Instead, the risk of the Dow-Jones portfolio is 2.8 times as high at 18% per year. In sum, this evidence suggests that these 30 stocks move together, i.e. that they tend to have mutual positive correlations. This reduces the effectiveness of diversification among them. This is actually a broader effect. When the U.S. stock market does well, most stocks do well at the same time (and vice versa).

IMPORTANT: The Dow-Jones 30 Market Index Portfolio has a risk of about 15% to 20% per year. (Broader U.S. stock market indexes, like the S&P500 Index, tend to have slightly lower risks.)
The 30 component stocks in the Dow-Jones 30 Index are mutually positively correlated, which limits the effectiveness of diversification—but not so much as to render diversification useless.

Make sure not to buy just tech stocks, or just growth stocks, or ...

The square root in the portfolio standard deviation formula suggests that most of the diversification typically comes from the first 10 to 50 stocks. Therefore, it is more important to be suitably diversified across different types of stocks (to avoid mutual positive covariances) than it is to add every single possible stock to a portfolio. If you holds the Dow-Jones 30 stocks if you want to further diversify using U.S. stocks, you should consider adding small or high-growth firm stocks to your portfolio (or, better, invest in a mutual fund that itself holds on many small firms).

A short digression: covariance matters more than variance!

↦ Unknown 11.2 on Page 244

What matters more in determining portfolio variance: covariances or variances? The number of covariance terms in the portfolio risk formula 11.2 increases roughly with the square of the number of securities—by $N^2 - N$ to be exact. The number of variances increases lineary—by N. For example, for 100 securities, there are 100 variance terms and 9,900 covariance terms (4,450 if you do not want to double-count the same pairwise covariance). It should come as no surprise that as the number of securities becomes large, the risk of a portfolio is determined more by the covariance terms than by the variance terms.

A short digression: covariance matters more than variance!

The Dow-Jones Index is not alone in benefiting from the effects of diversification. Academic research has shown that if you look at the average stock on the NYSE, about 75% of its risk can be diversified away (i.e., disappears!), while the remaining 25% of its risk cannot be diversified away. Put differently, undiversifiable co-movements among all stocks in the stock market are responsible for about one-quarter of the typical stocks' return variance; three-quarters are idiosyncratic day-to-day fluctuations, which average themselves away if you hold a highly diversified stock market index like portfolio.

Table 11.1: Risk and Reward for Dow Jones Constituents, Based on Monthly Rates of Returns, Then Annualized.

	about 10 years 1994/01-2002/10		about 5 years 1997/01-2002/10	
Asset	Mean	StdDev	Mean	StdDev
alcoa	18.4%	36.0%	15.1%	41.3%
american express	21.4%	27.9%	17.3%	31.4%
boeing	8.9%	30.1%	−2.9%	34.6%
citigroup	27.7%	33.6%	24.9%	37.4%
caterpillar	14.0%	31.2%	9.6%	34.0%
du pont	12.2%	26.2%	4.4%	29.0%
disney	6.7%	28.8%	0.3%	32.2%
eastman kodak	4.5%	30.0%	−5.5%	34.6%
general electric	17.1%	25.1%	12.8%	28.4%
general motors	5.2%	32.6%	5.1%	36.7%
home depot	18.5%	31.3%	22.8%	35.1%
honeywell	11.2%	37.9%	6.0%	44.5%
hewlett packard	14.8%	42.9%	4.0%	48.2%
ibm	25.8%	36.0%	20.2%	39.6%
intel	28.7%	46.7%	15.8%	53.1%
international paper	7.5%	31.6%	6.0%	36.0%
johnson and johnson	23.2%	23.9%	19.5%	26.4%
jp morgan	15.0%	36.8%	5.9%	42.5%
coca-cola	13.1%	26.5%	3.7%	30.7%
mcdonalds	6.2%	25.6%	0.6%	28.8%
3m	14.8%	23.2%	12.8%	26.0%
philip morris	13.5%	29.9%	6.9%	33.0%
merck	19.2%	28.6%	12.1%	32.0%
microsoft	35.8%	42.6%	28.4%	50.0%
proctor and gamble	16.7%	24.7%	13.7%	28.4%
sbc communications	9.4%	28.9%	8.1%	34.4%
att	−4.4%	36.5%	−5.7%	40.3%
united technologies	21.9%	29.2%	18.1%	34.1%
wal-mart	20.6%	28.8%	31.5%	31.1%
exxon	10.0%	16.8%	7.1%	18.3%
Average	15.3%	31.0%	10.6%	35.0%
Typical (Median)	14.8%	29.9%	8.1%	34.1%
dow jones 30 index	10.5%	16.5%	6.3%	18.7%

For comparison, the S&P 500 index had a mean of 8.6% (risk of 16.1%) from 1994–2002, and a mean of 4.8% (risk of 18.5%) from 1997—2002.

SIDE N

In Section a on Page 205, I claimed that historical standard deviations of rates of return tend to be relatively stable. This table shows that, although the riskiness of firms does change over time (it is different over 5 years and 10 years), it does change only slowly, even for the individual Dow-30 stocks. (There would be even more stability if you considered asset class portfolios instead of just stocks.) This stability gives us confidence in using historical risk measures (e.g., standard deviations) as estimates of future risk.

11·4.B. Mutual Funds

There are too many stocks for you to buy them all.

Diversification clearly reduces risk, but it can also be expensive to accomplish. How can you purchase 500 securities with a $50,000 portfolio? The transaction costs of purchasing $100 in each security would be prohibitive. With about 10,000 publicly traded equities in the U.S. stock market, even purchasing just $1,000 in every stock traded would require $10 million, well beyond the financial capabilities of most retail investors.

Mutual funds are investment vehicles to accomplish diversification.

Mutual funds, already mentioned in Section 8·3.B, come to the rescue. As already described in Chapter 8, a mutual fund is like a firm that consists of nothing but holdings in other assets, usually financial assets. In a sense, a mutual fund is a large portfolio that can be purchased as a bundle. However, there is one small catch—isn't there always? On the one hand, investing in a mutual fund rather than in its individual underlying assets can reduce your transaction costs, ranging from the time necessary to research stocks and initiate transactions, to the direct trading costs (the commission and bid-ask spread). But, on the other hand, mutual funds often charge a variety of fees and may force you to realize taxable gains in a year when they would rather not.

11·4.C. Alternative Assets

Other Assets Are Equally or More Important.

A common error committed by investors is that they focus only on the diversification among stocks traded on the major U.S. stock exchanges. But there are many other financial and non-financial instruments that can aid investors in diversifying their risk. Because these instruments are often less correlated with an investor's portfolio than domestic U.S. stocks, these assets can be especially valuable in reducing the portfolio risk. Among possible investment assets are:

- Savings accounts.
- Bonds.
- Commodities (such as gold).
- Other futures (such as agricultural commodities).
- Art.
- Real estate.
- Mortgage and corporate bonds.
- Labor income.
- International stocks.
- Hedge funds.
- Venture and private equity funds.
- Vulture and bankruptcy funds.

In addition, if you are a smart investor, you would not only consider the diversification within your stock portfolio, but across your entire wealth. Your wealth would include your house, your education, your job, etc. Many of these alternative investments could also have low covariation with your overall wealth.

Anecdote: Portfolios of Finance Professors

Many finance professors invest their own money into passively managed, low-cost mutual funds, often **Index Funds**, which buy-and-hold a wide cross-section of assets and avoid active trading. They also require minimal investment selection abilities by their managers, and usually incur minimal trading costs. Vanguard funds are particularly popular, because **Vanguard** is not only the largest mutual fund provider—though neck-in-neck with **Fidelity**—but it also does not even seek to earn a profit. It is a "mutual" mutual fund, owned by the investors in the funds themselves.

SIDE NOTE

Some of the above mentioned asset categories may be better held in modest amounts. Furthermore, some of these areas do not resemble the highly liquid, fair, and efficient financial markets that U.S. stock investors are used to. Instead, some are rife with outright scams. Therefore, it might be wise to hold such assets through sophisticated and dedicated professional investors (mutual funds), who have the appropriate expertise.

Solve Now!

Q 11.5 In Table 11.1, Exxon had the lowest standard deviation among the Dow-30 stocks. Why not just purchase Exxon by itself?

Q 11.6 How much does diversifying over all 500 stocks in the S&P500 help in terms of risk reduction relative to investing in the 30 stocks of the Dow Jones-30?

Q 11.7 Why do mutual funds exist?

Q 11.8 Should you just own U.S. stocks?

Q 11.9 Does the true value-weighted market portfolio just contain stocks?

11·5 Diversification Over Time

Explaining the Figure Illustrating Time Diversification.

Many investors think of diversification across securities within a portfolio, but do not realize that diversification can also work over time—although the sidenote below explains why academics are divided on this issue. Figure 11.4 illustrates this point by showing the risk and reward if you had invested in the S&P 500 from 1990–2002 for x consecutive trading days. The left graph shows your average daily rate of return; the right side your total compounded rate of return. For example, the right figure shows that if you had held an S&P500 portfolio during a random 25 day period (about a month), you would have earned a little less than 1%, but with a risk (standard deviation) of about 5%. Graph (a) quotes this in average daily terms: over 25 days, a 1% mean and 5% risk was an average reward of about 0.04%/day with a risk of about 0.2%/day.

Time Diversification at Work!

Note from graph (b) how the risk-reward trade-off changes with time. Over an investment horizon of a full year (255 trading days), you would have expected to earn about 10% with a risk of about 15%. The risk would have been 1.5 times the reward. In contrast, over an investment horizon of one day, you would have expected to earn about 0.04% with a risk of about 1.1%. Your risk would have been about 30 times your reward! If you stare at graph (b), you should notice that the reward goes up a little more than linearly (the compounding effect!), while the risk goes up like a parabola, i.e., a square root function. Indeed, this is the case, and the rest of this section shows why.

Over periods shorter than a few years, the ordinary portfolio return is roughly the sum of time period returns.

Recall that a portfolio that earns $\tilde{r}_{t=1}$ in period 1, $\tilde{r}_{t=2}$ in period 2, and so on until period T ($\tilde{r}_{t=T}$), will earn an overall rate of return of

$$\tilde{r}_{t=0,t=T} = (1+\tilde{r}_{t=0,1}) \cdot (1+\tilde{r}_{t=1,2}) \cdot \vdots \cdot (1+\tilde{r}_{t=T-1,T}) - 1$$
$$\approx \tilde{r}_{t=0,1} + \tilde{r}_{t=1,2} + \cdots + \tilde{r}_{t=T-1,T} + \text{many multiplicative } \tilde{r} \text{ terms}$$

The multiplicative terms reflect the power of compounding—which clearly matters over many years—but perhaps less so over periods of just a few months or years. After all, even monthly returns may typically be only on the order of 1%, so the multiplicative term would be on the order of 1% · 1% = 0.01%.

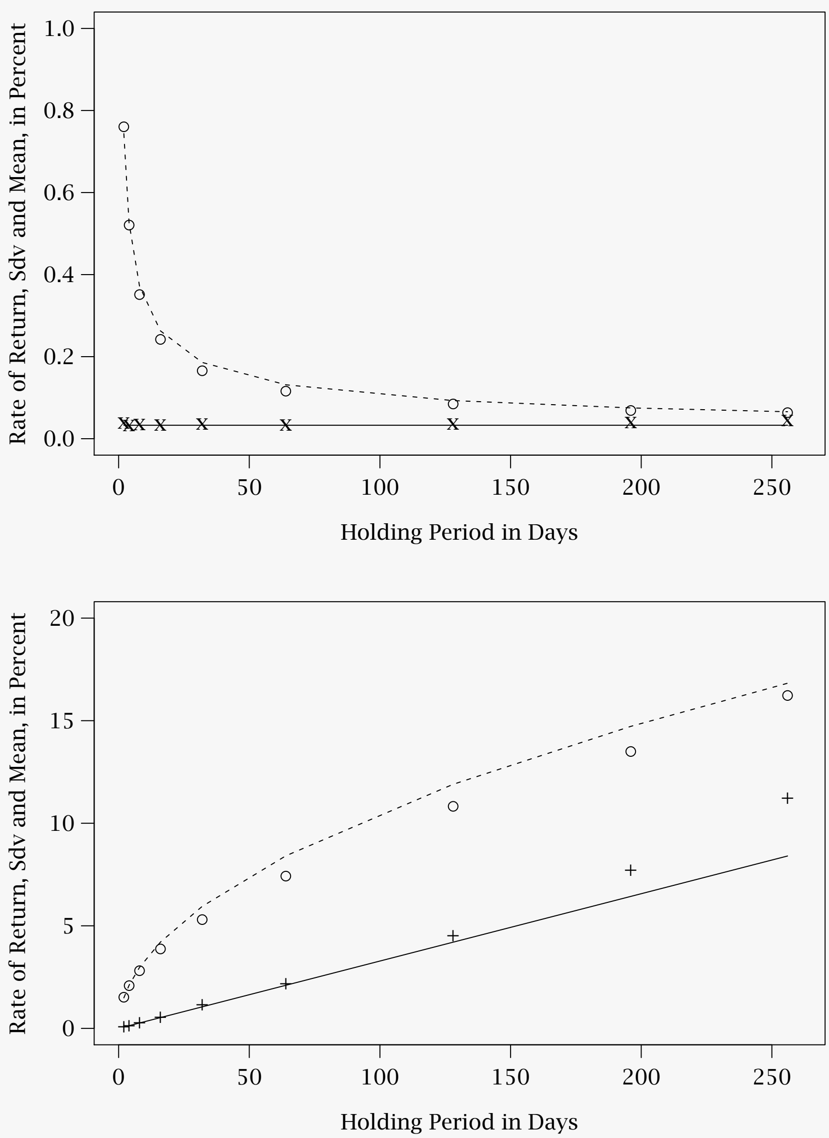
Stock returns have to be about independent across time-periods to avoid great money-making opportunities. This determines the riskiness of portfolios over multiple periods.

Now, if you forget about the small multiplicative terms, you already know that if you expect to earn 1% per period, then over x periods you expect to earn x%:

$$\mathcal{E}(\tilde{r}_{t=0,t=T}) = T \cdot \mathcal{E}(\tilde{r}_t)$$

For the variance, let us assume that the return variance in each time period is about the same and can just be called $Sdv(\tilde{r}_t)$. Further assume that the covariance terms among returns in different times (e.g., $\tilde{r}_{t=0,1}$ and $\tilde{r}_{t=1,2}$) are about zero. After all, if they were not, this would mean that you could predict future returns with past returns. Suffice it to say that this sort of

Figure 11.4: Average Risk and Return over Time in the S&P500, 1990–2002



(a)

(b)

The x-axis is the investment horizon, i.e., the number of consecutive investment days. The two top series (lines and dots) are the standard deviation, the two bottom series are the means. Lines are the theoretical values computed with the formulas below; circles and pluses are from simulated actual investment strategies.

prediction is not an easy task—if it were, you would quickly become rich! Put this all together and use the variance formula 11.3:

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_{t=0,T}) &= \mathcal{V}ar\,(\tilde{r}_{t=0,1} + \tilde{r}_{t=1,2} + \cdots + \tilde{r}_{t=T-1,T}) \\ &\quad + \text{ many covariance terms, all about zero} \\ &\approx T \cdot \mathcal{V}ar(\tilde{r}_t) \end{aligned} \tag{11.3}$$

It follows that

$$Sdv(\tilde{r}_{t=0,T}) \approx \sqrt{T} \cdot Sdv(\tilde{r}_t) \tag{11.4}$$

But this is just the relationship in the graph: The risk increases with the square root of time!

Annualization

Formula 11.4 is commonly used to "annualize" portfolio risk. For example, if an investment strategy has a monthly risk of 5% (i.e., the standard deviation of its rate of return), and the question is what kind of risk such an investment strategy would have per annum, you can compute the implied annual standard deviation to be about $\sqrt{12} \cdot 5\% \approx 17\%$. Conversely, if the five-year variance is 39%, then the annualized variance is 17%, because $\sqrt{5} \cdot 17\% \approx 39\%$.

IMPORTANT: A quick and dirty (and common) method to annualize portfolio risk (the standard deviation of the rate of return) is to multiply the single-period rate of return by the square root of the number of periods. For example, the annual portfolio risk is about $\sqrt{12}$ or 3.5 times as high the monthly risk.
The most common method, on Wall Street and in academia, is to compute risk and reward from monthly rates of return, but to report their annualized values.

The Sharpe-ratio is a measure of risk-reward performance.

The most commonly used measure of portfolio performance is the **Sharpe-ratio**, named after Nobel Prize Winner William Sharpe. It is the expected rate of return of a portfolio *above and beyond* the risk-free rate of return (r_F), divided by the standard deviation of the rate of the return,

$$\text{Sharpe Ratio} = \frac{\mathcal{E}(\tilde{r}_P - r_F)}{Sdv(\tilde{r}_P - r_F)} = \frac{\mathcal{E}(\tilde{r}_P) - r_F}{Sdv(\tilde{r}_P)}$$

Be aware that the Sharpe-ratio depends on the time interval that is used to measure returns. In the example with 1% monthly mean and 5% monthly standard deviation, if the risk-free rate were 6% per year, the Sharpe-Ratio of the portfolio would be about $^{(1\%\text{-}0.5\%)}/_{5\%} = 0.1$ if measured over a one month time interval; $^{(12\%\text{-}6\%)}/_{17\%} = 0.35$ if measured over a year; and $^{(60\%\text{-}30\%)}/_{39\%} = 0.77$ if measured over five years. Although the Sharpe ratio makes it clear how mean and standard deviation change with different time horizons, it is just a measure—it does not mean that you are better off if you hold stocks over longer time horizons. More importantly, you should be warned: although the Sharpe ratio is intuitively appealing and although it is in widespread use, it has the near-fatal flaw that it can easily be manipulated. You will find this out for yourself in Question 11.16.

SIDE NOTE

The overall division of assets between stocks and bonds is often called **asset allocation**. Many practitioners suggest that you should put more of your money into risky stocks when you are young. Over the very long run—and young people have naturally longer investment horizons—the expected rate of return versus risk relationship looks more favorable than it does over shorter investment horizons. After all, the mean goes up with time, while the risk goes up only with the square-root of time. This means that the *average* rate of return is less risky when you are young than when you are old.

But academics are divided on this advice. Some point out that the portfolio rate of return is the product of the individual returns:

$$\tilde{r}_{t=0,T} \;=\; (1+\tilde{r}_{0,1})\cdot(1+\tilde{r}_{1,2})\cdot\cdots\cdot(1+\tilde{r}_{T-1,T}) \;-1$$

It should not matter whether you choose risky stocks over safer bonds at time $t = 0$ (when you are young) or at time $t = T$ (when you are old). That is, even though it is true that the risk of the *average* rate of return declines over longer horizon, you should not be interested in the *average* rate of return, but in the *total* rate of return. This argument suggests that your time horizon should not matter to your asset allocation.

Even more sophisticated arguments take into account that you can adjust better (e.g., by working harder) when you are young if you experience a bad portfolio return; or that the stock market rate of return may be **mean reverting**. That is, the market rate of return may be negatively correlated with itself over very long time periods, in which case the long-run risk could be a little lower than the short-run risk.

Solve Now!

Q 11.10 What is a reasonable assumption for stock return correlations across different time periods?

Q 11.11 Table 10.4 on Page 228 shows that the S&P500 has an annual standard deviation of about 20% per year. As of November 2002, the S&P500 stood at a level of about 900. What would you expect the daily standard deviation of the S&P500 index to be? Assume that there are about 255 trading days in a year.

Q 11.12 Assume you have a portfolio that seems to have a monthly standard deviation of 5%. What would you expect its annual standard deviation to be?

Q 11.13 If the risk-free rate over the 1997-2002 sample period was about 3% per annum, what was the monthly and what was the annual Sharpe-ratio of the Dow-Jones 30 index?

Q 11.14 Assume you have a portfolio that seems to have had a daily standard deviation of 1%. What would you expect its annual standard deviation to be? Assume there are 255 trading days in a year.

Q 11.15 The S&P 500 was quoted in the first two weeks of June 2003 as

06/02/2003	967.00	06/06/2003	987.7606/12/2003	998.51
06/03/2003	971.56	06/09/2003	975.9306/13/2003	988.61
06/04/2003	986.24	06/10/2003	984.84	
06/05/2003	990.14	06/11/2003	997.48	

Compute the mean and standard deviation of annual returns of a portfolio that would have mimicked the S&P 500. Based on these returns, what would you expect to be the risk and reward if you held the S&P 500 for one year? Assuming a risk-free rate of return of 3%/annum, what would be the proper estimate for a Sharpe-ratio of daily, monthly, and annual returns? Is this consistent with the statistics in Table 10.4? Can you speculate why?

Q 11.16 Consider an investment strategy that has returned the following four rates of return: +5%, +10%, +5%, +20%. These are quoted above the risk-free rate (or equivalently assume the risk-free rate is 0%.)

(a) What was its Sharpe-ratio?

(b) Throw away 5% of the rate of return *in the final period only*. That is, if you had \$200 and you had ended up with \$240 (20%), you would now be throwing away \$10. You would end up with \$230 for a remaining rate of return of 15% only. (How easy is it to throw away money?) What is the Sharpe-ratio of this revised strategy?

(c) Which is the better investment strategy?

11·6 Summary

The chapter covered the following major points:

- Diversification—investment in different assets—reduces the overall risk (standard deviation).
- Diversification works better when assets are uncorrelated.
- The beta of a new asset (with respect to the existing portfolio) is a good measure of the marginal contribution of the new asset to the risk of the overall portfolio.
- Mutual funds invest in many assets, and thereby reduce retail investors' cost of diversification. However, they charge fees for this service.
- Diversification works over time, too. The portfolio reward (expected rate of return) grows roughly linearly over time, but the portfolio risk (standard deviation) grows roughly with the square-root of time. That is, if the expected rate of return is 1% per month, then the T month expected rate of return is approximately $T \cdot 1\%$. If the standard deviation of the rate of rate of return is 10% per month, then the T month standard deviation is approximately $\sqrt{T} \cdot 10\%$. The latter formula is often used to "annualize" risk.
- The Sharpe-ratio is the most common measure of investment strategy performance—although an awful one. One relatively minor problem is that it depends on the investment horizon on which it is quoted. A Sharpe-ratio based on annual returns is typically about $\sqrt{T}$ higher than a Sharpe-ratio based on monthly returns.

[No keyterm list for diversification-g.]

End of Chapter Problems

16 "Solve Now" Answers

1. The mean is 12%, the standard deviation is $30\%/\sqrt{N}$.
2. The mean is 12%. The variance now still has N variance terms, but $N \cdot (N-1)$ terms that are each ${}^1/_N \cdot {}^1/_N \cdot 0.001$. Thus, the variance is now

$$\mathcal{V}ar(\tilde{r}_P) = N \cdot ({}^1/_{N^2} \cdot (30\%)^2) + N \cdot (N-1) \cdot {}^1/_N \cdot {}^1/_N \cdot (0.0025)$$

Therefore, for 2 stocks, the variance is $1/2 \cdot .09 + 2 \cdot 1 \cdot (1/4 \cdot 0.0025) = 0.045 + 1/2 \cdot 0.0025 = 0.04625$. $Sdv = 21.5\%$. This is a little higher than the 21.2% from the previous question. For many stocks, $N \to \infty$ is

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= N \cdot ({}^1/_{N^2} \cdot (30\%)^2) + N \cdot (N-1) \cdot {}^1/_N \cdot {}^1/_N \cdot (0.0025) \\
&\approx N \cdot ({}^1/_{N^2} \cdot (30\%)^2) + 0.0025 \\
&= {}^1/_N \cdot (.09) + 0.0025 \\
\lim_{N\to\infty} Sdv(\tilde{r}_P) &= \sqrt{0.0025} = 5\%
\end{aligned}$$

3.

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= \mathcal{V}ar(\,1/2\cdot\tilde{r}_1 + 1/2\cdot\tilde{r}_2\,)\\
&= 1/2^2\cdot\mathcal{V}ar(\,\tilde{r}_1\,) + 1/2^2\cdot\mathcal{V}ar(\,\tilde{r}_2\,) + 2\cdot 1/2\cdot 1/2\cdot\mathcal{C}ov(\,\tilde{r}_1,\tilde{r}_2\,)\\
&= 1/4\cdot 0.01 + 1/4\cdot 0.01 + 2\cdot 1/2\cdot 1/2\cdot(-0.01)\\
&= 1/2\cdot 0.01 - 1/2\cdot 0.01 \quad = \quad 0.00\\
\mathcal{S}dv(\tilde{r}_P) &= \sqrt{\mathcal{V}ar(\tilde{r}_P)} \;=\; 0\% \qquad .
\end{aligned}$$

Therefore, with perfect negative correlation and equal investment weights, diversification works perfect!

4.

(a) The covariance is the correlation multiplied by the two standard deviations. For the Q portfolio,

$$\begin{aligned}
\mathcal{C}ov(\tilde{r}_Q,\tilde{r}_P) &= \mathcal{C}orrelation(\tilde{r}_Q,\tilde{r}_P)\cdot\mathcal{S}dv(\tilde{r}_Q)\cdot\mathcal{S}dv(\tilde{r}_P)\\
&= 20\%\cdot 20\%\cdot 50\% \quad = \quad 0.02
\end{aligned}$$

(b)

$$\beta_{Q,P} \;=\; \frac{\mathcal{C}ov(\tilde{r}_Q,\tilde{r}_P)}{\mathcal{V}ar(\tilde{r}_P)} \;=\; \frac{0.02}{0.04} \;=\; 0.5$$

(c) Therefore, the new portfolio with weights of 0.5 each is

$$\begin{aligned}
\tilde{r} &= 0.5\cdot\tilde{r}_P + 0.5\cdot\tilde{r}_Q\\
\mathcal{V}ar(\tilde{r}) &= w_P\cdot\mathcal{V}ar(\tilde{r}_P) + w_Q\cdot\mathcal{V}ar(\tilde{r}_Q) + 2\cdot w_Q\cdot w_P\mathcal{C}ov(\tilde{r}_Q,\tilde{r}_P)\\
&= 0.5^2\cdot(20\%)^2 + 0.5^2\cdot(50\%)^2 + 2\cdot 0.5\cdot 0.5\cdot 0.02\\
&= 0.01 + 0.0625 + 0.01 = 0.0825\\
\mathcal{S}dv(\tilde{r}) &= 28.7\%
\end{aligned}$$

(d) For the N portfolio,

$$\begin{aligned}
\mathcal{C}ov(\tilde{r}_N,\tilde{r}_P) &= \mathcal{C}orrelation(\tilde{r}_N,\tilde{r}_P)\cdot\mathcal{S}dv(\tilde{r}_N)\cdot\mathcal{S}dv(\tilde{r}_P)\\
&= 20\%\cdot 20\%\cdot 500\% \quad = \quad 0.2
\end{aligned}$$

(e)

$$\beta_{\tilde{r}_N,\tilde{r}_P} \;=\; \frac{\mathcal{C}ov(\tilde{r}_N,\tilde{r}_P)}{\mathcal{V}ar(\tilde{r}_P)} \;=\; \frac{0.2}{0.04} \;=\; 5$$

(f) Therefore, the new portfolio with weights of 0.5 each is

$$\begin{aligned}
\tilde{r} &= 0.5\cdot\tilde{r}_P + 0.5\cdot\tilde{r}_N\\
\mathcal{V}ar(\tilde{r}) &= w_P\cdot\mathcal{V}ar(\tilde{r}_P) + w_Q\cdot\mathcal{V}ar(\tilde{r}_Q) + 2\cdot w_Q\cdot w_P\mathcal{C}ov(\tilde{r}_Q,\tilde{r}_P)\\
&= 0.5^2\cdot(20\%)^2 + 0.5^2\cdot(500\%)^2 + 2\cdot 0.5\cdot 0.5\cdot 0.2\\
&= 0.01 + 6.25 + 0.1 = 6.36\\
\mathcal{S}dv(\tilde{r}) &= 252.2\%
\end{aligned}$$

(g) Even though N has an equal correlation with portfolio P, its diversification aid is overwhelmed by its scale: N is just a lot riskier than Q. Thus, the portfolio with N is definitely more risky than the portfolio with Q. This lesser diversification effect is reflected in the (higher) covariance and the (higher) beta with our portfolio P, but not in the (equal) correlation.

5. This is not a question you might necessarily know how to answer. However, it should get you to think. Exxon is indeed a low-risk stock. However, it is also a low mean stock. You would have done a little better purchasing the diversified Dow-30 index (remember: indexes do not count dividends!). However, it is reasonably likely that in the future, the diversified Dow-Jones 30 index will have lower risk than Exxon. More than likely, Exxon was just lucky over the sample period. For example, oil prices were relatively stable.

6. There is almost no difference in risk between these two indexes. To reduce risk further, you should instead invest in asset classes that are different from large corporate equities.

7. They allow investors with limited amounts of money to own large numbers of securities, without incurring large transaction costs.

8. No. You should hold all sorts of other assets that do not correlate too highly with your existing portfolio (wealth).

9. No. There are many alternative asset classes, such as commodities, art, and real estate, etc., which are part of the market portfolio.

10. That they are zero. If they were not zero, it would mean that you could use past returns to predict future returns. Presumably, this would allow you to earn extra returns.

11. The standard deviation is likely to be

$$\begin{aligned} Sdv_{\text{annual}}(\tilde{r}) &\approx \sqrt{T} \cdot Sdv_{\text{daily}}(\tilde{r}) \\ 20\% &\approx \sqrt{255} \cdot Sdv_{\text{daily}}(\tilde{r}) \\ \Rightarrow \quad Sdv_{\text{daily}}(\tilde{r}) &\approx 1.25\% \end{aligned}$$

This corresponds to a typical daily movement of about 11 points. In statistical terms, if the S&P500 is about normally distributed, about $2/3$ of all days, it should move up or down no more than 11 points. In about $9/10$ of all days, it should move up or down no more than 22 points.

12. $\sqrt{12} \cdot 5\% \approx 17.3\%$.

13. For the annual Sharpe-ratio, you can use the numbers in the Table 11.1:

$$\text{Annual Sharpe Ratio} = \frac{6.3\% - 3.0\%}{18.7\%} \approx 0.18$$

For the monthly Sharpe-ratio, you need to de-annualize the mean and standard deviation. The excess mean is $3.3\%/12 \approx 0.275\%$. The monthly standard deviation is about $18.7\%/\sqrt{12} \approx 5.4\%$. Therefore, the monthly

$$\text{Monthly Sharpe Ratio} = \frac{0.275\%}{5.4\%} \approx 0.05$$

14. $\sqrt{255} \cdot 1\% \approx 16.0\%$.

15. Over 10 real days, you would have earned a compound rate of return of $988.61/967 - 1 = 2.2\%$. With 36.5 such 10-day periods over the year, the compound annual return would have been 124%. As to the daily rates of return, they were 0.47%, 1.51%, 0.40%, –0.24%, –1.20%, 0.92%, 1.28%, 1.03%, –1.00%. The simple arithmetic average rate of return was an even higher 0.35%/day (3.5% over the 10 days). Clearly, this was a great ten days, not likely to repeat. You should not trust these 10 day means. The standard deviation is 0.97%/day. This indicates an annualized standard deviation of about $\sqrt{365} \cdot 0.97\% \approx 18.5\%$. (Over 10 days, the estimated standard deviation is 3.07%/(**10 days**).) As is fairly common, the annualized standard deviation is fairly reasonable. Finally a risk-free rate of return of 3% per annum is less than 0.01%./day. The Sharpe-ratio would therefore be approximately $(0.35\% - 0.01\%)/0.97\% \approx 0.35$. Quoted in annual terms, the Sharpe-ratio would be approximately $(0.35\% - 0.01\%) \cdot 365/(0.97\% \cdot \sqrt{365}) = \sqrt{365} \cdot 0.35 \approx 6.7$. Quoted in monthly terms, it would be $\sqrt{30} \cdot 0.35 \approx 1.9$.

16.
   (a) The average rate of return is 10%. The variance is 37.5%%. The standard deviation is 6.12%. Therefore, its Sharpe-ratio is 1.63.
   (b) The new strategy has rates of return of +5%, +10%, +5%, +15%. It is very easy to accomplish this—give me the money. The average rate of return has declined to 8.75%. The standard deviation has declined to 4.146%. Therefore, the Sharpe ratio is 2.11.
   (c) Obviously, the second strategy of throwing money away is terrible. The fact that the Sharpe-ratio comes out higher tells us that the Sharpe-ratio is an awful measure of portfolio performance. And, yes, the Sharpe-ratio is indeed the most common fund performance measure in practical use. Sharpe ratio manipulation is particularly profitable for portfolio managers whose returns until October or November were positive. In this case, managers who want to maximize Sharpe ratios should try to avoid really high positive rates of returns. If it “happens,” they can always bring down the return, e.g., by paying themselves more money, or by buying illiquid securities and then marking them down to less than their values.

All answers should be treated as suspect. They have only been sketched and have not been checked.

CHAPTER 12§

The Efficient Frontier—Optimally Diversified Portfolios

How much of each security?

This chapter appears in the Survey text only.

YOU already know the following:

1. Diversification reduces risk.
2. The covariance between investment asset returns determines the effectiveness of diversification. Beta is an equally good measure.
3. Because risk typically decreases approximately by the square root of the number of securities in the portfolio, it is especially important to have at least a handful and better a dozen of very different types of assets.
4. Publicly traded stocks are mutually positively correlated, but not perfectly so. This leaves diversification a useful tool, but not a perfect one.

But you do not yet know the optimal portfolio weights that you should assign to individual assets in their portfolios. For example, should you purchase equal amounts in every security? Should you purchase relatively more of securities with higher expected rates of return, lower variances, or lower covariances? This chapter answers these questions.

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12·1 The Mean-Variance Efficient Frontier

You know that diversification reduces risk. Therefore, you know that investors like diversification—but this does not tell you how much your investors should purchase in each security. It may be better to purchase 25% in A and 75% in B, rather than 50% in each. How can you determine generally good investment weights—and the best investment weights for *you*? This question is the primary subject of this chapter, and the optimal portfolio is the force that ultimately shapes the CAPM formula.

12·1.A. The Mean-Variance Efficient Frontier With Two Risky Securities

You can now tackle the problem of finding the best portfolio, starting with only two possible investment securities.

Graphing Portfolios in Mean-Standard Deviation Space

For two securities, compute the expected rate of return and risk computations for possible portfolios.

With two securities, you have just one decision to make: how much to invest in the first security. (The remainder is automatically allocated to the second security). For example, assume that each security has a risk (standard deviation) of 10%. The first security (call it A) has an expected rate of return of 10%, and the second security (call it B) has an expected rate of return of 5%. Should you ever choose to purchase the second security, which is so obviously inferior to the first? You already know that the second security helps you diversify risk—so the answer may be yes—but let us work out the requisite trade-offs.

Recap the formulas you need.

Recall the formulas for the portfolio risk and reward. The formula for the portfolio reward, or expected rate of return, is

$$\begin{aligned}
\mathcal{E}(\tilde{r}_P) &= w_A \cdot 10\% + (1 - w_A) \cdot 5\% \\
\mathcal{E}(\tilde{r}_P) &= w_A \cdot \mathcal{E}(\tilde{r}_A) + (1 - w_A) \cdot \mathcal{E}(\tilde{r}_B)
\end{aligned} \tag{12.1}$$

You have also worked out the formula for the portfolio risk

$$\begin{aligned}
Sdv(\tilde{r}_P) &= \sqrt{w_A^2 \cdot 0.01 + (1 - w_A)^2 \cdot 0.01 + 2 \cdot w_A \cdot (1 - w_A) \cdot Cov(\tilde{r}_A, \tilde{r}_B)} \\
Sdv(\tilde{r}_P) &= \sqrt{w_A^2 \cdot Var(\tilde{r}_A) + (1 - w_A)^2 \cdot Var(\tilde{r}_B) + 2 \cdot w_A \cdot (1 - w_A) \cdot Cov(\tilde{r}_A, \tilde{r}_B)}
\end{aligned} \tag{12.2}$$

where w_A is any fractional weight that you may choose to invest in the first security. These two formulas allow you to work with any combination portfolio between the two securities.

You can compute the possible portfolio choices.

Begin by assuming a particular covariance, say, $Cov(\tilde{r}_A, \tilde{r}_B)$ of -0.0075. (Given the standard deviations, this implies a correlation of -75%.) What reward can you achieve and at what risk? You have only one decision variable, the weight w_A, but you can choose any weight you desire. Table 12.1 shows various risk/reward "performance pairs" (i.e., the portfolio mean and standard deviation). For example, the portfolio with $w_A = 2/3$ (and thus $w_B = 1/3$) has an expected rate of return of 8.30% and a standard deviation of 4.7%.

You can also draw each portfolio in a coordinate system of risk versus reward. Each portfolio is one point in x-y space.

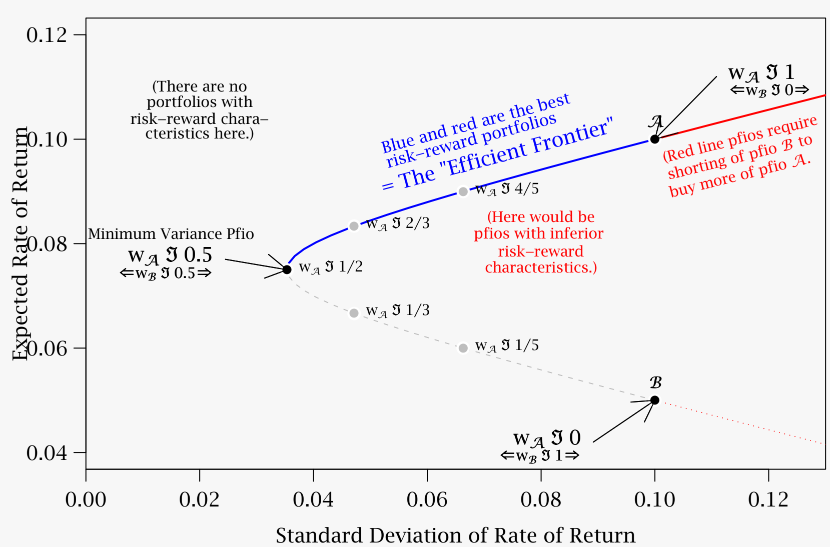
Figure 12.1 plots the data of Table 12.1 into a coordinate system, in which the overall portfolio standard deviation is on the x-Axis, and the overall portfolio expected rate of return is on the y-Axis. Each coordinate is a particular risk and reward characteristic. When you compute more risk-reward trade-offs (as in Table 12.1) that you can invest in, you find a curve that is called a **hyperbola**. The points on the hyperbola are the risk-reward combinations that you can achieve by varying w_A. For example, verify visually that your boldfaced portfolio ($w_A = 2/3, w_B = 1/3$) is on the hyperbola: it should be at $x = 4.7\%$ and $y = 8.3\%$. I have made it easier by drawing dashed lines to the X and Y axes for this portfolio. In this example, the smallest risk always obtains when the two securities are equally weighted. Above and below $w_A = 1/2$, the risk steadily increases.

Table 12.1: Portfolio Performance Under a Negative 75% Correlation.

w_A	w_B	$\mathcal{E}(\tilde{r}_P)$	$Sdv(\tilde{r}_P)$	w_A	w_B	$\mathcal{E}(\tilde{r}_P)$	$Sdv(\tilde{r}_P)$
1.0	0.0	10.0%	10.00%	0.5	0.5	7.5%	3.54%
0.9	0.1	9.5%	8.28%	0.4	0.6	7.0%	4.00%
0.8	0.2	9.0%	6.63%	0.3	0.7	6.5%	5.15%
0.7	0.3	8.5%	5.15%	0.2	0.8	6.0%	6.63%
2/3	1/3	**8.3%**	**4.71%**	0.1	0.9	5.5%	8.28%
0.6	0.4	8.0%	4.00%	0.0	1.0	5.0%	10.00%
Portfolios Involving Shorting							
−0.1	1.1	4.5%	11.77%	1.1	−0.1	10.5%	11.77%
−1	+2	0.0%	28.3%	+2	−1	15.0%	28.3%

The mean and standard deviation of the portfolio $w_A = 2/3$ (and thus $w_B = (1 - w_A) = 1/3$) are in bold. They were computed as follows: The portfolio has a mean of $2/3 \cdot 10\% + 1/3 \cdot 5\% \approx 8.3\%$, a variance of $(2/3)^2 \cdot 0.01 + (1/3)^2 \cdot 0.01 + 2 \cdot (2/3) \cdot (1/3) \cdot (-0.0075) \approx 0.00022$, and thus a risk of $\sqrt{0.00022} \approx 4.71\%$.

Figure 12.1: The Risk-Reward Trade-off With A Correlation of −75%.



The portfolios that invest fully in either A or B are marked with arrows and name. Other portfolio combinations are those on the curves. The fat blue curve is the mean-variance efficient frontier for portfolios that do not require shorting—portfolios that maximize reward given a desired level of risk. The fat red curve is also part of the efficient frontier, but consists of portfolios that require shorting B in order to purchase more of A. The lower part of the hyperbola is not mean-variance efficient—you would never purchase these. Incidentally, if you were allowed to throw away money, you could obtain any portfolio vertically below the efficient frontier—of course, you would never want to do this, anyway.

Similar portfolios are close to one another.

You know even more about where portfolios lie on the curves. Portfolios with similar weights must be close to one another in the graph: after all, both the expected rate of return and the standard deviations of a portfolio with 60% A and 40% B are very similar to the expected rate of return and the standard deviations of a portfolio with 61% A and 39% B. Therefore, the two portfolio points would have to be close. Thus, you can visually judge where a given portfolio combination must lie. For example, a portfolio with 95% weight on the A security and 5% weight on the B security would have to lie 95% of the way between the two on any of the curves, i.e., rather close to the $w_A = 1$ portfolio.

Where are portfolios with both long and short positions?

All portfolios that have positive weights in both securities lie between the two portfolios marked by arrows ($w_A = 1, w_B = 0$ and $w_A = 0, w_B = 1$). But Table 12.1 and Figure 12.1 also show portfolios that involve shorting (Section 8·2.B). For example, say you have \$200, but you want to invest \$220 in security A. This means that you have to short \$20 in security B, in order to obtain the money to have the full \$220 to invest in security A. Your investment weights are now $w_A = +110\%$ and $w_B = -10\%$. Please convince yourself that this portfolio—and any other portfolio that involves shorting security B—lie above the upper arrows in Figure 12.1. You can read the mean and standard deviation in Table 12.1, and then mark it in Figure 12.1.

Achievable Combinations, Unachievable Combinations, and The Efficient Frontier

Unobtainable and Inferior Risk-Reward Choices.

Now look again at Figure 12.1. There are some unobtainable regions—for example, combinations that would give you a portfolio with a 4% risk and a 10% reward. Similarly, you cannot purchase a portfolio with a 3% risk and a 7% reward. There are also some portfolios that are outright "ugly." You would never want to purchase more in security B than in security A—or you end up on the lower arm of the hyperbola. In general, you would also never want any portfolio in the red area (even if you could obtain it, though this is not possible in our example).

The highest achievable expected rates of return for each choice of risk is called the "Mean-Variance Efficient Frontier."

There is really only one set of portfolios that you would choose if you are sane—those on the upper boundary of the achievable set of portfolios. Only the portfolios on the upper arm of the hyperbola give you the highest reward for a given amount of risk. Which of these you would choose is a matter of your taste, however—if you are more risk-tolerant, you would purchase a portfolio farther to the right and top than if you are more risk-averse. The set of points/portfolios yielding the highest possible expected rate of return for a given amount of risk is called the **mean-variance efficient frontier**, or, in brief, the **efficient frontier**. This term will be used so often that we shall abbreviate "mean-variance efficient" as MVE.

Some confusasive namings.

Actually, mean-variance efficient frontier is a doubly bad name. First, the traditional graph draws the standard deviation on the "risk" or x axis. This graph should be called the "mean-standard deviation" graph, but it is commonly called the "mean-variance" graph. Fortunately, the two look fairly similar visually. And it does not matter whether you call it the mean-variance frontier or the mean-standard deviation frontier—they are really the same, in that a portfolio with lower variance also has lower standard deviation. Second, the efficient frontier here has nothing to do with the concept of **efficient markets** that you will learn in Chapter 16. Because the term *efficient* is so often used in Economics, it is often confusing.

You can solve for the functional form of all weighted combinations of two portfolios by substituting out w_A from Formula 12.1 (portfolio P's mean) and Formula 12.2 (portfolio P's standard deviation). The result is

$$\mathcal{Sdv}\left[\ \mathcal{E}(\tilde{r}_P)\ \right] = \pm\sqrt{a \cdot \mathcal{E}(\tilde{r}_P)^2 + b \cdot \mathcal{E}(\tilde{r}_P) + c}$$

where

$$\begin{aligned}
a &\equiv d^{-1} \cdot \mathcal{Var}(\tilde{r}_A) - 2 \cdot \mathcal{Cov}(\tilde{r}_A, \tilde{r}_B) + \mathcal{Var}(\tilde{r}_B) \\
b &\equiv d^{-1} \cdot 2 \cdot \mathcal{E}(\tilde{r}_A) \cdot [\mathcal{Cov}(\tilde{r}_A, \tilde{r}_B) - \mathcal{Var}(\tilde{r}_B)] \\
&\quad + d^{-1} \cdot 2 \cdot \mathcal{E}(\tilde{r}_B) \cdot [\mathcal{Cov}(\tilde{r}_A, \tilde{r}_B) - \mathcal{Var}(\tilde{r}_A)] \\
c &\equiv d^{-1} \cdot \mathcal{E}(\tilde{r}_B)^2 \cdot \mathcal{Var}(\tilde{r}_A) + \mathcal{E}(\tilde{r}_A)^2 \cdot \mathcal{Var}(\tilde{r}_B) \\
&\quad - d^{-1} \cdot 2 \cdot \mathcal{E}(\tilde{r}_B) \cdot \mathcal{E}(\tilde{r}_A) \cdot \mathcal{Cov}(\tilde{r}_A, \tilde{r}_B) \\
d &\equiv [\mathcal{E}(\tilde{r}_A) - \mathcal{E}(\tilde{r}_B)]^2
\end{aligned} \tag{12.3}$$

This formula states that the *variance* of the rate of return ($\mathcal{Sdv}(\tilde{r}_P)^2$) on an arbitrary weighted portfolio P is a quadratic formula in its expected rate of return ($\mathcal{E}(\tilde{r}_P)$): a **parabola**. Therefore, the MVE Frontier function is a hyperbola in a graph of the *expected rate of return* against their *standard deviations*.

Of course, the efficient frontier is only the upper arm of the hyperbola. Although this is by no means obvious, it turns out that this is the case even if there are more than 2 securities: the MVE Frontier is always the upper arm of a hyperbola—the combination of two particular portfolios. This only breaks down when there are additional constraints, such as short-sales constraints.

DIG DEEPER

12·1.B. Different Covariance Scenarios

Table 12.2: Portfolio Performance in Different Covariance Scenarios.

			Correlation is				
			−1.00	−0.75	0.00	+0.75	+1.00
			Risk ($\mathcal{Sdv}$) if Covariance is				
w_A	w_B	$\mathcal{E}(\tilde{r}_P)$	−0.01	−0.0075	0	+0.0075	+0.01
1	0	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%
0	1	5.0%	10.0%	10.0%	10.0%	10.0%	10.0%
1/5	4/5	6.0%	6.0%	6.6%	8.2%	9.6%	10.0%
1/3	2/3	6.8%	3.3%	4.7%	7.5%	9.4%	10.0%
1/2	1/2	7.5%	0.0%	3.5%	7.1%	9.3%	10.0%
2/3	1/3	**8.3%**	3.3%	**4.7%**	7.5%	9.4%	10.0%
4/5	1/5	9.0%	6.0%	6.6%	8.2%	9.6%	10.0%
Portfolios Involving Shorting:							
−1	+2	0.0%	30.0%	28.3%	22.4%	14.1%	10.0%
+2	−1	15.0%	30.0%	28.3%	22.4%	14.1%	10.0%

This figure shows the risk and reward under different assumptions about the covariance between A and B. The first two lines give the characteristics of the two base securities, A and B. (w_B is always $1 - w_A$, and thus could have been omitted.)

How do efficient frontiers look like with different correlations.
↦ Fig 12.2 on Page 267

How do different asset correlations between securities A and B change the shape of the efficient frontier. Table 12.2 is really the same as Table 12.1, except it works out different covariance scenarios. And Figure 12.1 is really the same as Figure 12.2. It plots the data into coordinate systems, in which the overall portfolio standard deviation is on the x-Axis, and the overall portfolio expected rate of return is on the y-Axis. The previous Figure 12.1 is now graph (b), just stretched due to the different axes.

Higher correlations "compress" the function.

Of course, regardless of covariance, when you choose only one or the other security ($w_A = 0$ or $w_A = 1$), the portfolio risk is 10% (as was assumed), and here noted with arrows. These two points are identical in all graphs. More interestingly, when you repeat the exercise for different covariance scenarios, shown in Table 12.2 and Figure 12.2, you can verify your earlier insight that lower covariance helps diversification. For example, if the two securities are perfectly negatively correlated, which implies a covariance of −0.01, then an equal-weighted portfolio of the two securities has zero risk. (When one security's value increases, the other security's value decreases by an equal amount, thereby eradicating any risk.) If the two securities are perfectly positively correlated, diversification does nothing, and the portfolio standard deviation remains at 10% no matter what weights are chosen. Thus, adding securities with low covariance to your existing portfolio lowers your overall portfolio risk particularly well; adding securities with high covariance to your existing portfolio is less effective.

Solve Now!

Q 12.1 In the example in Table 12.2, if the covariance is −0.01 (Figure 12.2e), what should you do?

Q 12.2 Assuming no transaction and shorting costs, does going short in a security and going long in the same security produce a risk-free investment?

Q 12.3 In the example, with risk of 10% for each security, compute the standard deviations of various portfolios' returns if the covariance between the two securities is +0.005. Graph the mean against the standard deviation.

Q 12.4 Security A has a risk (standard deviation) of 20% and an expected rate of return of 6%; Security B has a risk of 30% and an expected rate of return of 10%. Assume the two securities have +0.80 correlation. Draw the MVE Frontier.

12·1.C. The Mean-Variance Efficient Frontier With Many Risky Securities

With more securities, the computations become cumbersome...

The MVE Frontier is easy to compute when you can choose only one investment weight, which then determines your relative allocation between the two assets. In contrast, if you can choose from three securities, you must consider combinations of two portfolio weights (the third weight is one minus the other two). This is a two-dimensional choice problem. With four securities, you have three portfolio weights to optimize, and so on.

...but the principle remains the same.

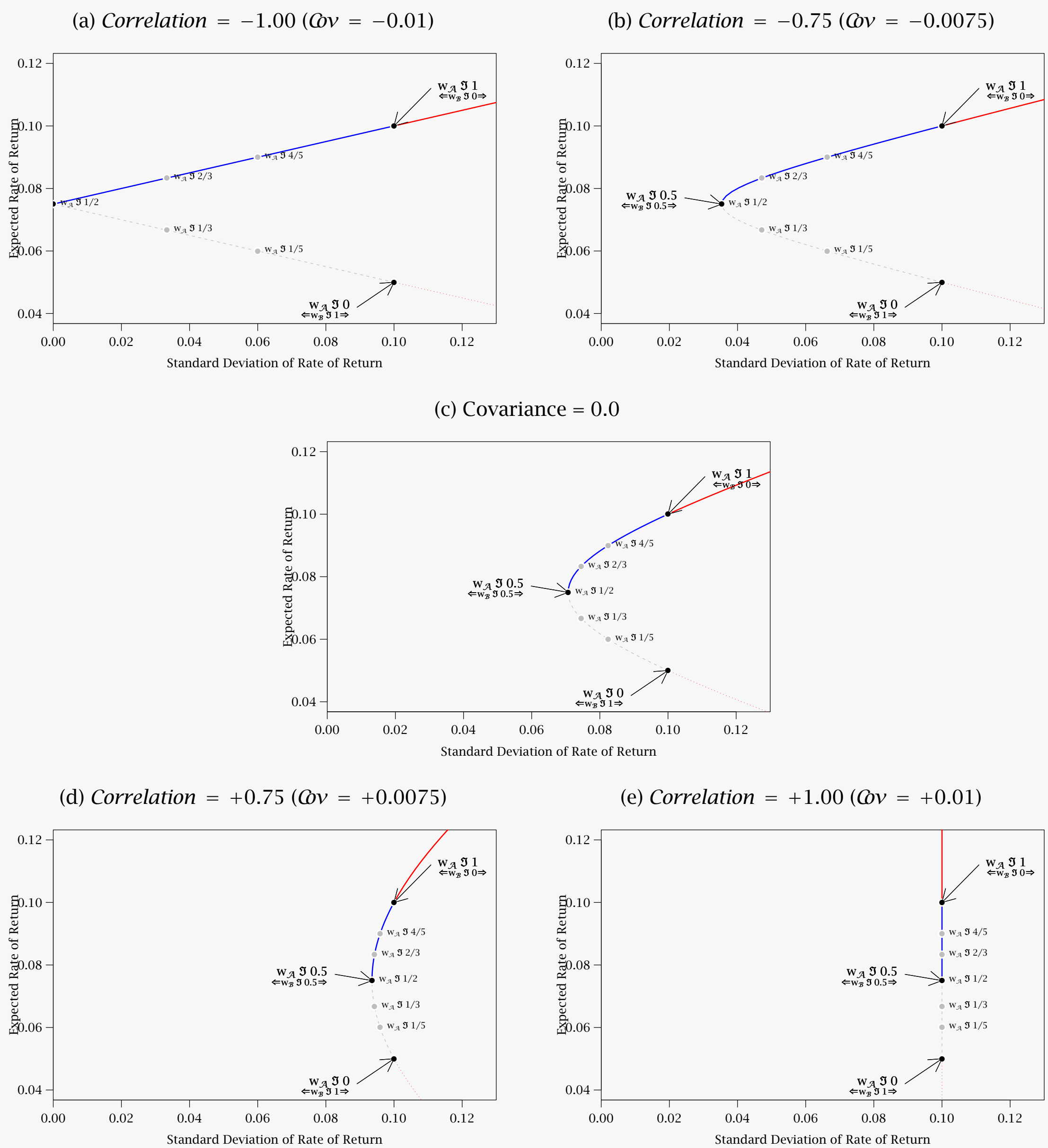
The good news is that the portfolio selection principle remains the same: each portfolio is a point in the graph of portfolio means (expected rates of return, i.e., reward) versus portfolio standard deviation (i.e., risk). After plotting all possible portfolios, you should choose from those points (portfolios) that lie on the MVE Frontier—which is still the upper half of the hyperbola.

Return to the three stocks, S&P 500, IBM, and Sony.

Try out many random investment portfolios from the three stocks from Page 189—S&P 500, IBM, Sony—and selecting the best ones. Of course, you would still assume that the historical annual rates of return from 1991 to 2002 are representative of the future, in the sense that we expect the historical means, variances, and covariances to apply in the future. (Otherwise, this would not be an interesting exercise!) These historical statistics were

		Covariance between $\tilde{r}_i$ and $\tilde{r}_j$		
Security	$\mathcal{E}(\tilde{r}_i)$	1=S&P 500	2=IBM	3=Sony
1=S&P 500	10.110%	3.6224%	3.2980%	4.7716%
2= IBM	15.379%	3.2980%	15.0345%	2.1842%
3= Sony	24.203%	4.7716%	2.1842%	81.4886%

Figure 12.2: The Risk-Reward Trade-off With Different Correlations



These figures repeat Figure 12.1, except that they consider different covariances between the portfolios. If the correlation is perfectly negative, you can manufacture a risk-free security by purchasing half of A and half of B. (If the correlation is perfectly positive, and if you can short B, you could even obtain a virtually infinite expected rate of return. This cannot happen in a reasonable market.)

The diagonal covariance elements are the variances, because of the way each is defined ($\mathcal{C}ov(x,x) \equiv \mathcal{V}ar(x)$). What would the risk-reward trade-offs of these portfolio combinations have looked like? Recall the portfolio statistics from Chapter 10. With three securities, the formulas to compute the overall portfolio mean and standard deviation are

$$\begin{aligned}
\mathcal{E}(\tilde{r}_P) &= w_1 \cdot \mathcal{E}(\tilde{r}_1) + w_2 \cdot \mathcal{E}(\tilde{r}_2) + w_3 \cdot \mathcal{E}(\tilde{r}_3) \\
\mathcal{V}ar(\tilde{r}_P) &= w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) + w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) + w_3^2 \cdot \mathcal{V}ar(\tilde{r}_3) \\
&\quad + 2 \cdot w_1 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) + 2 \cdot w_1 \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_3) \\
&\quad + 2 \cdot w_2 \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_3) \\
\text{and } w_3 &\equiv 1 - w_1 - w_2
\end{aligned}$$

Pick some portfolios and use the formulas.

Armed with formulas and statistics, you can now determine portfolio means and standard deviations for some portfolios (different weights of $w_{\text{S\&P 500}}, w_{\text{IBM}}, w_{\text{Sony}}$). For example, for the portfolio that invests 20% in S&P 500, 40% in IBM, and 40% in Sony, the expected rate of return is

$$\begin{aligned}
\mathcal{E}(\tilde{r}_P) &\approx 20\% \cdot 10.110\% + 40\% \cdot 15.379\% + 40\% \cdot 24.203\% \approx 17.85\% \\
\mathcal{E}(\tilde{r}_P) &= w_1 \cdot \mathcal{E}(\tilde{r}_1) + w_2 \cdot \mathcal{E}(\tilde{r}_2) + w_3 \cdot \mathcal{E}(\tilde{r}_3)
\end{aligned}$$

and the variance is

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &\approx (20\%)^2 \cdot 3.6224\% + (40\%)^2 \cdot 15.0345\% + (40\%)^2 \cdot 81.4886\% \\
&\quad + 2 \cdot (20\%) \cdot (40\%) \cdot 3.2980\% + 2 \cdot (20\%) \cdot (40\%) \cdot 4.7716\% \\
&\quad + 2 \cdot (40\%) \cdot (40\%) \cdot 2.1842\% \qquad \approx 0.1758 \\
\mathcal{V}ar(\tilde{r}_P) &= w_1^2 \cdot \mathcal{V}ar(\tilde{r}_1) + w_2^2 \cdot \mathcal{V}ar(\tilde{r}_2) + w_3^2 \cdot \mathcal{V}ar(\tilde{r}_3) \\
&\quad + 2 \cdot w_1 \cdot w_2 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_2) + 2 \cdot w_1 \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_1, \tilde{r}_3) \\
&\quad + 2 \cdot w_2 \cdot w_3 \cdot \mathcal{C}ov(\tilde{r}_2, \tilde{r}_3)
\end{aligned}$$

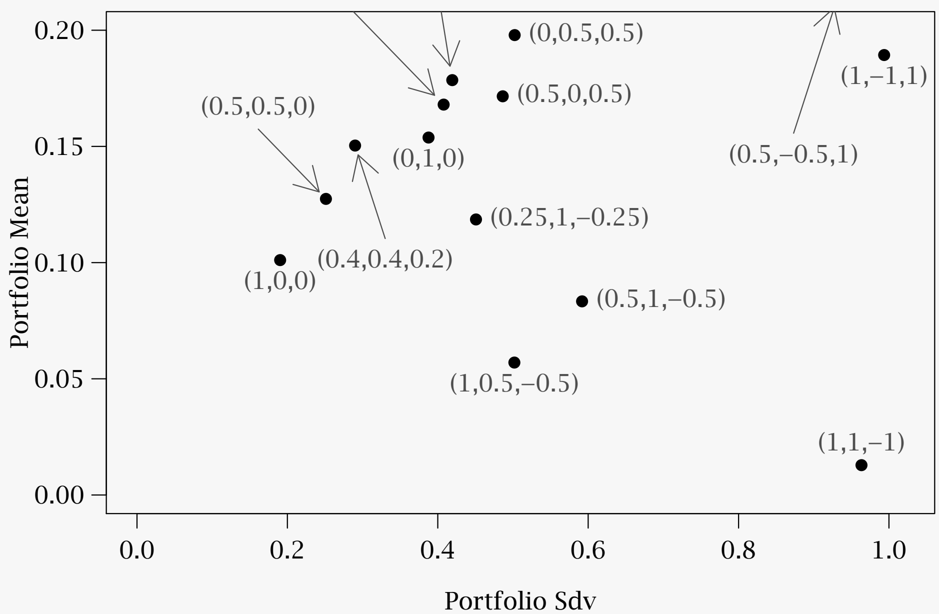
Therefore, the risk of the portfolio is $\mathcal{S}dv(\tilde{r}_P) \approx 41.93\%$.

Table 12.3: Risk and Reward of Hypothetical Portfolios Consisting Only of S&P 500, IBM, and Sony

Weights $(w_{\text{S\&P 500}}, w_{\text{IBM}}, w_{\text{Sony}})$	$\mathcal{E}(\tilde{r}_P)$	$\mathcal{S}dv(\tilde{r}_P)$	Weights $(w_{\text{S\&P 500}}, w_{\text{IBM}}, w_{\text{Sony}})$	$\mathcal{E}(\tilde{r}_P)$	$\mathcal{S}dv(\tilde{r}_P)$
(1,0,0)	0.1010	0.1903	(-1,1,1)	0.2947	0.9401
(0,1,0)	0.1537	0.3877	(1,-1,1)	0.1893	0.9936
(0,0,1)	0.2420	0.9027	(1,1,-1)	0.0128	0.9635
(0.5,0.5,0)	0.1274	0.2513	(1,0.5,-0.5)	0.0569	0.5018
(0.5,0,0.5)	0.1715	0.4865	(0.5,1,-0.5)	0.0833	0.5919
(0,0.5,0.5)	0.1979	0.5022	(0.5,-0.5,1)	0.2156	0.9332
(0.2,0.4,0.4)	0.1785	0.4193	(-0.5,0.5,1)	0.2683	0.9051
(0.4,0.2,0.4)	0.1680	0.4077	(0.4,0.4,0.2)	0.1503	0.2901
			(0.99,0.024, −0.014)	0.1004	0.1896

Portfolio means ($\mathcal{E}(\tilde{r}_P)$) and standard deviations ($\mathcal{S}dv(\tilde{r}_P)$) are quoted in percent per annum—and based on historical data. Of course, your computations are only useful if you assume that history is representative of the future, too.

Figure 12.3: The S&P 500, IBM, and Sony Portfolio Combinations From Table 12.3



Numbers in parentheses are the investment weights in the three portfolio assets (S&P 500, IBM, Sony). For example, a $200 portfolio that invests $200 in S&P 500, $200 in IBM, and –$200 in Sony is (+1,+1,-1), has a mean of about 1.3% and a standard deviation of about 96%, which places it in the lower right corner.

Table 12.3 lists some more randomly chosen portfolio combinations, and Figure 12.3 plots the data from this table. Looking at the set of choices, if you were extremely risk-averse, you might have chosen just to invest in a portfolio that was all S&P 500 (1,0,0). Indeed, from the set you already know, the portfolio with the absolute lowest variance among these 16 portfolios seems to be mostly an investment in the S&P 500. If you were more risk-tolerant, you might have chosen a portfolio that invested 20% in the S&P 500, 40% in IBM, and 40% in Sony (0.2,0.4,0.4). However, regardless of your risk aversion, the (0.5,0,0.5) portfolio, which invests 50% in S&P 500, zero in IBM, and 50% in Sony, would have been a poor choice: it would have had a lower portfolio mean *and* higher standard deviation than alternatives, such as the (0.2,0.4,0.4) portfolio.

Plot the mean and standard deviations of these portfolios.

Note also that again, portfolios with very similar portfolio weights have similar means and standard deviations, and therefore lie close to one another. For example, the portfolio that invests (0.5,0.5,0) is relatively close to the portfolio (0.4,0.4,0.2).

Similarity in portfolio weights means proximity in the Figure.

Figure 12.4 plots 10,000 further portfolios, randomly chosen. The possible investment choices are no longer the single curve of points that they were with two assets, but a cloud of points. Nevertheless, the MVE Frontier—the upper left portion of the cloud of points—looks remarkably similar to the hyperbolic shape it had with just two assets. You should never purchase a portfolio that is a point inside the cloud but not on the frontier: there are better portfolios with higher rewards and lower risks towards its upper left. The **minimum variance portfolio** is the portfolio with the lowest variance. It is mostly invested in the S&P 500, although a magnifying glass reveals that adding a little of IBM and shorting a little of Sony is slightly better—specifically, 99% of S&P 500, 2.4% of IBM, and −1.4% of Sony could reach as low a risk as 18.96%.

Plotting the mean and standard deviations of many random portfolios shows that the set of feasible portfolios produces a cloud of points, with a distinct border: the MVE Frontier.

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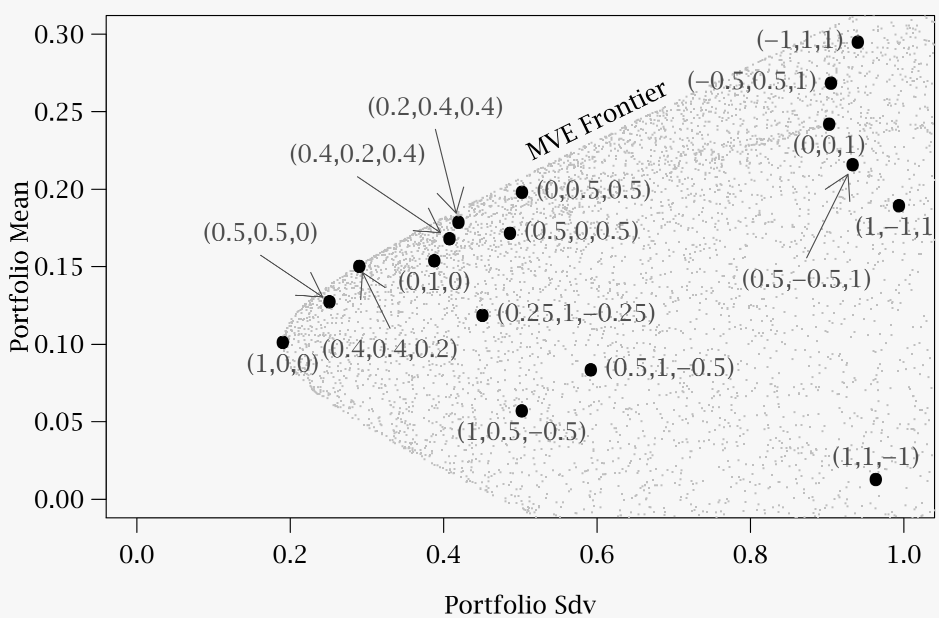
SIDE NOTE

The technique for obtaining the MVE Frontier can remain the same for more than three securities: select many possible portfolio combinations and plot the outcomes. However, the number of possible portfolio weights quickly becomes overwhelming. If the portfolio optimization is done through such trial-and-error, commonly called a **Monte-Carlo simulation**, it would be better to draw not just totally random portfolio weights, but to draw portfolio weights that lie closer to the best portfolio combinations already obtained. In this case, you can use a more convenient way to obtain the MVE Frontier, using matrix algebra. This is done in Nerd Appendix a.

Figure 12.4: 10,000 Randomly Chosen Portfolios Involving Only S&P 500, IBM, and Sony



Numbers in parentheses are the investment weights in the three portfolio assets (S&P 500, IBM, Sony). For example, a $200 portfolio that invests $100 in S&P 500, $200 in IBM, and −$100 in Sony (which is the point marked (0.5,1,−0.5)) has a mean of about 8% and a standard deviation of about 60%, which places it about in the middle of the gray area.

If you randomly drew another trillion portfolios, the entire region inside the hyperbola would be gray. That is, for any mean/standard deviation combination that is inside the hyperbola, you can find a set of portfolio weights that achieve it.

The historical MVE frontier, usually used, is not the same as what you really want to know: the future MVE frontier.

Forgive me for repeating myself, but you should keep in mind that this plot is *ex-post*, i.e., based on historical data. Your real interest is of course not the past, but the future. Unfortunately, you do not have a much better choice than to rely on history. In the real world, where you would use many more securities, the historical MVE Frontier would tend to be an *indicator* of the future MVE Frontier, but not a perfect *predictor* thereof. Use common sense!

SIDE NOTE

The historical portfolio mean estimates tend to be less reliable than the portfolio standard deviation estimates. Perhaps not surprisingly, academic research has found that the minimum variance portfolio (among broadly diversified portfolios) tends to do rather well—it often outperform other portfolio choices. That is, not only is its risk often lower than that of other portfolios (as might have been expected), but even its actual expected rate of return has often beaten its predicted expected rate of return (and other portfolios) out-of-sample.

Solve Now!

Q 12.5 For the three stocks, in a spreadsheet, randomly draw 1,000 random investment weights into the first security, 1,000 weights into the second security, and compute 1 minus these two weights to be the investment weight into the third security. Use the formulas and the following table to compute the risk and reward of each of the 1,000 portfolios in two separate columns.

		Covariance between $\tilde{r}_i$ and $\tilde{r}_j$		
Security	$\mathcal{E}(\tilde{r}_i)$	1	2	3
1	10.0%	0.010	0.001	0.003
2	5.0%	0.001	0.010	0.006
3	7.5%	0.003	0.006	0.010

Now plot the means and standard deviations of each portfolio in an X-Y graph.

Q 12.6 Continued: What is the **Minimum Variance Portfolio**?

Q 12.7 Continued: What is the best portfolio with an expected rate of return of about 11%?

Q 12.8 Continued: Would any investor purchase the portfolio (–12%,32%,81%) ?

12·2 Real-World Mean-Variance Efficient Frontier Implementation Problems

Figure 12.4 also hints at the main practical drawback of MVE portfolios. Many points on the MVE Frontier require shorting of securities (the weight is negative), but this may or may not be possible in the real world. Even if modest shorting is possible, the portfolio optimization often recommends strange portfolios that suggest shorting huge amounts in one security in order to go long a huge amount in another security.

Unfortunately, many portfolios on the MVE Frontier require shorting securities, and often massive amounts so.

Chances are that this is not because these are great portfolios, but because the historical covariance and mean estimates are not perfect predictors of future covariances and means. The historical covariance estimates cannot be easily relied upon, not so much because they are bad in themselves in an absolute sense, but because the optimization technique is very sensitive to *any* covariance estimation errors. This can be explained with a hypothetical bet optimizer: assume that you throw a coin 100 times, and observed 51 heads. A naïve portfolio optimizer relying on historical realizations would determine that the "heads bet" is much better than the "tails bet," and might recommend betting a million dollars on heads and against tails. This is not because the bet optimizer has made a mistake—after all, if it were truly 51%, this would be a pretty good bet. It is also not because 51% is a bad estimate of the true probability (of 50% if the coin is fair). Instead, it is the interaction: the bet optimizer is just very sensitive to historical data and therefore sampling error.

Here is why.

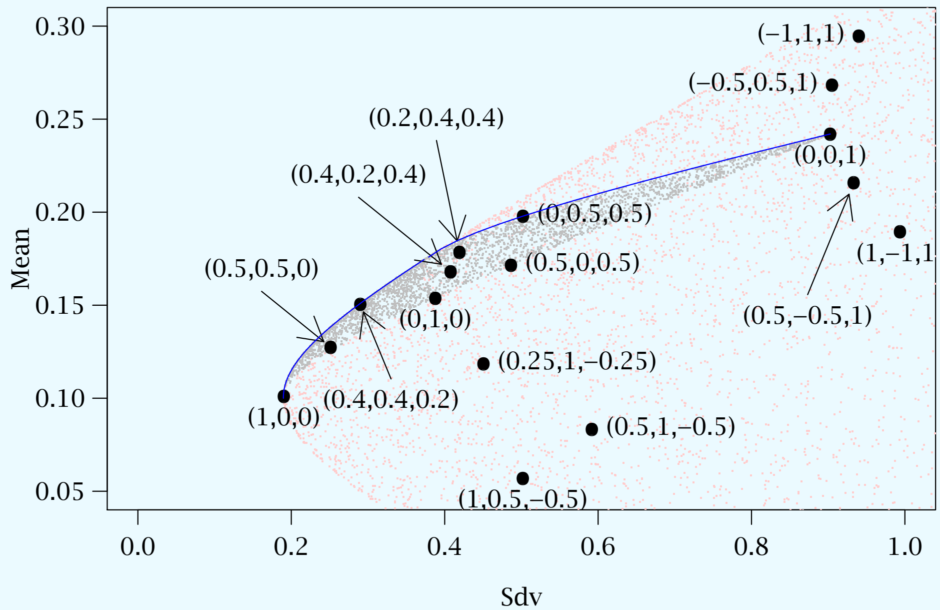
There are some methods which try to address this problem in order to make portfolio optimization a more useful tool. They rely on complicated statistical analysis, but simple ideas:

There are two solutions to the problem.

1. You can improve the estimates of future covariances and means, and not just rely blindly on their historical equivalents. In essence, these improvements rely on techniques that try to "pull in" extreme outlier returns. In the end, these techniques usually yield decent variance and covariance estimates.
2. You can use a model, like the CAPM (to be discussed in the next chapter) to better estimate means and variances.
3. You can use a portfolio optimizer that restricts the amount that can be shorted. This can be done by assuming a (high) cost of shorting, or by disallowing short-sales altogether.

↦ *Next Digging Deeper Box*

However, these techniques often still fail, especially when it comes to reliable *expected* rate of return estimates. In any case, you are warned—you should not blindly believe that the historical mean rates of return are representative of future mean rates of return. No one really knows how to estimate future expected rates of return *well.* In the real world, you must use your own judgment when the portfolio optimization result—your efficient frontier—seems reasonable and when it does not.



The portfolios with only positive investment weights (no shorting) are highlighted in gray in the graph on the right. It shows that an investor who is not permitted to short assets has a different MVE frontier. The short-constrained MVE frontier usually lies entirely inside the unconstrained MVE frontier.

DIG DEEPER

There is not enough data to estimate a good covariance matrix for 10,000 securities.

There is a closely related other problem with optimization. To obtain meaningful results, you need to have at least as many time periods (return observations) as there are terms in the covariance formula, just as you would need at least as many equations as you have unknowns to pin down a system of equations. Alas, with 10,000 securities, there are 50 million terms in the covariance formula. If you use daily returns, you would have to wait 196 thousand years to have just one data point per estimated covariance. (Mathematically, you could do the estimation with 10,000 daily data points—still 40 years. These would be fairly unreliable, of course.) Therefore, you cannot reliably estimate a good variance-covariance matrix with historical data for too many assets.

It is better to use portfolio optimization on asset classes than on many individual securities.

Consequently, mean-variance optimization can only be used when there are just a few portfolios—preferably broad asset-class portfolios—to choose from. Fortunately, such broad asset-class portfolios also tend to have low *and more reliable* historical variances and covariances estimates. Unfortunately, you must narrow down your investment choices into a small number of asset class portfolios *before* you can use the MVE Frontier toolbox. In sum, portfolio optimization is a very usable technique—as long as you restrict yourself to just a few big asset classes for which you have good historical data. You should not use mean-variance optimization with poor variance-covariance estimates. If you try to use it for individual stocks, and/or when you do not have long historical returns data, and/or if you apply it blindly, chances are that your results will not be very satisfying. Enhanced mean-variance optimization techniques, such as those discussed in this section, are indeed in common use among professional investors. For example, one class of **hedge fund** called a **fund of funds** invests itself in other hedge funds. Many fund of funds first determine their hedge fund investment candidates, and then allocate their money according to an estimated mean-variance frontier among their candidates.

12·3 Combinations of Portfolios on The Efficient Frontier

Any combination of MVE portfolios is itself also MVE.

If you purchase two different portfolios on the MVE Frontier, is the resulting portfolio still on the MVE Frontier? Put differently, when two MVE investors marry, is their total portfolio still MVE?

Table 12.4: Two Base MVE Portfolios and Two Portfolio Combinations

	Portfolio:	E1	E2	E3	E4
	Weight in E1:	+100%	0%	+50%	-25%
	Weight in E2 :	0%	+100%	+50%	+125%
(net	S&P 500	+99.448%	-90.620%	+4.414%	-138.138%
hol-	IBM	+2.124%	+135.770%	+68.910%	+169.088%
dings)	Sony	-1.573%	+54.925%	+26.676%	+69.049%
Reward	$\mathcal{E}(\tilde{r}_P)$	10.000%	25.000%	17.500%	28.750%
Risk	$\mathcal{Sdv}(\tilde{r}_P)$	18.969%	67.555%	37.498%	83.271%
	$\mathcal{Cov}(\tilde{r}_{E1}, \tilde{r}_{E2}) = 0.03508$				

Four specific MVE portfolios are defined by their relative investment weights into E1 = (99.448%, 2.124%, −1.573%) and E2 = (−90.620%, +135.770%, +54.925%) in S&P 500, IBM, and Sony, respectively. They are named E1, E2, E3, and E4 for convenience. P is a variable that can be any of these four portfolios.

The specific portfolio example (Figure 12.5).

Let's try it with our three security example. I tell you two MVE portfolios, and you get to compute the expected rates of return and standard deviations of portfolios that are combinations of these two. My part is in Table 12.4. Portfolio E1 is the minimum variance portfolio—you already knew that it would be mostly S&P 500, and we have already mentioned it on Page 269. Portfolio E2 has a higher risk (67.6%) but also a higher expected rate of return (25%). Looking at its components, E2 requires shorting a large amount in the S&P 500, but in your perfect world of zero transaction costs and under the assumption that the historical estimates are correct estimates of the future, this is not a problem. Now, because you know the investment weights, you can write down their twelve annual historical rates of return to compute the covariance between E1 and E2 —which would come to 0.035. You could also write down the historical realizations of any weighted combination portfolio between E1 and E2 in order to compute the new portfolio's mean and standard deviation—or you can use the portfolio formulas, which is much quicker. For example, the portfolio that invests half in E1 and half in E2 would have

$$\mathcal{E}(\tilde{r}_{E3}) = 50\% \cdot 10\% + 50\% \cdot 25\% = 17.5\%$$

$$\mathcal{Var}(\tilde{r}_{E3}) = (50\%)^2 \cdot (18.969\%)^2 + (50\%)^2 \cdot (67.555\%)^2 + 2 \cdot (50\%) \cdot (50\%) \cdot 0.03408 = 0.1406$$

$$\mathcal{Sdv}(\tilde{r}_{E3}) = \sqrt{0.1406} = 37.498\%$$

↦ *Middle Formula 10.4 on Page 217*

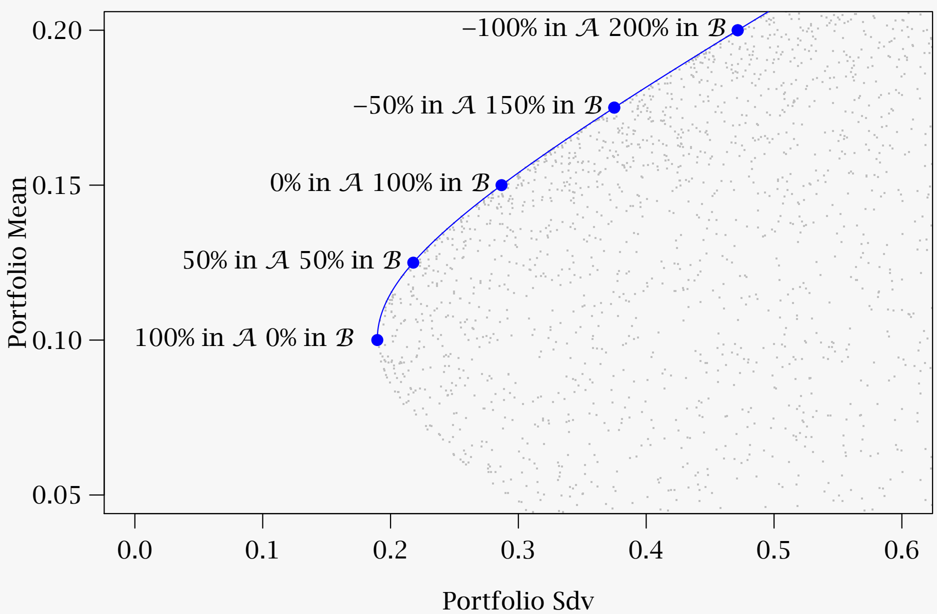
Please confirm the numbers for the E4 portfolio—or better yet, create a small spreadsheet that allows you to get quick mean/standard deviation values for any weight $w_{E1} = 1 - w_{E2}$ that you want to try.

It appears visually as if the combinations of two MVE portfolios is itself MVE.

After you have computed many such portfolio combinations, you can overlay their means and standard deviations onto Figure 12.4. This should give you something like Figure 12.5. Visually, it indeed appears as if combinations of the two MVE portfolios E1 and E2 "trace out" the entire MVE Frontier.

Visual confirmation is not a mathematical proof, so please trust me that this is more general—the answer to our original question is indeed yes.

Figure 12.5: S&P 500, IBM, and Sony Portfolio Combinations



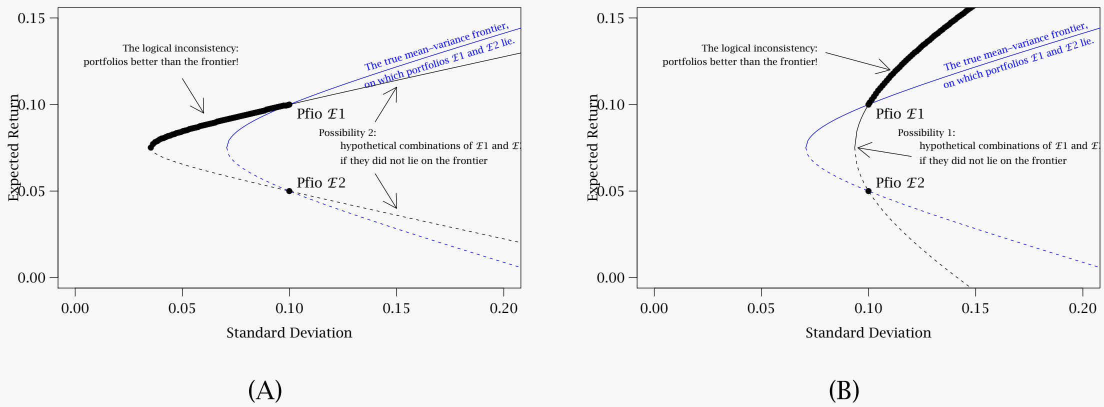
Portfolio E1 invests 99.448% in S&P 500, 2.124% in IBM, and -1.573% in Sony. Portfolio E2 invests -90.62% in S&P 500, +135.77% in IBM, and +54.925% in Sony. Other portfolios along the efficient frontier are combinations of E1 and E2 . Points to the right of the "100% in E2 " portfolio require shorting portfolio E1 in order to obtain the money to purchase more of portfolio E2.

IMPORTANT: *If* two investors hold portfolios that are mean-variance efficient, then the merged portfo also mean-variance efficient.

The non-graphical explanation.

Unfortunately, it requires Nerd Appendices b and c to prove that the combination of two MVE Frontier portfolios is also MVE. However, although the proof itself is not important, the following may give you an idea of how the proof works. You *must* believe me that the combination of two securities always forms a nice hyperbola without kinks—this is actually what Formula 12.3 states. There are also only three parameters that pin down the hyperbola, and limit how different inputs can stretch it. For convenience, my argument will work with the MVE-Frontier from Figure 12.2. Assume that this MVE-Frontier is the solid line in Figure 12.2, and contains the two MVE portfolios E1 and E2 . The question is whether combinations of E1 and E2 have to also lie along the solid line. Let's assume that they do not. The hyperbola can then extend beyond the efficient frontier, as in graph A—but then, the combination of E1 and E2 would best the original frontier. Or the hyperbola can lie inside the efficient frontier—but then shorting E2 to purchase E1 would extend the hyperbola above the original MVE frontier and best the original frontier. The only combinations of E1 and E2 that do not break the original MVE Frontier are the portfolios that lie on the MVE Frontier. This is what we stated: the combination of two MVE portfolios must itself be MVE.

Figure 12.6: Two MVE Portfolios



This figure illustrates that if E1 and E2 lie on the MVE frontier, then their combinations must be MVE. A hyperbola is defined by three points. Two points are already pinned down by E1 and E2. A hyperbola whose minimum variance portfolio is to the left of the current MVE would contradict the current MVE. A hyperbola whose minimum variance portfolio is to the right of the current MVE would contradict the current MVE if E2 is shorted to buy more of E1.

It is now easy to determine other portfolios on the MVE Frontier.

Because the combination of MVE portfolios is MVE, your task of finding the best portfolio—given your specific risk tolerance—is a lot easier. For example, if you want to know what the best portfolio P is that offers an expected rate of return of 15%, determine how much of portfolio E1 and how much of portfolio E2 you have to purchase in order to expect a rate of return of 15%. The answer is

$$\begin{aligned} w_{E1} \cdot 10\% + (1 - w_{E1}) \cdot 25\% &= 15\% \quad \Leftrightarrow \quad w = 2/3 \\ w_{E1} \cdot \mathcal{E}(\tilde{r}_{E1}) + (1 - w_{E1}) \cdot \mathcal{E}(\tilde{r}_{E2}) &= \mathcal{E}(\tilde{r}_P) \end{aligned}$$

Portfolio P should consist of $2/3$ of portfolio E1 and therefore $1/3$ of portfolio E2 . If you wish, you can think of portfolios E1 and E2 as two funds, so you can use the method from Section 8·3.B to compute the underlying net investment weights of portfolio E3:

Portfolio:	E1	E2	E3
Weight in E1	100%	0%	2/3
Weight in E2	0%	100%	1/3
S&P 500	+99.448%	-90.620%	+36.092%
IBM	+2.124%	+135.770%	+46.648%
Sony	-1.573%	+54.925%	+17.260%
$\mathcal{E}(\tilde{r})$	10.000%	25.000%	15.000%
$\mathcal{S}dv(\tilde{r})$	18.969%	67.555%	28.684%

P, which invests 36% in S&P 500, 47% in IBM, and 17% in Sony, is the best portfolio that offers an expected rate of return of 15% per annum; it requires you to take on 29% risk per year.

Solve Now!

Q 12.9 Among the three stocks, what is the portfolio that yields an expected rate of return of 12.5% per year?

Q 12.10 Among the three stocks, what is the portfolio that yields an expected rate of return of 20% per year?

12·4 The Mean-Variance Efficient Frontier With A Risk-Free Security

Risk-free securities do exist.

If a risk-free security is available, it turns out that the portfolio optimization problem becomes a lot easier. From a practical perspective, you can always assume that you can find a Treasury bond that is essentially risk-free. You can proceed under the assumption that you have access to one such security. This also means that you know already one MVE portfolio—after all, you should not be able to get a higher interest rate for a risk-free investment—or you could become rich.

SIDE NOTE

> The special role applies *only* to the risk-free security, not to any other kind of bond. Although we did discuss these *risky* bonds in the preceding parts of the book (Chapters 6), you should look at them the same way you would look at any equity stock or fund you might purchase. They are just other risky components inside your larger investment portfolio.

12·4.A. Risk-Reward Combinations of Any Portfolio Plus the Risk-Free Asset

It is very easy to compute the expected rate of return and risk for a portfolio in which one of the two securities is risk-free.

What are the achievable risk-reward combinations if there is a risk-free security? For example, consider portfolios from the combination of a risk-free security (called F) which pays 6% for sure, with only one risky security (called R) which pays a mean rate of return of 10.56% and has a standard deviation of 12%. The formulas for portfolio mean and standard deviation are

$$\begin{aligned}
\mathcal{E}(\tilde{r}_P) &= (1-w_R)\cdot 6\% &&+ \quad w_R \cdot 10.56\% \\
&= 6\% &&+ \quad w_R \cdot 4.56\% \\
\mathcal{E}(\tilde{r}_P) &= (1-w_R)\cdot \mathcal{E}(r_F) &&+ \quad w_R \cdot \mathcal{E}(\tilde{r}_R) \\
&= r_F &&+ \quad w_R \cdot [\mathcal{E}(\tilde{r}_R) - r_F]
\end{aligned} \tag{12.4}$$

Easy. More interestingly,

$$Sdv(\tilde{r}_P) = \sqrt{w_F^2 \cdot Var(r_F) + w_R^2 \cdot Var(\tilde{r}_R) + 2 \cdot w_F \cdot w_R \cdot Cov(\tilde{r}_R, r_F)}$$

But a risk-free rate of return is just a constant rate of return, so it has no variance and no covariance with anything else—it is what it is. Now combine your portfolio R with the risk-free rate, purchasing w_F in the risk-free asset and $w_R = (1-w_F)$ in R.

$$\begin{aligned}
Sdv(\tilde{r}_P) &= \sqrt{\cancel{w_F^2 \cdot Var(r_F)} + w_R^2 \cdot Var(\tilde{r}_R) + \cancel{2 \cdot w_F \cdot w_R \cdot Cov(\tilde{r}_R, r_F)}} \\
&= \sqrt{w_R^2 \cdot Var(\tilde{r}_R)} = w_R \cdot Sdv(\tilde{r}_R) = w_R \cdot 12\%
\end{aligned} \tag{12.5}$$

The combinations are a straight line in the mean-standard deviation graph.

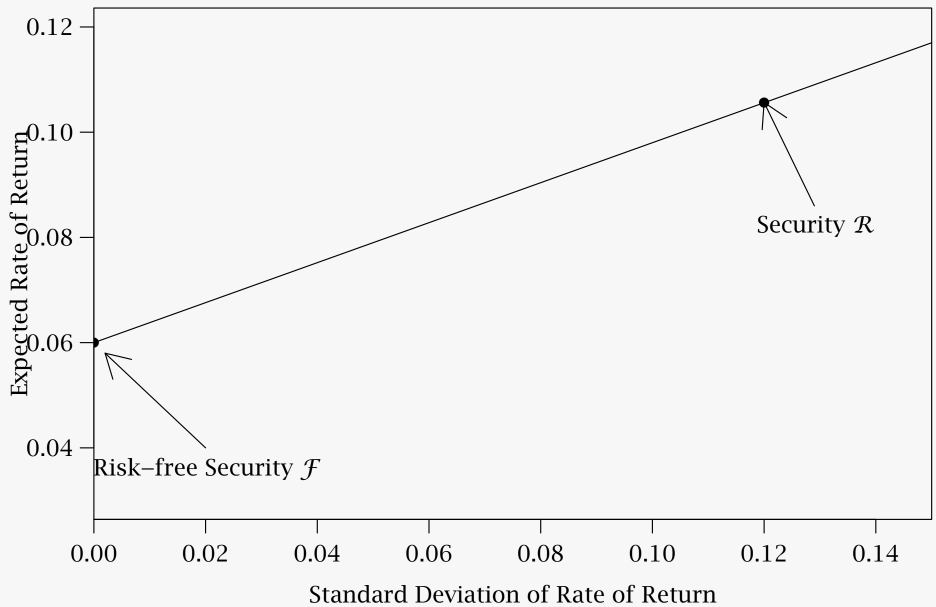
This is nice: the standard deviation of the rate of return on any portfolio is simply the weight times the standard deviation of the risky asset. This makes the calculations much easier. Figure 12.7 uses Formulas 12.4 and 12.5 to plot the mean and standard deviation of portfolio pairs for your particular security R. Note how all portfolio combinations lie on a straight line, which starts at the risk-free rate (at *Sdv*=0%, $\mathcal{E}$=6%) and goes through the risky portfolio R (at *Sdv*=12%, $\mathcal{E}$=10.56%).

You can prove the linearity, too!

It is easy to show that the combination of portfolios that consist of the risk-free security and a risky security have risk/reward trade-offs that lie on a straight line. Solve for w_R (=

Figure 12.7: Combinations of the Risk-Free Security (F) and a Risky Asset (R)

Risky Weight	0	1/4	1/2	3/4	1	3/2
Mean (Expected Rate of Return)	6%	7.14%	8.28%	9.42%	10.56%	12.84%
Standard Deviation	0%	3%	6%	9%	12%	18%



$Sdv(\tilde{r}_P)/Sdv(\tilde{r}_R)$) in the standard deviation formula 12.5, and substitute it into the expected return formula 12.4,

$$\begin{aligned}
\mathcal{E}(\tilde{r}_P) &= r_F + w_R \cdot [\mathcal{E}(\tilde{r}_R) - r_F] = r_F + \left[\frac{Sdv(\tilde{r}_P)}{Sdv(\tilde{r}_R)}\right] \cdot [\mathcal{E}(\tilde{r}_R) - r_F] \\
&= r_F + \left[\frac{\mathcal{E}(\tilde{r}_R) - r_F}{Sdv(\tilde{r}_R)}\right] \cdot Sdv(\tilde{r}_P) \\
&= 6\% + \left[\frac{10.56\% - 6\%}{12\%}\right] \cdot Sdv(\tilde{r}_P) \\
&= 6\% + 38\% \cdot Sdv(\tilde{r}_P) \\
&= a + b \cdot Sdv(\tilde{r}_P)
\end{aligned} \tag{12.6}$$

This is the formula for a line, where a (= r_F = 6%, the risk-free rate) is the intercept and b (= $[\mathcal{E}(\tilde{r}_R) - r_F]/Sdv(\tilde{r}_R)$ = 38%) is the slope. You have actually already encountered the slope in Formula 12.6 in Formula 11·5 It is the **Sharpe Ratio**,

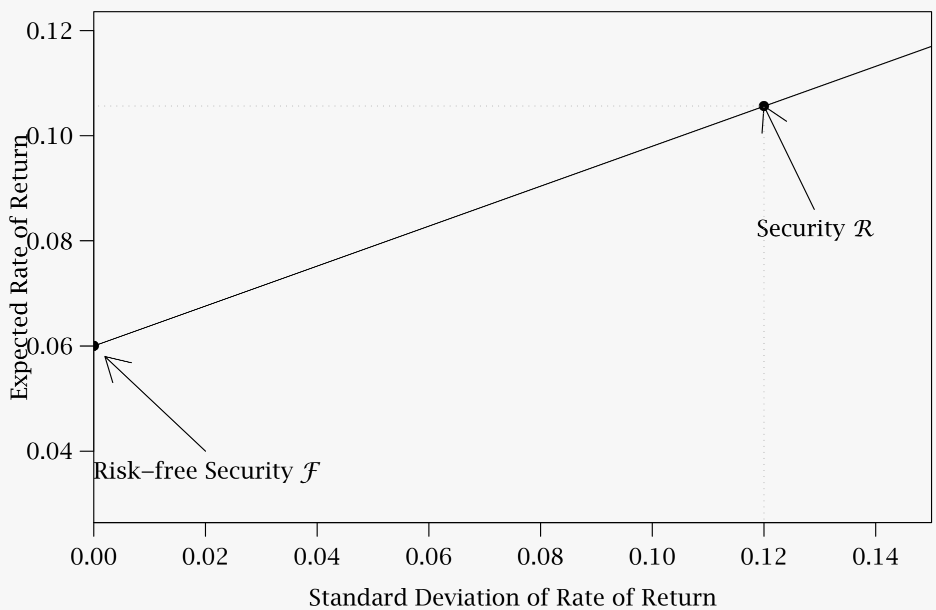
↦ *Sharpe Ratio Formula 11·5 on Page 255*

$$b = \text{Sharpe Ratio} \equiv \frac{\mathcal{E}(\tilde{r}_R) - r_F}{Sdv(\tilde{r}_R)}$$

For a given portfolio R, $[\mathcal{E}(\tilde{r}_R) - r_F]/Sdv(\tilde{r}_R)$ is the ratio of the expected reward over the required risk. (Theoretically, you can also compute the Sharpe ratio by first subtracting the risk-free rate from all returns, called excess returns, and then dividing its mean by its standard deviation, because $Sdv(\tilde{r}_R) = Sdv(\tilde{r}_R - r_F)$. This is the more common practical way to compute the Sharpe ratio.) Please be aware that although the Sharpe ratio is a common measure of the risk-reward trade-off of individual portfolios, it is not a good one.

12·4.B. The Best Risk-Reward Combinations With A Risk-Free Asset

Figure 12.8: Combinations of The Risk-Free Security (F) And a Risky Asset (R)



The MVE among risky assets is based on the example from Table 12.2 with a covariance of −0.0075. (This means the first security has a reward of 5% and a risk of 10%; the second has a reward of 10% and a risk of 10%.)

Would you still purchase a combination of R, if you could instead purchase any other combination of E1 and E2 ?

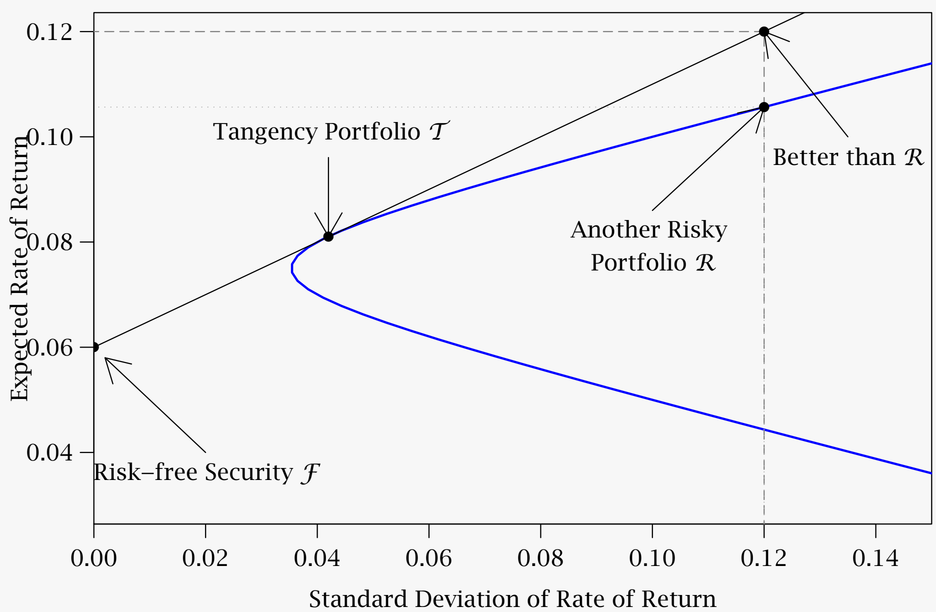
Now return to the efficient frontier from Table 12.2 and Figure 12.2, and assume the covariance is −0.0075. Portfolio R happens to be among the portfolios that you can purchase, combining E1 and E2 . Figure 12.8 repeats the last graph, but also plots the other securities you could purchase, putting together two base securities, E1 and E2 . This raises an interesting question: Assume that you have drawn the MVE Frontier using all risky assets. (It will always looks like a sideways hyperbola, so the drawn figure is accurate even for the more general case.) Now a risk-free asset becomes available, so that you can draw a line between the risk-free rate and whatever portfolio you want to purchase. How does your new achievable MVE Frontier look like? Would you ever purchase portfolio R in Figure 12.8, which is on the previous MVE Frontier (or any combination of the risk-free security and portfolio R)?

The true MVE Frontier is a line from the risk-free rate to the "tangency" portfolio.

No! You can do better than this. Figure 12.9 draws the "tangency portfolio," T. By combining portfolio T and the risk-free asset, you can do better than purchasing even a tiny bit of R. In fact, the figure shows that you would draw a line from the risk-free asset to this **Tangency Portfolio** (T)—which is the true MVE frontier—and you would only purchase combinations of these two. No risky portfolio other than T would ever be purchased by *any* rational investor.

IMPORTANT: If there is a risk-free security, every investor would purchase a combination of the risk-free security and the Tangency Portfolio. No combination of risky assets other than the Tangency Portfolio T would ever be purchased by any investor.

Figure 12.9: The Mean-Variance Efficient Frontier With a Risk-Free Asset



Note how you can do better if you buy a combination of T and the risk-free asset instead of portfolio R. It follows that you would never purchase risky assets in any other weight combination than that packed into the tangency portfolio T.

12·4.C. The Formula to Determine the Tangency Portfolio

It is not difficult to figure out the tangency portfolio. It is the portfolio on the MVE Frontier that has the maximum slope, i.e., the highest Sharpe ratio. Let us find the tangency portfolio for our three by-now familiar stocks. Recall the first three MVE portfolios for S&P 500, IBM, and Sony from Table 12.4 Recall that

↦ *MVE Pfio Table 12.4 on Page 273*

Portfolio:	E1	E2	E3
Weight in E1:	+100%	0%	+50%
Weight in E2 :	0%	+100%	+50%
S&P 500	+99.448%	-90.620%	+4.414%
IBM	+2.124%	+135.770%	+68.910%
Sony	-1.573%	+54.925%	+26.676%
$\mathcal{E}(\tilde{r}_P)$	10.000%	25.000%	17.500%
$\mathcal{S}dv(\tilde{r}_P)$	18.969%	67.555%	37.498%
Sharpe Ratio @ $r_F = 6\%$	$\frac{10\% - 6\%}{19\%} = 0.21$	$\frac{25\% - 6\%}{67.6\%} = 0.28$	$\frac{17.5\% - 6\%}{37.5\%} = 0.31$

The Agenda. Work out the Sharpe ratio, so mean first, covariance next, and standard deviation last.

First, you should determine the covariance between two efficient frontier portfolios—here between E1 and E2 . You can do this either by computing the twelve historical portfolio returns and compute it, or you can read the nerd note below, or you can just take it as given that $\mathcal{C}ov(\tilde{r}_{E1}, \tilde{r}_{E2}) = 0.03505$. You also know that you can trace out the MVE Frontier by purchasing any combinations of these three portfolios. Your goal is to find, for a given risk-free interest rate, the tangency portfolio. First, write down again the risk and reward of portfolios of E1 and E2. For any portfolio P itself comprised of E1 and E2,

$$\begin{aligned}
\mathcal{E}(\tilde{r}_P) &= w_{E1}\cdot 10\% + (1-w_{E1})\cdot 25\% = 25\% - 15\%\cdot w_{E1} \\
&= w_{E1}\cdot \mathcal{E}(\tilde{r}_{E1}) + (1-w_{E1})\cdot \mathcal{E}(\tilde{r}_{E2})
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}ar(\tilde{r}_P) &= w_{E1}^2\cdot(18.969\%)^2 + w_{E2}{}^2\cdot(67.555\%)^2 + 2\cdot w_{E1}\cdot w_{E2}\cdot 3.505\% \\
\mathcal{V}ar(\tilde{r}_P) &= w_{E1}^2\cdot \mathcal{V}ar(\tilde{r}_{E1}) + w_{E2}{}^2\cdot \mathcal{V}ar(\tilde{r}_{E2}) + 2\cdot w_{E1}\cdot w_{E2}\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2})
\end{aligned}$$

With a risk-free rate of 6%, the Sharpe Ratio for P is therefore defined to be

$$\text{SR}(w_{E1}) = \frac{\mathcal{E}(\tilde{r}_P) - r_F}{\mathcal{S}dv(\tilde{r}_P)} = \frac{25\% - 15\%\cdot w_{E1} - 6\%}{\sqrt{w_{E1}^2\cdot(18.969\%)^2 + (1-w_{E1})^2\cdot(67.555\%)^2 + 2\cdot w_{E1}\cdot(1-w_{E1})\cdot 3.505\%}}$$

You can now use a computer spreadsheet (or calculus) to determine the w_{E1} that maximizes this fraction. The solution is $w_{E1} \approx 68\%$, which means that the net portfolio weights in this tangency portfolio are

$$\begin{aligned}
w_{\text{S\&P 500}} &= 0.68\cdot(+99.448\%) + 0.32\cdot(-90.620\%) = 38.791\% \\
w_{\text{IBM}} &= 0.68\cdot(+2.124\%) + 0.32\cdot(135.770\%) = 44.751\% \\
w_{\text{Sony}} &= 0.68\cdot(-1.573\%) + 0.32\cdot(54.925\%) = 16.458\%
\end{aligned}$$

The mean rate of return of this tangency portfolio is 14.8%, the standard deviation is 28%, and the Sharpe ratio is 0.3138. You should henceforth only purchase a combination of the risk-free security and this tangency portfolio, with relative weights determined by your taste for risk.

DIG DEEPER

There are two methods to extract the covariance: one is brute force to manipulate the component security covariances. The more clever approach recognizes that you already know

$$\begin{aligned}
(18.969\%)^2 &= 1^2\cdot \mathcal{V}ar(\tilde{r}_{E1}) + 0^2\cdot \mathcal{V}ar(\tilde{r}_{E2}) + 2\cdot 1\cdot 0\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2}) \\
(67.555\%)^2 &= 0^2\cdot \mathcal{V}ar(\tilde{r}_{E1}) + 1^2\cdot \mathcal{V}ar(\tilde{r}_{E2}) + 2\cdot 0\cdot 1\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2}) \\
(37.498\%)^2 &= 0.5^2\cdot \mathcal{V}ar(\tilde{r}_{E1}) + 0.5^2\cdot \mathcal{V}ar(\tilde{r}_{E2}) + 2\cdot 0.5\cdot 0.5\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2}) \\
\mathcal{V}ar(\tilde{r}_P) &= w_{E1}^2\cdot \mathcal{V}ar(\tilde{r}_{E1}) + w_{E2}{}^2\cdot \mathcal{V}ar(\tilde{r}_{E2}) + 2\cdot w_{E1}\cdot w_{E2}\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2})
\end{aligned}$$

which simplifies into

$$\begin{aligned}
(18.969\%)^2 &= \mathcal{V}ar(\tilde{r}_{E1}) \\
(67.555\%)^2 &= \mathcal{V}ar(\tilde{r}_{E2}) \\
(37.498\%)^2 &= 0.25\cdot \mathcal{V}ar(\tilde{r}_{E1}) + 0.25\cdot \mathcal{V}ar(\tilde{r}_{E2}) + 2\cdot 0.5\cdot 0.5\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2}) \\
&= 0.25\cdot(18.969\%)^2 + 0.25\cdot(67.555\%)^2 + 0.5\cdot \mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2})
\end{aligned}$$

Solving, you find that $\mathcal{C}ov(\tilde{r}_{E1},\tilde{r}_{E2}) = 0.03505$.

12·4.D. Combining The Risk-Free Security And the Tangency Portfolio

The special case with a risk-free security.

If a risk-free security exists, so that the "true" MVE Frontier is the straight line tracing out all combinations of the risk-free security and the tangency portfolio T, then the fact that the combination of two MVE portfolios is still MVE is easy to understand. Say, we want to marry two portfolios. The groom owns \$100 in the tangency portfolio T, and \$900 in the risk-free asset. The bride owns \$2,000 in T, purchased partly with \$500 worth of debt. The couple then owns $\$100 + \$2,000 = \$2,100$ in T and $\$900 - \$500 = \$400$ in the risk-free asset. This is still an MVE portfolio, because it consists only of the risk-free asset and the risky tangency portfolio T. The fact that the combination of MVE portfolios is itself MVE is the key to the CAPM, as you shall learn in the next section.

DIG DEEPER

However, the opposite statement—*if the merged portfolio is mean-variance efficient, then the individual components are mean-variance efficient*—is not true. It is possible that E1 and E2 do not invest in T, but that their merged investment-weighted stock market portfolio is T. For example, allow the tangency portfolio to consist itself of two securities, X and Y, say in proportions $1/3$ and $2/3$, respectively. If the groom holds \$300 in X and the bride holds \$600 in Y, then the couple's portfolio is MVE. But each individual's portfolio was not. Who says marriage does not pay?

Solve Now!

Q 12.11 For the three securities, S&P 500, IBM, and Sony, what is the Sharpe ratio of a portfolio that invests 50% in the risk-free security and 50% in the tangency portfolio T?

Q 12.12 What is the tangency portfolio if the risk-free rate were not 6%, but 3%? Before solving it, think where it should lie geometrically.

Q 12.13 What is the tangency portfolio if the risk-free rate were not 6%, but 9%? Before solving it, think where it should lie geometrically.

12·5 What does a Security need to offer to be in an Efficient Frontier Portfolio?

12·5.A. What if the Risk-Reward Relationship is Non-Linear?

Some Portfolios.

↦ *Fig 12.4 on Page 270*

Return to Figure 12.4, which graphs the risk and reward for many portfolios. Among them are

Portfolio	$w_{\text{S\&P 500}}$	w_{IBM}	w_{Sony}	$\mathcal{E}(\tilde{r}_P)$	$Sdv(\tilde{r}_P)$
G	0.4000	0.2000	0.4000	16.80%	40.77%
E3	0.04414	0.6891	0.2668	17.50%	37.50%
I	0.000	1.000	0.000	15.38%	38.77%

Portfolio E3 is better than G. The investor should buy less in "expensive" stocks and buy more of "cheap" stocks.

The G portfolio is clearly inefficient: it has a lower expected rate of return and a higher standard deviation than the E3 portfolio. Comparing the two portfolios, it appears as if the G portfolio has too much of the S&P 500, too little of IBM, and too much of Sony. Put differently, if you were holding portfolio G, then S&P 500 and Sony would really be too expensive for you, while IBM would be a bargain. You would be better off rebalancing away from the S&P 500 and Sony securities, and towards IBM.

Whether a stock is too expensive or too cheap depends not just on the stock itself, but on the portfolio for which it is begin considered.

Whether a stock is too expensive depends on your current portfolio. You can choose an example of a portfolio in which IBM appears too expensive, instead. For example, say that you are holding portfolio I—IBM only. Again, you would be better off with portfolio E3 instead, but it has smaller holdings in IBM (compared to I), not more. From your perspective as an owner of portfolio I who wants to get the better portfolio E3, the S&P 500 and Sony would look like bargains, because they would help you to reduce your portfolio risk, while IBM would look too expensive.

IMPORTANT: The risk contribution of a security depends on the particular portfolio. For some portfolios, a security may appear like a bargain, for others like a gouge. The process by which you can improve your portfolio is to purchase securities that appear like a bargain—high reward given their risk contribution to *your* portfolio; and to divest securities that appear too expensive—low reward given their risk contribution to *your* portfolio.

The annual portfolio returns, including portfolio E1:

It turns out that this insight is the basis for an important method to measure the risk/reward characteristics of an individual security, i.e., whether this individual security is a bargain, given your specific overall portfolio. Assume you are holding portfolio G. Its annual rates of return would have been

Year	S&P 500	IBM	Sony	Pfio G	Year	S&P 500	IBM	Sony	Pfio G
1991	0.263	−0.212	−0.103	0.022	1997	0.310	0.381	0.391	0.356
1992	0.045	−0.435	−0.004	−0.070	1998	0.267	0.762	−0.203	0.178
1993	0.071	0.121	0.478	0.244	1999	0.195	0.170	2.968	1.299
1994	−0.015	0.301	0.135	0.108	2000	−0.101	−0.212	−0.511	−0.287
1995	0.341	0.243	0.105	0.227	2001	−0.130	0.423	−0.348	−0.107
1996	0.203	0.658	0.077	0.244	2002	−0.234	−0.357	−0.081	−0.197
					$\mathcal{E}$	0.101	0.154	0.242	0.168
					$\mathcal{V}ar$	0.037	0.150	0.815	0.166
					$\mathcal{S}dv$	0.190	0.388	0.903	0.408

Compute the beta of each stock with respect to portfolio G.

↦ *Covariance Formula: 9.2 on Page 203*

Now compute your measure of how each security helps diversifying portfolio G—the covariance between each security and portfolio G. These covariances may be tedious to compute, but they are not difficult, and you have done this before

$$\begin{aligned}
\mathcal{C}ov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{G}}) &= \frac{(+0.2631 - 0.101)\cdot(0.0217-0.168) + \cdots + (-0.2337-0.101)\cdot(-0.1971-0.168)}{11} \\
&= 0.0402 \\
\mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{G}}) &= \frac{(-0.2124 - 0.154)\cdot(0.0217-0.168) + \cdots + (-0.3570-0.154)\cdot(-0.1971-0.168)}{11} \\
&= 0.0520 \\
\mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{G}}) &= \frac{(-0.1027 - 0.242)\cdot(0.0217-0.168) + \cdots + (-0.0808-0.242)\cdot(-0.1971-0.168)}{11} \\
&= 0.3494
\end{aligned}$$

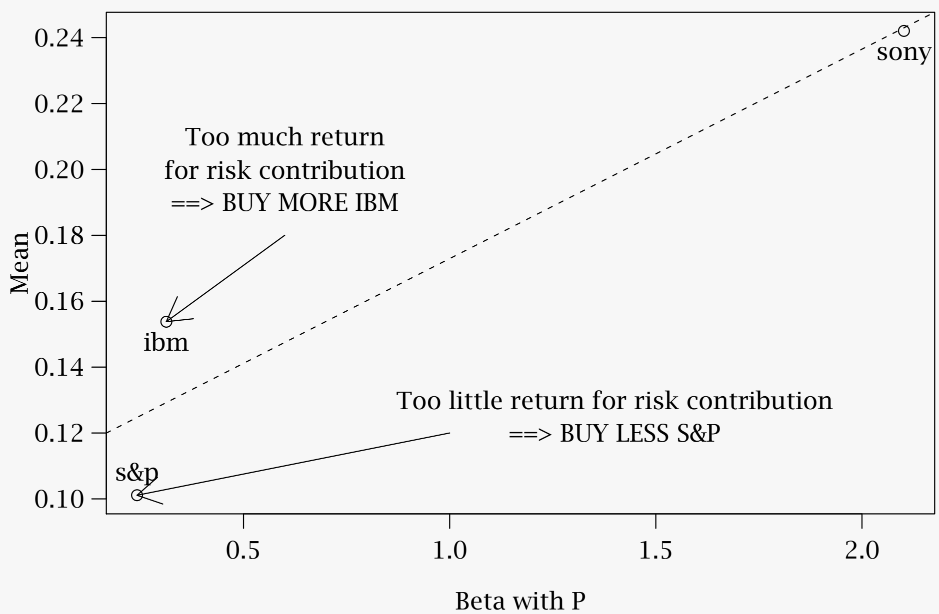
(We are dividing by 11 and not 12, because these are draws from a sample, not from the population.) As the measure for how each stock contributes to the risk of portfolio G, let us adopt the beta of each stock *with respect to* portfolio G. To obtain this beta, divide each of these securities' covariances by the same number, the variance of G:

$$\begin{aligned}
\beta(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{G}}) &= \frac{\mathcal{C}ov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{G}})}{\mathcal{V}ar(\tilde{r}_{\text{G}})} = \frac{0.0402}{0.1662} = 0.2417 \\
\beta(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{G}}) &= \frac{\mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{G}})}{\mathcal{V}ar(\tilde{r}_{\text{G}})} = \frac{0.0520}{0.1662} = 0.3129 \\
\beta(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{G}}) &= \frac{\mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{G}})}{\mathcal{V}ar(\tilde{r}_{\text{G}})} = \frac{0.3494}{0.1662} = 2.1023
\end{aligned}$$

Now, plot the expected rates of return against the betas (for portfolio G, of course), and try to draw a line through these three points. Figure 12.10 shows that the three points do not lie on one line. IBM seems to have too high an expected rate of return for its beta (it lies above the line), while the S&P 500 has too low an expected rate of return. Therefore, as you already knew from comparing portfolio G and E3, it appears that good advice to portfolio G owners would be to rebalance away from the expensive S&P 500 and towards the cheaper IBM. (Thereafter, you should recheck the relative expected pricing of these securities relative to your new portfolio.)

Now graph each beta against its expected rate of return.

Figure 12.10: Means of Stocks versus Betas with respect to non-MVE G portfolio



G is a poor portfolio, because it bought too little of IBM, given its risk contribution to G; and too much of S&P 500, given its risk contribution to G. Portfolios that have either higher mean and the same risk, or the same mean and lower risk, could have been achieved by buying more IBM and less of S&P 500.

Indeed, as it turns out, the fact that the three securities do not perfectly lie on one line is proof-positive that portfolio G can be improved upon, i.e., that portfolio G is not on the MVE Frontier. (Incidentally, although you could plot covariances instead of betas, we used the latter because they are a little easier to interpret. The slope of the line would change in this case, but not the fact that all points either lie on a line or do not lie on a line.)

MVE of a pfio means its securities must lie on a line in a beta versus exp.ret. graph.

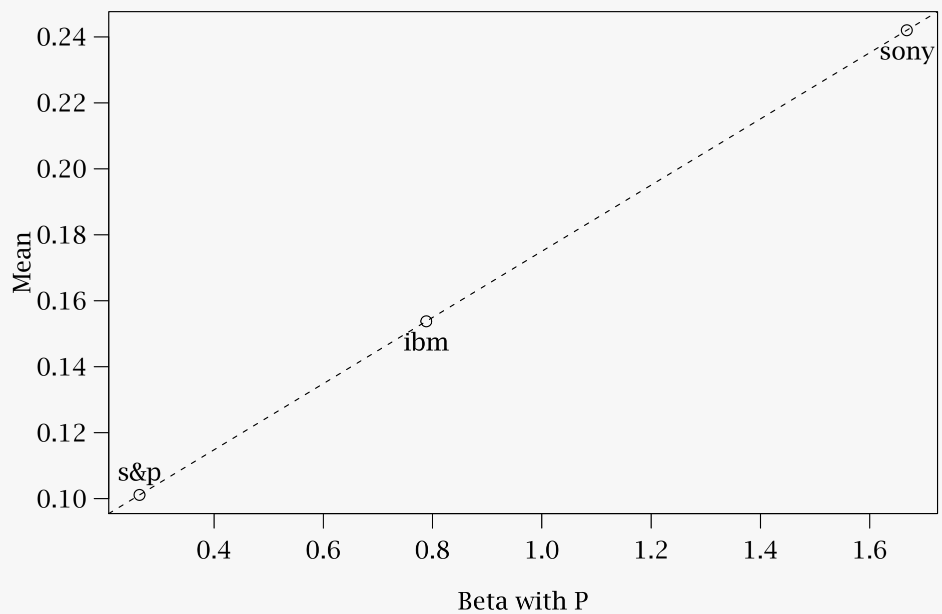
12·5.B. What if the Risk-Reward Relationships is Linear?

If you repeat the same exercise for portfolio E3, which you will confirm in Exercise 12.14, you will find that

For the MVE portfolio E3, when graphing expected rate of return against beta (with respect to portfolio E3), all stocks lie exactly on one straight line.

$$
\begin{aligned}
\beta(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{E3}}) &= \frac{\mathcal{C}ov(\tilde{r}_{\text{S\&P500}}, \tilde{r}_{\text{E3}})}{\mathcal{V}ar(\tilde{r}_{\text{E3}})} = \frac{0.03706}{0.1406} \approx 0.2635 \\
\beta(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{E3}}) &= \frac{\mathcal{C}ov(\tilde{r}_{\text{IBM}}, \tilde{r}_{\text{E3}})}{\mathcal{V}ar(\tilde{r}_{\text{E3}})} = \frac{0.1109}{0.1406} = 0.7885 \qquad (12.7) \\
\beta(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{E3}}) &= \frac{\mathcal{C}ov(\tilde{r}_{\text{Sony}}, \tilde{r}_{\text{E3}})}{\mathcal{V}ar(\tilde{r}_{\text{E3}})} = \frac{0.2346}{0.1406} = 1.6680
\end{aligned}
$$

Figure 12.11: Means of Stocks versus Betas with respect to MVE E3 portfolio



E3 is an MVE portfolio. You cannot do better by purchasing more or less of S&P 500, IBM, or Sony in terms of your mean/variance trade-off. (This also means that to obtain more mean, you would also have to accept more risk.)

An example of an MVE portfolio: each stock offers the right expected return for its beta.

Figure 12.11 plots the beta of each stock with respect to portfolio E3 against the expected rate of return of each stock. All securities lie on a single line. This actually means that if you hold portfolio E3 , each security offers an expected rate of return (reward) that is appropriate for its beta (risk). There is no way to improve upon portfolio E3 by changing these portfolio weights. Therefore, portfolio E3 is MVE.

Preview: This will be the CAPM formula.

This linear relationship between the expected rate of return of each security with its beta with a portfolio, *if the portfolio is mean-variance efficient*, will play a profound role in the CAPM, which will be discussed in the next chapters.

IMPORTANT: If all securities' expected rates of return lie on a line when plotted against the beta with respect to some portfolio, then this portfolio is MVE. If they do not lie on a line, then this portfolio is not MVE.

To enter the tangency portfolio, each stock must follow a particular relationship

Reflect on the example more generally. You know that the security holdings are optimized for each security i in a mean-variance efficient portfolio E—or else E would not be MVE! The exact relationship if all the components in Portfolio E are optimized is

$$\text{Portfolio E is MVE} \iff \mathcal{E}(\tilde{r}_i) = a + b \cdot \beta_{\tilde{r}_i,\tilde{r}_E}, \tag{12.8}$$

where i is an index naming each and every stock. (Nerd Appendix d proves Formula 12.8.) The formula contains two constants, a and b, which are numbers that depend on the portfolio E. Formula 12.8 relates how the expected rate of return for each and every single available security must be related to its beta with respect to E, i.e., lie along a straight line.

Here is a different way to think of the relationship. If you purchase an MVE portfolio, the relationship between each security's expected rate of return and its beta with respect to your own portfolio comes about automatically. It could even be stated that you purchase a best possible (MVE) portfolio, *if and only if* you always add more of stocks that seem relatively cheap for your portfolio (securities with too high an expected rate of return for their betas, i.e., that lie above your portfolio's line); and always reduce stocks that seem relatively expensive (securities with too low an expected rate of return for their betas, i.e., that lie below your portfolio's line).

A different way to think about portfolio optimization: given a portfolio, rebalance from more expensive towards cheaper stocks.

12·5.C. The Line Parameters

You can yet learn more about the "Line Formula 12.8": you can even determine the two constant numbers a and b. If $\beta_{i,E}$ (the covariance) is zero for a particular investment i, then a is the expected rate of return of this investment. If a risk-free security is available it would have such zero covariance, so

You can determine the numbers a and b.

$$\text{Portfolio E is MVE} \iff \mathcal{E}(\tilde{r}_i) = a + b \cdot 0 = r_F$$

You can replace a with r_F,

$$\text{Portfolio E is MVE} \iff \mathcal{E}(\tilde{r}_i) = r_F + b \cdot \beta_{\tilde{r}_i,\tilde{r}_E}$$

You can also consider an investment in portfolio E itself. (Naturally, buying more or less of it does not improve the portfolio performance.) The beta of portfolio E with respect to itself is 1. Therefore

$$\text{Portfolio E is MVE} \iff \mathcal{E}(\tilde{r}_E) = r_F + b \cdot 1 = r_F + b \tag{12.9}$$

You can solve Formula 12.9 for b and find $b = \mathcal{E}(\tilde{r}_E - r_F)$. Putting this all together, the line equation must be

$$\text{Portfolio E is MVE} \iff \mathcal{E}(\tilde{r}_E) = r_F + [\mathcal{E}(\tilde{r}_E) - r_F] \cdot \beta_{i,E} \tag{12.10}$$

Now, for any portfolio on the (upper half) of the MVE frontier, b is positive. This gives the line formula in 12.10 a nice intuitive interpretation. In order to find its way into your MVE portfolio, each security has to offer a reward (expected rate of return) that is appropriate for its contribution to your portfolio's risk:

The formula states that inside any MVE portfolio, each stock must offer an appropriate expected rate of return for its risk.

- ➤ A stock that fluctuates a lot with your E portfolio itself (having high rates of return when your portfolio E has high rates of return) has a high beta with respect to your portfolio. Such a stock does not help much in diversifying away the risk inside E. Therefore, this stock must offer you a relatively high expected rate of return to enter your E portfolio.
- ➤ A stock whose returns tend to move in the opposite direction from E's returns has a negative beta. Therefore, this stock can offer a relatively low expected rate of return and it would still be sufficiently desirable to enter into your MVE portfolio.

This "appropriateness" relationship between risk and reward is a peculiar property of MVE portfolios. If you hold a portfolio that is not MVE, some of your stocks offer too low or too high an expected rate of return for their risk contributions to your portfolio.

This is not necessarily the case for non-MVE portfolios.

IMPORTANT: All securities inside an MVE portfolio have to offer a fair reward for their risk-contribution. If even a single security offered either too much or too little reward for its risk, there would be a better portfolio, and the original portfolio would not have been MVE to begin with.

Offering a fair risk/reward means that when the expected rates of return are graphed against the beta (with respect to the MVE portfolio E) for each and every security, the relationship must be a straight line,

$$\mathcal{E}(\tilde{r}_{\text{i}}) = a + b \cdot \beta_{\tilde{r}_{\text{i}},\tilde{r}_{\text{E}}}$$

where a is the risk-free rate r_{F} and b is $\mathcal{E}(\tilde{r}_{\text{E}}) - r_{\text{F}}$, so

$$\mathcal{E}(\tilde{r}_{\text{i}}) = r_{\text{F}} + [\mathcal{E}(\tilde{r}_{\text{E}}) - r_{\text{F}}] \cdot \beta_{\text{i,E}}$$

Keep The Forest in Mind: The goal is to find the best overall portfolio.

Be warned: we have covered a lot of variances and covariances in many different aspects now. Do not confuse them. To recap, the efficient frontier graphs in Figures 12.2–12.4 had standard deviation on the x-Axis. The lines in Figures 12.10 and 12.11 had β on the x-Axes. It is important to remember that *you are only interested in minimizing your overall portfolio variance, given your desired overall portfolio expected rate of return.* You are not interested in the expected rate of return of an individual stock per se. You are not interested in the standard deviation of an individual stock per se. You are not interested in the beta of an individual stock per se. The individual stock statistics—means, betas, etc.—are useful information only insofar as they help you in determining your best overall portfolio.

DIG DEEPER

If there is no risk-free security, then r_{F} is simply replaced by the expected rate of return of a zero-beta stock. For Portfolio E from Figure 12.11, this means that you could only state that the line is

$$\mathcal{E}(\tilde{r}_{\text{E}}) = a + [\mathcal{E}(\tilde{r}_{\text{E}}) - a] \cdot \beta_{\text{i,E}} \qquad (12.11)$$

This is sometimes called the zero-beta CAPM or Black CAPM, after Fischer Black who first derived it.

An Example to Illustrate This Formula.

How can you use the linear risk-reward relationship? Easy. Say, you own a mean-variance efficient portfolio E that has a mean $\mathcal{E}(\tilde{r}_{\text{E}})$ of 8% and a standard deviation of 4% (variance of 0.0016). Further, the risk-free rate of return is 6%. What is the relationship between the expected rate of return of each security and the beta of each security that is available to you? The answer is that Formula 12.8 states that every security should have an expected rate of return of

$$\mathcal{E}(\tilde{r}_{\text{i}}) = 6\% + [8\% - 6\%] \cdot \beta_{\text{i,E}}$$

You did not need to use the information about the 4% risk, although it might come in handy when you have to compute $\beta_{\text{i,E}}$. For example, if a particular security—call it Oscar—has an expected rate of return of 10%, then it must have a beta with respect to E of 2. Another security, Meyer, which has a beta with respect to E of 3, must have an expected rate of return of 12%.

Solve Now!

Q 12.14 Portfolio E3 contains 4.41% S&P 500, 68.9% IBM, and 26.7% Sony. Confirm Figure 12.11—that is, first compute and then plot the beta of each security with respect to E3 against each security's expected rate of return. Note that you can either put together a historical data table, similar to how you computed it on Page 281; or you can use the covariance tools from Table 10.5 on Page 231, and the covariances themselves, e.g., from Page 266 or Table 10.1 on Page 212. Use the standard deviation of the rate of return on E3 as given on Page 281 as 37.5%.

Q 12.15 Assume that you are told that portfolio E is MVE. Further, you are told that E has an expected rate of return of 14%. Finally, you are told that security 1 has an expected rate of return of 8% and a beta with respect to E of 0.5. What expected rate of return would a security i with a beta with respect to E of $\beta_{\text{i,E}} = 2$ have to have, in order for portfolio E to be MVE (i.e., not able to be improved upon)?

Q 12.16 A new security, E5, has appeared. Recall also the mean-variance efficient portfolio E2 :

Portfolio	$w_{\text{S\&P 500}}$	w_{IBM}	w_{Sony}	$\mathcal{E}(\tilde{r}_P)$	$Sdv(\tilde{r}_P)$
E5	−0.146	0.734	0.286	18.00%	39.38%
E2	+99.448%	+2.124%	−1.573%	10.00%	18.969%

E5's variance-covariance is

	$\tilde{r}_{\text{S\&P500}}$	$\tilde{r}_{\text{IBM}}$	$\tilde{r}_{\text{Sony}}$	$\tilde{r}_{\text{E5}}$	$\tilde{r}_{\text{E2}}$
$\tilde{r}_{\text{S\&P500}}$	3.622%	3.298%	4.772%	4.362%	3.698%
$\tilde{r}_{\text{IBM}}$	3.298%	15.034%	2.184%	4.866%	10.587%
$\tilde{r}_{\text{Sony}}$	4.772%	2.184%	81.489%	57.956%	22.124%
$\tilde{r}_{\text{E5}}$	4.362%	4.866%	57.956%	41.978%	17.974%
$\tilde{r}_{\text{E2}}$	3.698%	10.587%	22.124%	17.974%	12.707%

The new stock has a mean rate of return of 21.029%. Recall that the mean rate of return of the MVE portfolio E2 was 17%.

(a) Compute the beta of portfolio E5 with respect to portfolio E2 .

(b) Plot this new stock into Figure 12.11. Does it lie on the line? What does this mean?

(c) Add a little bit of stock E5 to portfolio E2 . That is, buy 99% of E2 and 1% of E5. Compute the risk and reward of this combination portfolio. Does this combination portfolio offer superior risk/reward characteristics than portfolio E2 alone?

(d) Subtract a little bit of stock E5 from portfolio E2 . That is, buy 101% of E2 and −1% of E5. Compute the risk and reward of this combination portfolio. Does this combination portfolio offer superior risk/reward characteristics than portfolio E2 alone?

Q 12.17 Repeat this question, but assume that the expected rate of return on stock E5 is only 10%.

12·6 Summary

This was a long and complex chapter, so this summary is more detailed than our usual chapter summaries:

1. Each portfolio is a point in a graph of overall portfolio expected rate of return (reward) against overall portfolio standard deviation (risk).
   Portfolios with similar compositions are close to one another.
2. For two securities, the possible risk versus reward choices are a curve (called a hyperbola).
   For more than two securities, the possible risk versus reward choices are a cloud of points.
3. The mean-variance efficient frontier is the upper left envelope of this cloud of points. You should not purchase a portfolio that is not on the MVE frontier.
4. In real life, computing the historical MVE Frontier is useful especially when choosing among a small set of asset classes. Otherwise, the historical standard deviations and expected rates of return are too unreliable to produce good forecasts of the future MVE Frontier.
5. If there is a risk-free security, the combination of the risk-free security and any other portfolio is a straight line.
6. If there is a risk-free security, the MVE Frontier is the line between the risk-free rate and the Tangency portfolio. You should not purchase a portfolio that is not on this line.

7. The combination of two MVE portfolios is also MVE.
8. If two MVE portfolios are known, finding other MVE portfolios is a matter of taking weighted combinations of these two MVE portfolios.

9. MVE portfolios have an important property: Each security *inside* the MVE portfolio offers a fair reward for its risk contribution.
   - If any single security were too cheap, you could do better by buying more of it—which means that the portfolio would not have been MVE to begin with.
   - If any single security were too expensive, you could do better by buying less of it—which means that the portfolio would not have been MVE to begin with.
10. The reward of each security is suitably measured by its expected rate of return. The risk of each security inside the MVE portfolio is suitably measured by its beta *with respect to* your specific MVE portfolio.
    To be fairly priced, every security in the portfolio must lie on a straight line when the expected rates of return are plotted against these betas. Algebraically, this means that the relation is

$$\mathcal{E}(\tilde{r}_{i}) = a + [\mathcal{E}(\tilde{r}_{E}) - a] \cdot \beta_{i,E}$$

    for every security i inside the MVE portfolio E. If there is a risk-free security offering an expected (actual) rate of return r_F, then this line becomes

$$\mathcal{E}(\tilde{r}_{i}) = r_F + [\mathcal{E}(\tilde{r}_{E}) - r_F] \cdot \beta_{i,E}$$

    This relationship will play a prominent role in the subsequent CAPM chapters.

Deeply Dug Appendix

A Excessive Proofs

a. The Optimal Portfolio Weights Formula

Warning: Linear Algebra.

This section relies on algebra (matrices and vectors). This is necessary, because it is simply too messy to deal with 10,000 individually named securities, with their requisite 10,000 expected rates of return and the requisite 50 million covariance terms. Unless you already know linear algebra inside out, and/or you want to write a computer program to determine the MVE Frontier, please ignore this section. It is not necessary to an understanding of this book.

Variable Definitions and the Problem Setup.

Using matrix notation, the portfolio mean and variance can be expressed as

$$\mathcal{E}(\tilde{r}_{P}) \equiv \vec{w}'\vec{m} \equiv \sum_{i=1}^{N} w_i \cdot Exp(\tilde{r}_i)$$

$$\mathcal{V}ar(\tilde{r}_{P}) \equiv \vec{w}'\mathbf{S}\vec{w} \equiv \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \cdot w_j \cdot \sigma_{i,j}$$

where $\mathbf{S}$ is the matrix of variance-covariances, $\vec{m}$ is the vector of expected rates of return on each security (which is easier to write than $\overline{\mathcal{E}(\tilde{r})}$), and $\vec{1}$ is a vector containing only the scalar 1 in each of N positions. A prime denotes a row rather than a column vector. The problem to solve is therefore

$$\begin{aligned} \min_{\vec{w}} \quad \mathcal{V}ar(\tilde{r}_P) &\equiv \vec{w}'\mathbf{S}\vec{w} && \text{minimize portfolio variance,} \\ \text{subject to} \quad \vec{1}'\vec{w} &= 1 && \text{portfolio weights add to 1,} \\ \mathcal{E}(\tilde{r}_P) &\equiv \vec{m}'\vec{w} && \text{user wants mean return } \mathcal{E}(\tilde{r}_P). \end{aligned}$$

This problem is usually solved via Lagrangian Optimization techniques. Introduce λ and γ as two new Lagrangian multiplier parameters, and optimize

$$\min_{\vec{w},\lambda,\gamma} L(\vec{w},\lambda,\gamma) \equiv \vec{w}'\mathbf{S}\vec{w} - \lambda\cdot\left(1-\vec{1}'\vec{w}\right) - \gamma\cdot\left[\mathcal{E}(\tilde{r}_P) - \vec{m}'\vec{w}\right]$$

The MVE set is obtained by differentiating with respect to the weights $\vec{w}$ and the two Lagrangian constants.

DIG DEEPER

If you are not familiar with Lagrangian techniques, differentiating with respect to the two Lagrangian multipliers produces the two original constraints, thereby ensuring that the optimization really works across the constrained set only. For example, $\partial L/\partial\lambda = 1 - \vec{1}'\vec{w}$, which if set to zero (as it is in the first-order condition) reimposes the constraint $1 = \vec{1}'\vec{w}$.

After some manipulation, the solution is

The solution are the optimal portfolio weights, $\vec{w}^*$.

$$\begin{aligned} \vec{w}^* &= \lambda^*\cdot\mathbf{S}^{-1}\vec{1} + \gamma^*\cdot\mathbf{S}^{-1}\vec{m} \\ \lambda^* &= \frac{C - \mathcal{E}(\tilde{r}_P)\cdot B}{D} \\ \gamma^* &= \frac{\mathcal{E}(\tilde{r}_P)\cdot A - B}{D}, \end{aligned} \tag{12.12}$$

where A, B, C, and D are simple numerical constants (scalars):

$$\begin{aligned} A &\equiv \vec{1}'\mathbf{S}^{-1}\vec{1} & B &\equiv \vec{1}'\mathbf{S}^{-1}\vec{m} \\ C &\equiv \vec{m}'\mathbf{S}^{-1}\vec{m} & D &\equiv A\cdot C - B^2 \end{aligned}$$

Using this formula, you can obtain the formula relating the mean and variance, the parabola

$$\begin{aligned} \mathcal{V}ar(\tilde{r}_P) &= \vec{w}'^*\mathbf{S}\vec{w}^* = \vec{w}'^*\mathbf{S}\left(\lambda^*\mathbf{S}^{-1}\vec{1} + \gamma^*\mathbf{S}^{-1}\vec{m}\right) \\ &= \lambda^* + \gamma^*\cdot\mathcal{E}(\tilde{r}_P) = \frac{A\cdot\mathcal{E}(\tilde{r}_P)^2 + 2\cdot B\cdot\mathcal{E}(\tilde{r}_P) + C}{D} \end{aligned}$$

This formula generalizes Formula 12.3.

↦ *Formula 12.3 on Page 265*

The minimum variance portfolio.

The weights of the minimum variance portfolio are worth writing down. They are the solution when the mean constraint $\vec{m}'\vec{w} = \mathcal{E}(\tilde{r}_P)$ is not imposed. The minimum variance portfolio is then

$$\vec{w}^*_{\substack{\text{min}\\\text{var}}} = \frac{\mathbf{S}^{-1}\vec{1}}{A}$$

Note that this portfolio does not depend on any security's expected rate of return.

b. The Combination of MVE Portfolios is MVE — With Risk-Free Security.

Combining two MVE portfolios is MVE again.

It is difficult to prove that the combination of MVE portfolios is MVE—except in the special case in which a risk-free security exists. Call the MVE tangency portfolio "T." Because T is MVE,

$$\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_T) - r_F] \cdot \beta_{i,T} \qquad \forall i$$

Now invest w_F in the risk-free security (F) and $(1 - w_F)$ in T. Call this portfolio E4.

$$\tilde{r}_{E4} = w_F \cdot r_F + (1 - w_F) \cdot \tilde{r}_T$$

Both F and T are MVE portfolios. The question is whether portfolio E4 is also MVE. The expected rate of return of E4 is

$$\mathcal{E}(\tilde{r}_{E4}) = w_F \cdot r_F + (1 - w_F) \cdot \mathcal{E}(\tilde{r}_T) \tag{12.13}$$

The beta of any security i with respect to this portfolio E4 is

$$\begin{aligned}
\beta_{i,E4} &= \frac{\mathcal{Cov}(\tilde{r}_i, \tilde{r}_{E4})}{\mathcal{Var}(\tilde{r}_{E4})} = \frac{\mathcal{Cov}[\tilde{r}_i,\ w_F \cdot r_F + (1 - w_F) \cdot \tilde{r}_T]}{\mathcal{Var}[w_F \cdot r_F + (1 - w_F) \cdot \tilde{r}_T]} \\
&= \frac{(1 - w_F) \cdot \mathcal{Cov}(\tilde{r}_i, \tilde{r}_T)}{(1 - w_F)^2 \cdot \mathcal{Var}(\tilde{r}_T)} = \left(\frac{1}{1 - w_F}\right) \cdot \left[\frac{\mathcal{Cov}(\tilde{r}_i, \tilde{r}_T)}{\mathcal{Var}(\tilde{r}_T)}\right] \\
&= \frac{\beta_{i,T}}{1 - w_F}
\end{aligned} \tag{12.14}$$

because the risk-free rate has neither a variance nor a covariance with any other return series. The new portfolio E4 is also MVE, if and only if

$$\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_{E4}) - r_F] \cdot \beta_{i,E4} \qquad \forall i$$

(A security i can have weight zero in the portfolio, so this really means for all securities.) Substitute Formula 12.13 for $\mathcal{E}(\tilde{r}_{E4})$, and Formula 12.14 for $\beta_{i,E4}$:

$$\begin{aligned}
\mathcal{E}(\tilde{r}_i) &= r_F + \{[w_F \cdot r_F + (1 - w_F) \cdot \mathcal{E}(\tilde{r}_{E4})] - r_F\} \cdot \left(\frac{\beta_{i,T}}{1 - w_F}\right) \\
&= r_F + [(1 - w_F) \cdot \mathcal{E}(\tilde{r}_{E4}) - (1 - w_F) \cdot r_F] \cdot \left(\frac{\beta_{i,T}}{1 - w_F}\right) \\
&= r_F + [\mathcal{E}(\tilde{r}_{E4}) - r_F] \cdot \beta_{i,T} \qquad \forall i
\end{aligned}$$

This is true if and only if a portfolio is MVE. Therefore, the new portfolio E4 is also MVE.

c. The Combination of Mean-Variance Efficient Portfolios is Mean-Variance Efficient — Without Risk-Free Security.

The combination of any two risky portfolios is a parabola in mean-variance space.

There are many ways to prove that the combination of two MVE portfolios is also MVE. The brute-force method is the linear algebra solution from the previous section. In Formula 12.12, $\mathbf{S}^{-1}\vec{1}$ is one portfolio (a vector of [unnormalized] portfolio weights), $\mathbf{S}^{-1}\vec{m}$ is another portfolio (another vector of portfolio weights). λ and μ determine the relative investment proportions in each.

Start with an "intuitive" proof.

The rest of this subsection is an alternative conceptual argument which may or may not help you to get some more intuition. Ignore it if you wish. Return to Formula 12.3, which shows that the relationship between two risky investment assets is

$$\mathcal{Var}[\ \mathcal{E}(\tilde{r}_P)\ ] = a \cdot \mathcal{E}(\tilde{r}_P)^2 + b \cdot \mathcal{E}(\tilde{r}_P) + c$$

where a, b, and c are constants that depend on the particular expected rate of return between two securities, their individual variances, and their mutual covariance.

The question you should ponder is whether the combination of two MVE portfolios is itself MVE. You already saw the argument in Figure 12.6: it must be so. Imagine two portfolios on the MVE frontier—call them portfolios E1 and E2. Portfolio E1 could be on the upper half of the parabola, portfolio E2 could be its equal-variance counterpart on the lower part of the parabola. Name the actual minimum variance portfolio by the letter E0. The question is whether the combination of the E1 and E2 portfolios traces out the MVE Frontier. You already know that the combination of E1 and E2 is a quadratic equation. The previous section showed that the MVE Frontier is a quadratic equation, too. The question is whether the quadratic frontier between E1 and E2 can define a set other than the MVE Frontier. In a quadratic equation, three points define the parabola. Thus, if E1 and E2 are already chosen as two portfolios with equal risk, then all possible combinations of these portfolios would have to lie on a parabola, and the minimum variance portfolio based only on E1 and E2 would have to lie on the same horizontal line that the actual minimum variance portfolio E0 lies on. What would happen if the combination of E1 and E2 produced a minimum variance portfolio with lower variance than E0? Then the original statement that E0 is the minimum variance portfolio would be false. What would happen if the combination of E1 and E2 produced a minimum variance portfolio with higher variance than E0? Then, the parabola would overshoot the MVE Frontier when either E1 or E2 is shorted. However, then the real MVE Frontier would not be a simple quadratic equation, but one quadratic equation to the left of E1 and E2 and another quadratic equation to the right of E1 and E2. You already know that this cannot be the case, either.

Being only a parabola means that the combination portfolio of any two MVE Frontier portfolios has to curve at the same rate as the MVE Frontier.

Therefore, the combinations of E1 and E2 trace out the MVE frontier. In other words, the combination of the two MVE risky portfolios E1 and E2 is the full and sole MVE. Any other portfolio, say E4, that is a weighted combination of E1 and E4, can be represented by a unique weight on E1 versus E4. Combining many arbitrary portfolios consisting of E1 and E2 is still a weighted combination of E1 and E2, and therefore still MVE.

Therefore, any combination of any two risky MVE portfolios traces out the MVE Frontier.

d. Proof of the Linear Beta versus Expected Rate of Return Relationship for MVE Frontier Portfolios

Our intent is to prove that all securities in an MVE portfolio must follow a linear beta-expected return relationship. It is the mathematical relationship that is required for every stock to be at just its optimal level, so that you are indifferent between purchasing one more or one less cent of this security. The derivation relies on the envelope theorem, which states that adding a little, little bit or subtracting a little, little bit at the optimal investment choice $w_i^\star$ should make only a minimal difference *for the optimal portfolio*. If the expected rate of return were too high or too low for a security, given its beta, you could do better than E—either buying a little bit more or a little bit less of this security. But then, E would not be MVE.

Consider the combination of the MVE Portfolio and a new security.

Figures 12.12 shows a MVE portfolio called E. You are considering adding w_i of a security i (security i *alone* does not have to be MVE). Could it be that the combination portfolio were not tangent to the frontier, i.e. where $w_i \approx 0$ and $w_E \approx 1$? The answer must be no—or it would be possible to purchase a combination portfolio that outperforms the MVE Frontier. Perturbing the weight in the E by adding security i (where $w_i = 0$) has to result in a mean-variance trade-off that is tangent to the MVE Frontier, as shown in the graph in Figure 12.12. You can use this insight to prove the linear relationship between the expected rate of return on security i and its beta with respect to E.

If you combine an MVEportfolio with a little bit of another portfolio, you should not be able to perform better.

1. By inspection, the slope of the MVE Frontier at E is

$$\text{Slope} = \frac{\mathcal{E}(\tilde{r}_E) - \mathcal{E}(\tilde{r}_0)}{Sdv(\tilde{r}_E) - Sdv(\tilde{r}_0)} = \frac{\mathcal{E}(\tilde{r}_E) - \mathcal{E}(\tilde{r}_0)}{Sdv(\tilde{r}_E)} \tag{12.15}$$

   because $Sdv(\tilde{r}_0)$ is 0.

2. Consider an arbitrary combination portfolio (call it P). It is defined by

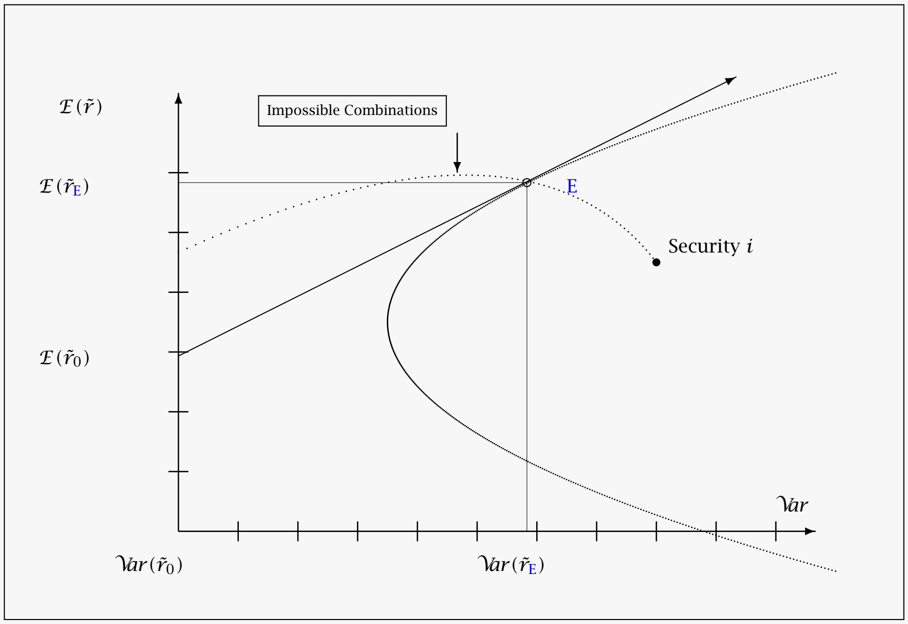
$$\tilde{r}_P = w_i \cdot \tilde{r}_i + (1 - w_i) \cdot \tilde{r}_E$$

   The characteristics of this portfolio P are

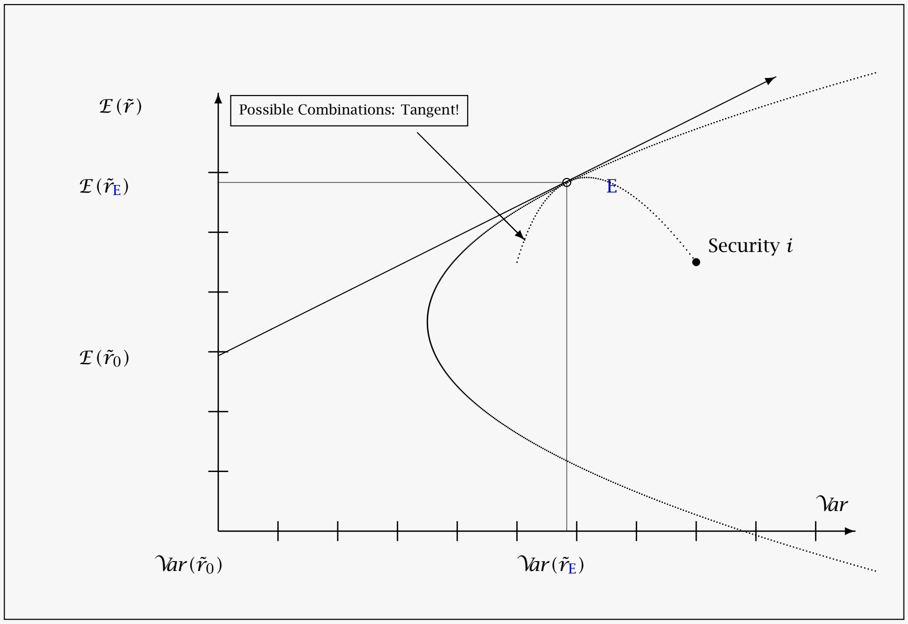
$$\begin{aligned} \mathcal{E}(\tilde{r}_P) &= w_i \cdot \mathcal{E}(\tilde{r}_i) + (1 - w_i) \cdot \mathcal{E}(\tilde{r}_E) \\ Sdv(\tilde{r}_P) &= \sqrt{w_i^2 \cdot \mathcal{V}ar(\tilde{r}_i) + (1 - w_i)^2 \cdot \mathcal{V}ar(\tilde{r}_E) + 2 \cdot w_i \cdot (1 - w_i) \cdot \mathcal{C}ov(\tilde{r}_i, \tilde{r}_E)} \end{aligned}$$

Figure 12.12: Possible and Impossible Combinations of a New Security with a Mean-Variance Efficient Portfolio

Impossible Combinations of a New Security with a Mean-Variance Efficient Portfolio:



Possible Combinations of a New Security with a Mean-Variance Efficient Portfolio:



How do the mean and standard deviation of P change with w_i?

$$
\begin{aligned}
\frac{\partial \mathcal{E}(\tilde{r}_P)}{\partial w_i} &= \mathcal{E}(\tilde{r}_i) - \mathcal{E}(\tilde{r}_E) \\
\frac{\partial Sdv(\tilde{r}_P)}{\partial w_i} &= \left\{ \frac{Cov(\tilde{r}_i,\tilde{r}_E) - Var(\tilde{r}_E) + w_i \cdot [Var(\tilde{r}_i) - 2 \cdot Cov(\tilde{r}_i,\tilde{r}_E) + Var(\tilde{r}_E)]}{\sqrt{w_i^2 \cdot Var(\tilde{r}_i) + (1-w_i)^2 \cdot Var(\tilde{r}_E) + 2 \cdot w_i \cdot (1-w_i) \cdot Cov(\tilde{r}_i,\tilde{r}_E)}} \right\}
\end{aligned}
\tag{12.16}
$$

Now, compute the slope of the curve that defines the sets of combination portfolios P, i.e., how its expected rate of return changes as its standard deviation is changed. Because you can only change w_i, you need to compute this slope by dividing the two derivatives in Formula 12.16:

$$
\frac{\partial \mathcal{E}(\tilde{r}_P)}{\partial Sdv(\tilde{r}_P)} = \frac{\frac{\partial \mathcal{E}(\tilde{r}_P)}{\partial w_i}}{\frac{\partial Sdv(\tilde{r}_P)}{\partial w_i}}
$$

What you really need to know is the slope when $w_i = 0$, i.e., when the weight is solely on the MVE Frontier portfolio E. Substitute $w_i = 0$ into this messy expression, and you get

$$
\begin{aligned}
\lim_{w_i \to 0} \left[\frac{\partial \mathcal{E}(\tilde{r}_P)}{\partial Sdv(\tilde{r}_P)} \right] &= \frac{\lim_{w_i \to 0} \left[\frac{\partial \mathcal{E}(\tilde{r}_P)}{\partial w_i} \right]}{\lim_{w_i \to 0} \left[\frac{\partial Sdv(\tilde{r}_P)}{\partial w_i} \right]} \\
&= \frac{\mathcal{E}(\tilde{r}_i) - \mathcal{E}(\tilde{r}_E)}{\left[\frac{Cov(\tilde{r}_i,\tilde{r}_E) - Var(\tilde{r}_E)}{Var(\tilde{r}_E)} \right]}
\end{aligned}
\tag{12.17}
$$

3. You can now use the fact that the slope of portfolio P at $w_i = 0$ has to be the same slope as the MVE Frontier. That is, the line defined in equation 12.15 and the curve defined in equation 12.17 must be tangent.

$$
\begin{aligned}
\text{Slope of Tangent Line} &= \text{Slope of Curve at } w_i = 0 \\
\frac{\mathcal{E}(\tilde{r}_E) - \mathcal{E}(\tilde{r}_0)}{Sdv(\tilde{r}_E)} &= \frac{\mathcal{E}(\tilde{r}_i) - \mathcal{E}(\tilde{r}_E)}{\left[\frac{Cov(\tilde{r}_i,\tilde{r}_E) - Var(\tilde{r}_E)}{Var(\tilde{r}_E)} \right]}
\end{aligned}
$$

Use $Var(\tilde{r}_E) = Sdv(\tilde{r}_E) \cdot Sdv(\tilde{r}_E)$ to simplify this equation to

$$
\begin{aligned}
\mathcal{E}(\tilde{r}_i) &= \mathcal{E}(\tilde{r}_0) + \left[\frac{Cov(\tilde{r}_i,\tilde{r}_E)}{Var(\tilde{r}_E)} \right] \cdot [\mathcal{E}(\tilde{r}_E) - \mathcal{E}(\tilde{r}_0)] \\
&= \mathcal{E}(\tilde{r}_0) + \beta_i \cdot [\mathcal{E}(\tilde{r}_E) - \mathcal{E}(\tilde{r}_0)]
\end{aligned}
$$

which is Formula 12.8 that we wanted to prove! This formula has to hold for every available security i. Indeed, a portfolio is MVE *if and only if* this equation holds for all available investment opportunities i.

DIG DEEPER

Figure 12.12 is drawn for better illustration, not for accuracy. To avoid eventually intersecting the MVE Frontier, the possible combinations would necessarily have to intersect E from below, not from above (as drawn).

[No keyterm list for optimalpfio-g.]

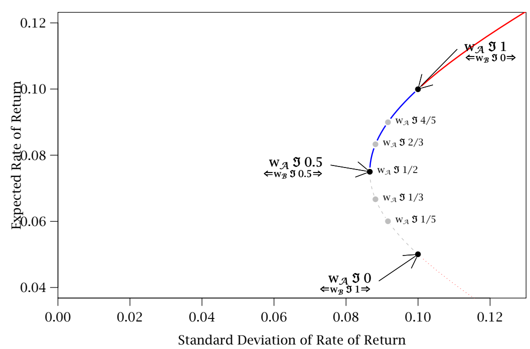
End of Chapter Problems

17 "Solve Now" Answers

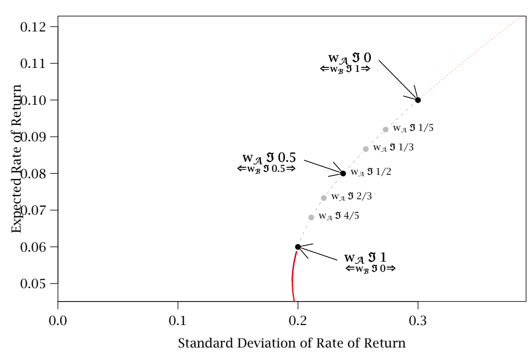
1. The risk is always 10%. By shorting the security with lower expected rate of return and buying the security with higher expected rate of return, you can invest in an opportunity with an infinite expected rate of return. This is *expected*, but it is not certain. (If you are curious: by choice of portfolio, you can guarantee yourself any positive rate of return, but you have still standard deviation, because you do not know whether you will earn a very large amount or a very large amount plus or minus a little. The only risk is now how much you will earn, not whether you will earn a positive or a negative return.)
2. Indeed, going long and going short is risk-free. However, it is also a zero-investment and zero-payoff strategy (except for frictions, of course).
3. The portfolios have the following risks:

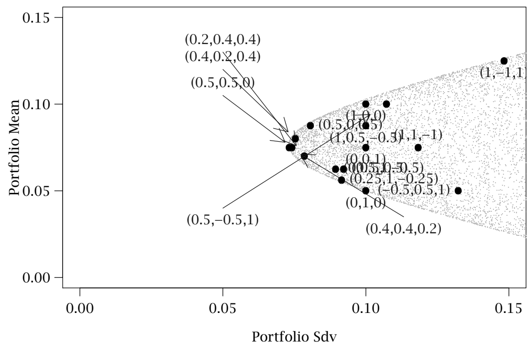
Weight w_A	-1	0	$1/3$	$1/2$	$2/3$	1
Risk (σ_P)	17.3%	10%	8.8%	8.7%	8.8%	10%

A graph of these portfolios (and weights in between) is:



4. First, compute the covariance: $Correlation_{A,B} = Cov_{A,B}/(Sdv_A \cdot Sdv_B) \Rightarrow Cov(A,B) = 20\% \cdot 30\% \cdot 80\% = 0.048$. The MVE frontier is:



5. 

6. The Minimum Variance Portfolio in this world is a portfolio that invests about 45% into security 1, about 38% into security 2, and the rest (about 17%) into security 3.
7. An investor with more taste for risk would choose a portfolio higher up along the MVE Frontier. The least risky portfolio with an expected rate of return of 11% invests about 97% into security 1, shorts 42% in security 2, and invests 45% into security 3.
8. No, because it is inside the main "cloud" of points. There are better portfolios that offer higher expected rates of return for the same amount of risk. The three "pure" portfolios, which invest in only one stock are not on the MVE Frontier, either. However, the portfolio that invests 100% in security 1 and nothing in the other two securities comes fairly close *by accident*.

9. You can save time if you work directly off combinations of portfolio E1 and portfolio E3.

	E1	E3	$^1/_2\cdot$E1 + $^1/_2\cdot$E3
S&P 500	99.5%	36.1%	67.770%
IBM	2.1%	46.6%	24.386%
Sony	−1.6%	17.2%	7.844%
$\mathcal{E}(\tilde{r})$	10.0%	15.0%	12.500%
$Sdv(\tilde{r})$	19.0%	67.6%	30.949%

10. You can save time if you work directly off combinations of portfolio E2 and portfolio E3.

	E2	E3	$^1/_2\cdot$E1 + $^1/_2\cdot$E3
S&P 500	−90.6%	36.1%	−27.264%
IBM	135.7%	46.6%	91.172%
Sony	54.9%	17.2%	36.092%
$\mathcal{E}(\tilde{r})$	25.0%	15.0%	20.000%
$Sdv(\tilde{r})$	19.0%	67.6%	59.266%

11. Still the same: 0.35. Indeed, this is the case for any MVE portfolio now.
12. If you sketch it, the intuition is that when the risk-free rate is lower, the tangency portfolio should have less risk and less reward. With $r_F = 3\%$, the weight on portfolio E1 is now 81.606%. The tangency portfolio has a mean rate of return of 12.759% (excess return, 9.759%), and a standard deviation of 22.344%. The Sharpe ratio is about 0.4368.
13. If you sketch it, the intuition is that when the risk-free rate is higher, the tangency portfolio should have more risk and more reward. With $r_F = 3\%$, the weight on portfolio E1 is now −23.949%. The tangency portfolio has a mean rate of return of 28.592% (excess return, 19.592%), and a standard deviation of 82.607%. The Sharpe ratio is about 2371.

14.

$$\begin{aligned}
\mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{E3}) &= \mathcal{C}ov(\tilde{r}_{S\&P500}, 4.414\% \cdot \tilde{r}_{S\&P500} + 68.91\% \cdot \tilde{r}_{IBM} + 26.68\% \cdot \tilde{r}_{Sony}) \\
&= 4.414\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{S\&P500}) + 68.91\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{IBM}) + 26.68\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) \\
&= 4.414\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{S\&P500}) + 68.91\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{IBM}) + 26.68\% \cdot \mathcal{C}ov(\tilde{r}_{S\&P500}, \tilde{r}_{Sony}) \\
&= 4.414\% \cdot 0.0362 + 68.91\% \cdot 0.0330 + 26.68\% \cdot 0.0477 \\
&\approx 0.0371
\end{aligned}$$

so the beta of S&P 500 with respect to E3 (the opposite of what we usually would compute, which is a beta of something with respect to the market) is $0.0371/0.375^2 \approx 0.2635$, which confirms the computations in Formula 12.7. The other betas are correct, too.

15.

$$8\% = \alpha + (14\% - \alpha) \cdot 0.5 \qquad \Longleftrightarrow \qquad \alpha = 2\%$$

Therefore,

$$\mathcal{E}(\tilde{r}_i) = 2\% + (14\% - 2\%) \cdot 2 = 26\%$$

16.

(a) $\beta_{E5,E2} = \mathcal{C}ov(\tilde{r}_{E5}, \tilde{r}_{E2})/\mathcal{V}ar(\tilde{r}_{E2}) = 17.974\%/12.707\% = 1.414$.
(b) Yes, stock E5 does lie exactly on the line. This means that portfolio E2 is still MVE: you cannot do better.
(c) The mean rate of return of this combination portfolio is $\mathcal{E}(\tilde{r}) = 99\% \cdot 17\% + 1\% \cdot 21.029\% = 17.040\%$. For E2 to remain MVE, the combination portfolio must not be more risky. It is $\mathcal{V}ar(\tilde{r}) = (99\%)^2 \cdot 12.707\% + (1\%)^2 \cdot 41.978\% + 2 \cdot 1\% \cdot 99\% = 12.814\%$. This is higher than the variance of

$$\tilde{r}_{E2}$$

alone. Therefore, this combination portfolio is not superior to

$$\tilde{r}_{E2}$$

.
(d) The mean rate of return of this $\mathcal{E}(\tilde{r}) = 101\% \cdot 17\% + (-1\%) \cdot 21.029\% = 16.960\%$. This portfolio already has lower mean, so it cannot dominate portfolio E2. You do not even have to compute the risk. (The *Sdv* happens to be $\sqrt{0.126}$.)

17.

(a) Beta has not changed. $\beta_{E5,E2} = \mathcal{C}ov(\tilde{r}_{E5}, \tilde{r}_{E2})/\mathcal{V}ar(\tilde{r}_{E2}) = 17.974\%/12.707\% = 1.414$.
(b) No, stock E5 does not lie on the line. This means that portfolio E2 is no longer MVE: you can do better.
(c) The mean rate of return of this combination portfolio is $\mathcal{E}(\tilde{r}) = 99\% \cdot 17\% + 1\% \cdot 10.0\% = 16.93\%$. This portfolio already has lower mean, so it cannot dominate portfolio E2 .
(d) The mean rate of return of this combination portfolio is $\mathcal{E}(\tilde{r}) = 101\% \cdot 17\% + (-1\%) \cdot 10.0\% = 17.07\%$. The variance of the combination portfolio is $\mathcal{V}ar(\tilde{r}) = (101\%)^2 \cdot 12.707\% + (-1\%)^2 \cdot 41.978\% + 2 \cdot (-1\%) \cdot 99\% = 12.603\%$. Thus, this combination portfolio has both higher mean and lower risk than portfolio E2 . Therefore, portfolio E2 is no longer MVE!.

All answers should be treated as suspect. They have only been sketched and have not been checked.

CHAPTER 13§

The CAPM: A Cookbook Recipe Approach

How much expected return should investments offer?

This chapter appears in the Survey text only.

The previous chapters explored *your* best investment choices. This chapter explores the economy-wide consequences if *all* important investors in the economy were to engage in such good investment choices—a fair relationship between expected rates of return and market beta, called the CAPM.

This chapter offers the "recipe version" of the CAPM. That is, it will show you how to use the model even if you do not know why it is the right model or where it comes from. Chapter 14 will explain and critically evaluate the theory behind the CAPM.

Warning—this chapter is long and covers a lot of ground.

capmrecipe-g.tex: Last file change: Jan 6, 2008 (13:19h). Latexed on: (15:38h).

13·1 The Opportunity Cost of Capital

Knowing what (all) other investors do is less important to an investor, than it is to a firm that wants to sell to investors.

Why is it important what portfolios other investors are buying? It is not important if you just want to invest and *buy* your own portfolio. Regardless of what other investors are holding, you can make your own investment decisions. But it is very important for corporations who want to *sell* investments. They need to know what investors in the aggregate will want.

The firm must know the opportunities available to investors to know what they like and dislike.

Put yourself into the shoes of a manager that wants to act in the best possible way on behalf of its corporate owners. The capital that investors have given the firm has an opportunity cost—investors could put their money into alternative projects elsewhere. Thus, acting on behalf of the investors means that you need to determine what projects investors find worthwhile and what projects investors do not find worthwhile—which projects investors would want to buy and which projects they would want to pass up. If the firm's projects are not good enough, it is better for you to not adopt these projects and instead return the money to investors.

This is why the appropriate expected rate of return is called the cost of capital.

↦ *Hurdle Rate Section 17·1 on Page 444*

Thinking about opportunity costs is how managers should determine the price at which their investors—representative investors in the economy—are willing to purchase projects. This is how managers should make the decision of what projects to take and what projects to avoid. This task of "finding a fair price" is the process of determining an appropriate "cost of capital" (expected rate of return) for a project. The **Capital Asset Pricing Model** (or CAPM) provides this number.

IMPORTANT: The CAPM stipulates an **opportunity cost of capital**. This opportunity cost helps corporate managers determine whether a particular project is beneficial or detrimental for the corporate investors—whether the corporation should take the project or instead return cash to its investors for better uses elsewhere.

13·2 The CAPM

Higher Risk can mean higher reward...but not always.

For the most important CAPM insight, you do not need a formal model. After the principle that diversification reduces risk, the second most important principle of investments is that it takes extra compensation to get investors to accept extra risk. This does not mean that, as an investor, you can expect to receive extra compensation for risks that you do not need to take. You can easily avoid betting in Las Vegas, so you are unlikely to receive extra compensation—a high expected rate of return—when betting there. But it does mean that when a firm wants investors to accept risks that they cannot easily diversify away, the firm must offer a higher expected rate of return in order to get investors to accept this risk. The CAPM is a model that translates this principle into specific costs of capital.

13·2.A. The Premise and Formula

The CAPM Formula is similar to the Formula from Section 12·5, except the efficient portfolio is M.

The basic premise of the Capital Asset Pricing Model is that investors—to whom the firm wants to sell its projects—are currently holding a well-diversified market portfolio. The CAPM further stipulates that this market portfolio is also mean-variance efficient. For a new project to be desired by investors and become part of their mean-variance efficient portfolio, it must offer a fair risk/reward trade-off. You already know from Section 12·5 what fair risk/rewards trade-offs mean in specific terms—a relationship between the project's market-beta and the project's expected rate of return. Recall that the project's risk contribution is measured by beta with respect to the investors' portfolio, here assumed to be the stock market M,

$$\beta_i \;\equiv\; \beta_{i,M} \;\equiv\; \frac{\mathcal{C}ov(\tilde{r}_i, \tilde{r}_M)}{\mathcal{V}ar(\tilde{r}_M)}$$

It is common to call this the beta, rather than the market-beta. Now, this beta is a parameter which you are assumed to know—though in real life, you can only estimate it. Stocks that strongly move together with the market (high beta) are relatively risky, because they do not help investors—already holding the widely diversified market portfolio—to diversify. The project's reward contribution is measured by its expected rate of return, $\mathcal{E}(\tilde{r}_i)$. In the CAPM, securities that contribute more risk have to offer a higher expected rate of return.

IMPORTANT: To estimate an appropriate CAPM expected rate of return for a project or firm, i.e., the cost of capital, you need three inputs:

1. The risk-free rate of return, r_F.
2. The expected rate of return on the market, $\mathcal{E}(\tilde{r}_M)$.
3. A firm's or project's beta with respect to the market, β_i.

The CAPM states that the relationship between risk contribution (β_i) and reward ($\mathcal{E}(\tilde{r}_i)$) is

$$\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i \qquad (13.1)$$

where i is the name of your project, and $\mathcal{E}(\tilde{r}_i)$ is your project's expected rate of return (the tilde indicates that the return is unknown). Through its choice of market-beta as the risk-measure, this formula in effect assumes that the relevant portfolio held by investors is the overall market portfolio.
$[\mathcal{E}(\tilde{r}_M) - r_F]$ is also called the **Equity Premium** or **Market Risk Premium**.

The CAPM Formula is about the suitable risk/reward trade-off for an investor holding the market portfolio.

The CAPM specifically places little importance on the standard deviation of an individual investment (a stock, project, fund, etc.). A stock's own standard deviation is only a measure of how risky it is *by itself*, which would only be of relevance to an investor who holds just this one stock and nothing else. But our CAPM investors are smarter. They care about their overall portfolio risk, and keep it low by holding the widely diversified overall stock market portfolio. To them, the one new stock matters only to the extent that it alters their market portfolio's risk—which is best measured by the stock's beta.

SIDE NOTE

Market-beta has an interesting use outside the CAPM. For example, if you believe IBM will outperform the stock market, but you do not know whether the stock market will do well or do poorly, you may want to **hedge** out the market risk. If β is 2, for example, it would mean that for each 10% rate of return in the stock market, IBM would move 20%. If you purchase \$100 in IBM and go short $\beta \cdot \$100 = \$200$ in the stock market, then you have neutralized the systematic stock market risk and therefore are left only with the idiosyncratic risk of IBM.

Solve Now!

Q 13.1 If the stock market has gone up by 16%, and the risk-free rate of return is 4%, then what would you expect a firm i with a stock-market beta of 1.5 to have returned?

The CAPM has three inputs. We will cover them in detail.

↦ *Will history repeat?, Section 7·1.E on Page 161*

For the three CAPM inputs, as always, you are really interested in the future: the future expected rate of return on the market and the future beta of your firm/project with respect to the market, not the past average rates of return or the past market betas. And, as usual, you have no choice other than to rely on estimates that are based at least partly on historical data. In Section 13·4, you will learn how you can estimate each CAPM input. But let's explore the model itself first, assuming that you know all the inputs.

13·2.B. The Security Market Line (SML)

Example of CAPM rates of return that individual securities should offer.

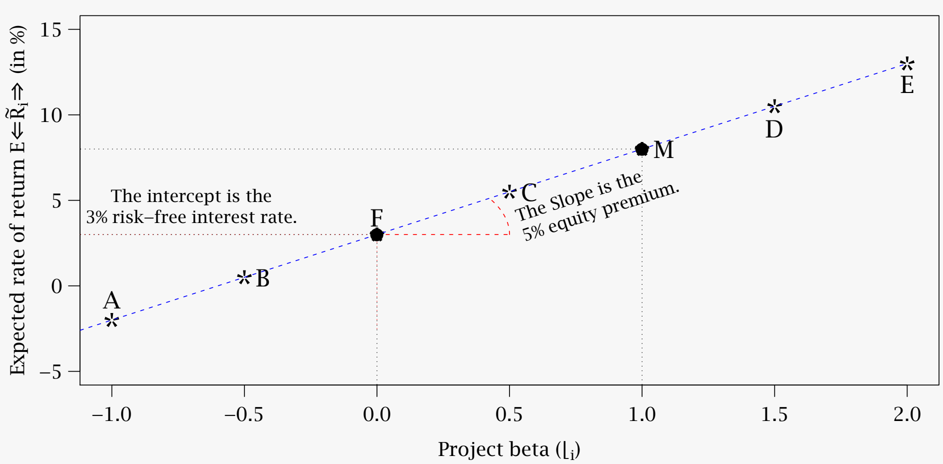
Let's apply the CAPM in a specific example. Assume that the risk-free rate is 3% per year and that the stock market offers an expected rate of return of 8% per year. The CAPM formula then states that a stock with a beta of 1 should offer an expected rate of return of 3% + (8% – 3%) · 1 = 8% per year; that a stock with a beta of 0 should offer an expected rate of return of 3% + (8% – 3%) · 0 = 3% per year; that a stock with a beta of $1/2$ should offer an expected rate of return of 3% + (8% – 3%) · 0.5 = 5.5% per year; that a stock with a beta of 2 should offer an expected rate of return of 3% + (8% – 3%) · 2 = 13% per year; and so on.

The SML is just the CAPM formula.

The CAPM formula is often graphed as the **security market line** (**SML**), which shows the relationship between the expected rate of return of a project and its beta. Figure 13.1 draws a first security market line for seven assets. Each stock (or project) is a point in this coordinate system. Because all assets properly follow the CAPM formula in our example, they must lie on a straight line. In other words, the SML is just a graphical representation of the CAPM formula. The slope of this line is the equity premium, $\mathcal{E}(\tilde{r}_M) - r_F$, and the intercept is the risk-free rate, r_F.

Figure 13.1: The Security Market Line

Note—I changed the sml figures a little bit, so that they all have the same scale. I added vertical and horizontal dot-line indicators for the rf and rm.



the symbols, $\mathcal{E}(\tilde{r}_i)$ and β_i should be in parentheses in the axes titles, not the weird symbols that are there now. I don't know how to fix this in my graphing program right now.

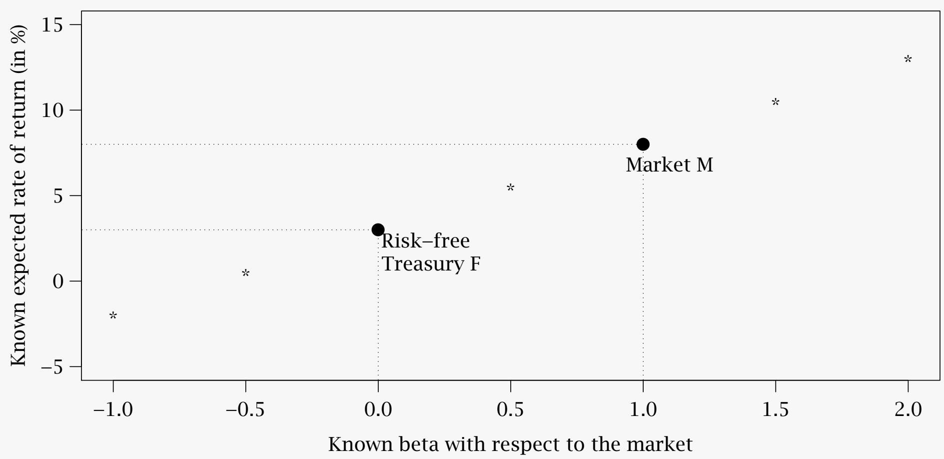
	Stock	A	B	F	C	M	D	E
Market Beta	β_i	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
Expected Rate of Return	$\mathcal{E}(\tilde{r}_i)$	−2.0%	0.5%	3.0%	5.5%	8.0%	10.5%	13.0%

This graph plots the CAPM relation $\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 3\% + (8\% - 3\%) \cdot \beta_i$, where β_i is the beta of an individual asset with respect to the market. In this graph, we assume that the risk-free rate is 3%, and the equity premium is 5%. Each point is one asset (such as a stock, a project, or a mutual fund). The point M in this graph could also be any other security with a $\beta_i = 1$. F could be the risk-free asset or any other security with a $\beta_i = 0$.

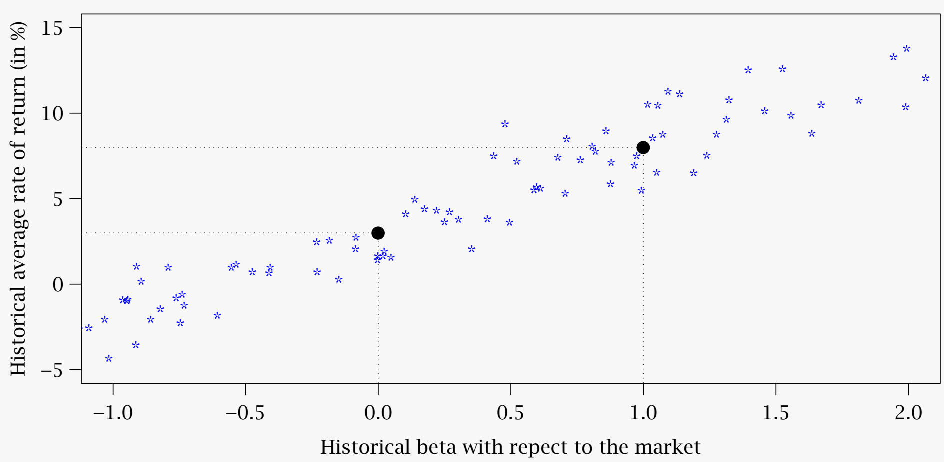
Figure 13.2: The Security Market Line in an Ideal CAPM World

Note—I changed the sml figures a little bit, so that they all have the same scale. I added vertical and horizontal dot-line indicators for the rf and rm.

(a) The Relationship among Unobservable Variables



(b) The Relationship among Observable Variables



The lower panel shows what we are usually confronted with: Historical average returns and historical betas are just estimates from the data. We hope they are representative of the true underlying mean returns and true betas, which in turn would mean that they will also be indicative of the future mean returns and betas.

If you know the inputs, the SML is a sharp line; if you estimate them, it is a scatterplot.

Alas, in the real world, even if the CAPM holds, you would not have the data to draw Figure 13.1. The reason is that you do not know true expected returns and true market betas. Figure 13.2 plots two graphs in a perfect CAPM world. Graph (a) repeats Figure 13.1 and assumes you know CAPM inputs—the true market betas and true expected rates of return—although in truth you really cannot observe them. This line is perfectly straight. In graph (b), you have to rely only observables—estimates of expected returns and betas, presumably based mostly on historical data averages. Now you can only fit an "estimated security market line," not the "true security market line." Of course, you hope that your historical data provide good, unbiased estimates of true market beta and true expected rates of return (and this is a big if), so that your fitted line will look at least approximately straight. (Section 9·1 already discussed some of the pitfalls.) A workable version of the CAPM thus can only state that there should roughly be a linear relationship between the data-estimated market beta and the data-estimated expected rate of return, just as drawn here.

↦ Section 9·1 on Page 188

Solve Now!

Q 13.2 The risk-free rate is 4%. The expected rate of return on the stock market is 7%. What is the appropriate cost of capital for a project that has a beta of 3?

Q 13.3 The risk-free rate is 4%. The expected rate of return on the stock market is 12%. What is the appropriate cost of capital for a project that has a beta of 3?

Q 13.4 The risk-free rate is 4%. The expected rate of return on the stock market is 12%. What is the appropriate cost of capital for an asset that has a beta of −3? Does this make economic sense?

Q 13.5 Is the real-world security market line a line?

Q 13.6 The risk-free rate is 4%. The expected rate of return on the stock market is 7%. A corporation intends to issue publicly traded bonds which *promise* a rate of return of 6%, and offer an *expected* rate of return of 5%. What is the implicit beta of the bonds?

Q 13.7 Draw the security market line if the risk-free rate is 5% and the equity premium is 4%.

Q 13.8 What is the equity premium, both mathematically and intuitively?

13·3 The CAPM Cost of Capital in the Present Value Formula

We usually use the CAPM output, the expected rate of return, as our discount rate.

For a corporate manager, the most important need for the CAPM arises in the denominator of the NPV formula:

$$\text{NPV} = C_0 + \frac{\mathcal{E}(\tilde{C}_1)}{1+\mathcal{E}(\tilde{r}_1)} + \frac{\mathcal{E}(\tilde{C}_2)}{1+\mathcal{E}(\tilde{r}_2)} + \cdots$$

The CAPM gives you an estimate for the opportunity cost of capital, $\mathcal{E}(\tilde{r}_1)$. This form again tells you that cash flows that correlate more with the overall market are of less value to your investors and therefore require a higher expected rate of return ($\mathcal{E}(\tilde{r})$) in order to pass muster (well, the hurdle rate).

13·3.A. Deconstructing Quoted Rates of Return—Risk Premiums

Reminder: Stated bond yields contain time and default premiums.

↦ Time and default premiums, Section 6·2.C on Page 125

Let me return to the subject of Section 6·2.C. You learned that in a perfect and risk-neutral world, stated rates of return consist of a time premium and a default premium. On average, the default premium is zero, so the expected rate of return is just the time premium.

The CAPM extends the expected rate of return to a world in which investors are risk averse. It gives you an expected rate of return that adds a **risk premium** (as a reward for your willingness to absorb risk) to the time premium.

M gives you me and risk premium.

$$\begin{aligned} \text{Promised Rate of Return} &= \text{Time Premium} + \text{Default Premium} + \text{Risk Premium} \\ \text{Actual Earned Rate} &= \text{Time Premium} + \text{Default Realization} + \text{Risk Premium} \\ \underbrace{\text{Expected Rate of Return}}_{\text{provided by the CAPM}} &= \text{Time Premium} + \text{Expected Risk Premium} \end{aligned}$$

In the risk-neutral perfect world, there were no differences in *expected* rates of returns across assets. There were only differences in *stated* rates of returns. The CAPM changes all this—different assets can now also have different *expected* rates of returns.

However, the CAPM does *not* take default risk into account, much less give you an appropriate stated rate of return. You should therefore wonder: How do you find the appropriate quoted rate of return in the real world? After all, it is this stated rate of return that is usually publicly posted, not the expected rate of return. Put differently, how do you put the default risk and CAPM risk into one valuation?

Important: *The CAPM totally ignores default risk and thus does not provide a default premium. You must take care of it yourself!*

Here is an example. Say you want to determine the PV of a corporate zero bond that has a beta of 0.25 and promises to deliver \$200 next year. This bond pays off 95% of the time, and 5% of the time it totally defaults. Assume that the risk-free rate of return is 6% per annum and that the expected rate of return on the market is 10%. Therefore, the CAPM states that the expected rate of return on your bond must be

A specific bond example: The price necessary to make you "even" relative to the Treasury gives you the time premium and default premium.

$$\begin{aligned} \mathcal{E}(\tilde{r}_{\text{Bond}}) &= 6\% + 4\% \cdot 0.25 = 7\% \\ &= r_{\text{F}} + [\mathcal{E}(\tilde{r}_{\text{M}}) - r_{\text{F}}] \cdot \beta_{\text{Bond}} \end{aligned}$$

This takes care of the time and risk premiums. To take the bond's default risk into account, you must still find the numerator. You cannot use use the promised payment. You must adjust it for the probability of default. You expect to receive not \$200, but

$$\begin{aligned} \mathcal{E}(\tilde{C}_{\text{Bond}}) &= 95\% \cdot \$200 + 5\% \cdot 0 = \$190. \\ &= \mathcal{P}rob(\text{No Default}) \cdot \text{Promise} + \mathcal{P}rob(\text{Default}) \cdot \text{Nothing} \end{aligned}$$

Therefore, the present value formula states that the value of the bond is

$$\text{PV}_{\text{Bond}} = \frac{\mathcal{E}(\tilde{C}_{\text{Bond}})}{1 + \mathcal{E}(\tilde{r}_{\text{Bond}})} = \frac{\$190}{1 + 7\%} \approx \$177.57$$

Given this price, you can now compute the promised (or quoted) rate of return on this bond:

$$\begin{aligned} \text{Promised rate of return} &= \frac{\$200 - \$177.57}{\$177.57} \approx 12.6\% \\ &= \frac{\text{Promised cash flow} - \text{PV}}{\text{PV}} \end{aligned}$$

You can now quantify the three components in this example. For this bond, the time premium of money is 6% per annum—it is the rate of return that an equivalent-term Treasury offers. The time premium plus the risk premium is provided by the CAPM, and it is 7% per annum. Therefore, 1% per annum is your "average" compensation for your willingness to hold this risky bond instead of the risk-free Treasury. The remaining 12.6% − 7% ≈ 5.6% per annum is the default premium: you do not expect to earn money from this default premium "on average." You only earn it if the bond does not default.

The risk premium is above and beyond the time and default premiums. On average, risky investments earn more than risk-free investments now.

$$\begin{aligned} 12.6\% &= 6\% + 5.6\% + 1\% \\ \text{Promised Interest Rate} &= \text{Time Premium} + \text{Default Premium} + \text{Risk Premium} \end{aligned}$$

In the real world, most bonds have fairly small market betas (often much smaller than 0.25) and thus fairly low risk premiums. Instead, most of the premium that ordinary corporate bonds quote above equivalent risk-free Treasury rates is not due to the risk premium, but due to the default premium (and some imperfect market premiums that you will learn in Chapter 15). For corporate projects and equity, however, the risk premium can loom quite large.

In sum, in this section you learned the following:

IMPORTANT:

- The CAPM provides an expected rate of return.
- This return is not a stated (promised, quoted) rate of return, because it does not include a default pren
- The probability of default must be handled in the NPV numerator (through the expected cash flow), anc in the NPV denominator (through the expected rate of return).

Solve Now!

Q 13.9 A corporate bond with a beta of 0.2 will pay off next year with 99% probability. The risk-free rate is 3% per annum, and the risk premium is 5% per annum.

(a) What is the price of this bond?

(b) What is its promised rate of return?

(c) Decompose the bond's quoted rate of return into its components.

Q 13.10 Going to your school has total additional and opportunity costs of $30,000 *this year and up-front*. With 90% probability, you are likely to graduate from your school. If you do not graduate, you have lost the entire sum. Graduating from the school will increase your 40-year lifetime annual salary by roughly $5,000 per year, but more so when the stock market rate of return is high than when it is low. For argument's sake, assume that your extra-income beta is 1.5. Assume the risk-free rate is 3%, the equity premium is 5%. What is the value of your education?

13·4 Estimating the CAPM Inputs

How can you obtain reasonable estimates of the three inputs into the CAPM formula $\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i$?

13·4.A. The Equity Premium

Jaqui, please change reference from rf-in-capm to eqprem-in-capm:
comparables.tex: estimate in Section mgaref[Equity Premium Estimates,
comparables.tex: Section]sect:rf-in-capm was that a reasonable equity premium seems to be
comparables.tex: around 3-5%. To estimate the expected rate of return on the stock market,

You must provide the CAPM with the equity premium. *Good luck!*

The input that is most difficult to estimate is the equity premium. It measures the extra expected rate of return that risky projects are offering above and beyond what risk-free projects are offering. Worse: it is not only difficult to estimate, but the value you choose can also have a tremendous influence over your estimated cost of capital. Of course, the theoretical CAPM model assumes that you know the *expected* rate of return on the market perfectly, and not that you have to estimate it. Yet, in real life, the equity premium is not posted anywhere, and *no one really knows the correct number.* There are a number of methods to guesstimate it—but unfortunately they do not tend to agree with one another. This leaves me with two choices: I can either throw you one estimate and pretend it is the only one, or I can tell you about the different methods that lead to different estimates. I prefer the latter, if only because the former would eventually leave you startled to discover that your boss has used another number and has therefore come up with another cost-of-capital estimate. I will explain the intuition behind each of five methods and describe the estimates their respective intuitions suggest. In this way, you can make up your own mind as to what you deem to be an appropriate equity premium estimate.

Method 1: historical averages.

1. **Historical averages I:** The first course of action is to assume that whatever the equity premium was in the past will continue in the future. In this case, you can rely on historical average equity premiums as good indicators of future risk premiums.

   Please note—I just learned that Morningstar may hold us up for $1,500 over the use of their few numbers. I may just recompute these numbers myself if they are copyrighted. Someone at AWP must check this. Morningstar appears a second time in the invmotive chapter.

   As of 2006, Morningstar reported the arithmetic average equity premium to be 12.3 − 3.8% = **8.5**% per annum if you start the data in 1926 and 12.5 – 6.9% = **5.6**% per annum if you start in 1970. (The buy-and-hold geometric equivalent averages were **6.5%** and **4.9%**.) However, if you start computing the average in 1869, even the arithmetic equity premium estimate drops to around **6.0%**. Maybe you should start in 1771? Or 1980? Which is the best estimation period? And is the United States the right country to consider, or should you take a more global and longer-term perspective? (A recent paper suggests that over many countries and more than 100 years, the average is more like **4.0%**. The United States may have had a lucky streak, not indicative of the future.) No one really knows what the right start date and set of countries should be for judging future U.S. performance. If you choose too few years, your sample average could be unreliable. For example, what happened over the last 20 or 30 years might just have been happenstance and not representative of the statistical process driving returns. Such an estimate would carry a lot of uncertainty. Although your estimate can be more reliable if you use more years, you are then leaning more heavily on a brave assumption that the world has not changed. That is, if you choose too many years, the data in the earlier part of your sample period may be so different from those today that they are no longer relevant. Do you really want to argue that the experience of 1880 still has relevance today?

↦ *Geometric vs. arithmetic averages, Section 7·1.A on Page 154*

Method 2: inverse historical averages.

2. **Historical averages II:** The second estimation method looks at historical equity premiums in the opposite light. You can draw on an analogy about bonds—if stocks become more desirable, perhaps because investors have become less risk averse, then more investors compete to own them, drive up the price, and thereby lower the future expected rates of return. High historical rates of return would then be indicative of low future expected rates of returns.

   An even more extreme version of this argument suggests that high past equity returns could have been due not just to high ex-ante equity premiums, but due to historical **bubbles** in the stock market. The proponents of the bubble view usually cannot quantify the appropriate equity premium, but they do argue that it is lower after recent market run ups—exactly the opposite of what proponents of the *Historical Averages I* argue.

SIDE NOTE

> A bubble is a run-away market, in which rationality has temporarily disappeared. There is a lot of debate as to whether bubbles in the stock market ever occurred. A strong case can be made that technology stocks experienced a bubble from around 1998 to 2000. It is often called the **dot-com bubble**, the **internet bubble**, or simply the **tech bubble**. There is no convincing explanation based on fundamentals that can explain *both* why the Nasdaq Index had climbed from 2,280 in March 1999 to 5,000 on March 27, 2000, *and* why it then dropped back to 1,640 on April 4, 2001.

Method 3: dividend or earnings yields.

3. **Current predictive ratios:** The third method tries to actively predict the stock market rate of return with historical dividend yields (i.e., the dividend payments received by stockholders). Higher dividend yields should make stocks more attractive and therefore predict higher future equity premiums. The equity premium estimation is usually done in two steps: First, you must estimate a statistical regression that predicts next year's equity premium with this year's dividend yield; then, you substitute the currently prevailing dividend yield into your estimated regression to get a prediction. Unfortunately, as of 2007, current dividend yields were so low that the predicted equity premiums were negative—which is not a sensible number. Variations of this method have used interest rates or earnings

yields, typically with similar results. In any case, the empirical evidence suggests that this method would have yielded poor predictions—for example, it predicted low equity premiums in the 1990s, which was a period of superb stock market performance.

Method 4: introspection and philosophy.

4. **Philosophical prediction:** The fourth method wonders how much rate of return is required to entice reasonable investors to switch from bonds into stocks. Even with an equity premium as low as 3%, over 25 years, an equity investor would end up with more than twice the money of a bond investor. Naturally, in a perfect market, nothing should come for free, and the reward for risk-taking should be just about fair. Therefore, equity premiums of 8% just seem too high for the amount of risk observed in the stock market. This philosophical method generally suggests equity premiums of about **1%** to **3%.**

Method 5: just ask?

5. **Consensus survey:** The fifth method just asks investors or experts (or people who don't know either) what they deem reasonable. The ranges can vary widely, and they seem to correlate with very recent stock market returns. For example, in late 2000, right after a huge run up in the stock market, surveys by *Fortune* or *Gallup/Paine Webber* had investors expect equity premiums as high as 15% per year. (They were acutely disappointed: The stock market dropped by as much as 30% over the following two years. Maybe they just got the sign wrong?!) The consulting firm McKinsey uses a standard of around 5% to 6%. The Social Security Administration settled on a standard of around 4%. A joint poll by Graham and Harvey (from Duke) and CFO magazine found that the 2005 average equity premium estimate of CFOs was around 3% per annum. And in a survey of finance professors in late 2007, the most common equity premium estimate was **5%** for a 1-year horizon and 6% for a 30-year horizon.

Analysts' estimates are all over the map, too. Estimates between 2% and 6% per annum seem reasonable.

What to choose? Welcome to the club! No one knows the true equity premium. On Monday, February 28, 2005, the Wall Street Journal reported the following average after-inflation forecasts from then to 2050 (per annum):

		Inflation Adjusted			
Name	Organization	Stocks	Govmnt bonds	Corp. bonds	Forecast Equity Premium
William Dudley	Goldman Sachs	5.0%	2.0%	2.5%	3.0%
Jeremy Siegel	Wharton	6.0%	1.8%	2.3%	4.2%
David Rosenberg	Merrill Lynch	4.0%	3.0%	4.0%	1.0%
Ethan Harris	Lehman Brothers	4.0%	3.5%	2.5%	0.5%
Robert Shiller	Yale	4.6%	2.2%	2.7%	2.4%
Robert LaVorgna	Deutsche Bank	6.5%	4.0%	5.0%	2.5%
Parul Jain	Nomura	4.5%	3.5%	4.0%	1.0%
John Lonski	Moody's	4.0%	2.0%	3.0%	2.0%
David Malpass	Bear Stearns	5.5%	3.5%	4.3%	2.0%
Jim Glassman	J.P. Morgan	4.0%	2.5%	3.5%	1.5%
				Average	2.0%

It does not matter that these numbers are inflation adjusted. Because the equity premium is a difference, inflation cancels out. However, it matters whether you quote the equity premium with respect to a short-term or a long-term interest rate. It is more common to use a short rate, because these are typically safer and therefore closer to the risk-free rate that is in the spirit of the CAPM. This is why you may want to add another 1% to the equity premium estimates in this table—the long-term government bonds used in the table usually carry higher interest rates than their short-term counterparts. On the other hand, if your project is longer term, you may want to adopt a risk-free bond whose duration is more similar to that of your project. You would then even prefer the equity premium estimates in this table. In addition, these are arithmetic rates of return. You already know that they are higher than geometric rates of return.

↦ *Duration, Section c on Page 107*

↦ *Geometric vs. arithmetic averages, Section 7·1.A on Page 154*

(A +20% rate of return followed by a −20% rate of return gives you a 0% arithmetic average, but leaves you with a 2-year loss of 4%.) Thus, if your project is long horizon, don't expect your project to offer geometric returns that can be compared to arithmetic returns on the market. It would be an unfair benchmark.

Jacqui: my latex macro screws up on section number references when the section reference is in an appendix. The above "duration" reference should read something like Section "4.App.c" and not just "c".

Remain consistent: Don't use different equity premium estimates for different projects.

You now know that no one can tell you the authoritative number for the equity premium. It does not exist. Everyone is guessing, but there is no way around it—you have to take a stance on the equity premium. I cannot insulate you from this problem. I could give you the arguments that you should contemplate when you are picking *your* number. Now I can also give you my own take: First, I have my doubts that equity premiums will return to the historical levels of 8% anytime soon. (The twentieth century was the American Century for a good reason: There were a lot of positive surprises for American investors.) I personally prefer equity premiums estimates between 2% and 4%. (Incidentally, it is my impression that there is relatively less disagreement about equity premiums forecasts today than there was just 5 to 10 years ago.) But realize that reasonable individuals can choose equity premiums estimates as low as 1% or as high as 8%. Of course, I personally find their estimates less believable the farther they are from my own personal range. And I find anything outside this 1% to 8% range just too tough to swallow. Second, whatever equity premium you do choose, *be consistent*. Do not use 3% for investing in one project and 8% for investing in another. Being consistent can sometimes reduce your relative mistakes in choosing one project over another.

The equity premium is a most important premium even absent the CAPM.

Yes, the equity premium may be difficult to estimate, but there is really no way around taking a stance. Even if you had never heard of the CAPM, you would still consider the equity premium to be the most important number in finance (together with the risk-free rate, the other CAPM input). If you believe that the equity premium is high, you would want to allocate a lot of your personal assets to stocks. Otherwise, you would allocate more to bonds. You really do need it for investing purposes, too. No escape possible.

Anecdote: Was the 20th Century really the "American Century?"

The compound rate of return in the United States was about 8% per year from 1920 to 1995. Adjusted for inflation, it was about 6%. In contrast, an investor who would have invested in Romania in 1937 would have experienced not only the German invasion and Soviet domination, but also a real annual capital appreciation of about −27% per annum over the 4 years of the stock market existence (1937–1941). Similar fates befell many other Eastern European countries—but even countries not experiencing political disasters often proved to be less than stellar investments. For example, Argentina had a stock market from 1947 to 1965, even though its only function seems to have been to wipe out its investors. Peru tried three times: From 1941 to 1953 and from 1957 to 1977, its stock market investors lost *all* their money. But three times was a charm: From 1988 to 1995, its investors earned a whopping 63% real rate of return. India's stock market started in 1940 and offered its investors a real rate of return of just about −1% per annum. Pakistan started in 1960 and offered about −0.1% per annum. Even European countries with long stock market histories and no political trouble did not perform as well as the United States. For example, Switzerland and Denmark earned nominal rates of return of about 5% per annum from 1921 to 1995, while the United States earned about 8% per annum.
The United States stock market was indeed an unusual above-average performer in the twentieth century. Will the twenty-first century be the Chinese century?
Source: Goetzmann and Jorion, 1999.

The CAPM is about relative pricing, not absolute pricing.

In a corporate context, like everyone else, you cannot let your limited knowledge of the equity premium stop you from making investment decisions. In order to use the CAPM, you do need to judge the appropriate reward for risky projects relative to risk-free projects. Indeed, you can think of the CAPM as telling you the *relative* expected rate of return for projects, not their *absolute* expected rate of return. Given *your* estimate of how much risky projects should earn relative to safe projects, the CAPM can tell you the right costs of capital for projects of riskiness "beta." But the basic judgment of the appropriate spread between risky and safe projects is left up to you.

No way around it: you must guesstimate the equity premium.

The S&P 500 is usually used as an approximation for the market.

Finally, I have been deliberately vague about the "market." In CAPM theory, the market should be all investable assets in the economy, including real estate, art, risky bonds, and so on. In practice, we typically use only a stock market index. And among stock market indexes, it often does not matter too much which index is used—whether it is the value-weighted stock market index, the **Dow Jones 30** (another popular market index consisting of 30 large stocks in different industries), or the S&P 500. The S&P 500 is perhaps the most often used stand-in for the stock market, because its performance is posted everywhere, and because historical returns are readily downloadable. From the perspective of a corporate executive, it is a a reasonable simplification to use the S&P 500 as the market.

Solve Now!

Q 13.11 What are appropriate equity premium estimates? What are not? What kind of reasoning are you relying on?

13·4.B. The Risk-Free Rate and Multiyear Considerations

Which risk-free rate?

The second input into the CAPM formula is the risk-free rate of return. It is relatively easily obtained from Treasuries. There is one small issue, though—which one? What if the yield curve is upward sloping and Treasuries yield 2% per year over 1 year, 4% per year over 10 years, and 5% per year over 30 years? How would you use the CAPM? Which interest rate should you pick in a multiyear context?

Advice: Pick the interest rate for a Treasury that is "most similar" to your project.

Actually, the CAPM offers no guidance, because it has no concept of more than one single time period or a yield curve (Chapter 5). However, from a practical perspective, it makes sense to match projects and Treasuries:

To estimate one benchmark required expected rate of return (e.g., for benchmarking your project's one IRR), you should probably use the yield on Treasuries that seem to take similarly time to come to fruition as your own project. A good rule of thumb is to pick the risk-free rate closest by some measures (maturity or duration) to your project. For example, to value a machine that produces for 3 years, it could make sense to use an average of the 1-year, 2-year, and 3-year risk-free zero interest rates, perhaps 2.5% per annum. On the other hand, if you have a 10-year project, you would probably use the 10-year Treasury rate of 4% as your risk-free rate of return. You may think this is a pretty loose method to handle an important question (and it is), but it is also a very reasonable one. Think about the opportunity cost of capital for an investment with a beta of 0. If you are willing to commit your money for 10 years, you could earn the 10-year Treasury rate of return. It would be your opportunity cost of capital. If you are willing to commit your money only for 3 months, you could only earn the 3-month Treasury rate—a lower opportunity cost for your capital.

To estimate multiple required expected rates of return (e.g., for an NPV analysis with cash flows occurring at many different times), you should probably use different zero-bond rates, each corresponding to the timing of the cash flow in the numerator.

There is universal agreement that you should use a risk-free rate that is similar to the duration of your project in the first part of the CAPM formula (where it appears by itself). Thus, if your project has a beta of 0, you should expect to offer the same rate of return as the duration-equivalent risk-free Treasury. If your project takes longer to complete, and if the yield curve is upward sloping, then your project would have to offer a higher expected rate of return.

But should you also use a different risk-free rate in the second part of the formula (where the risk-free rate is part of the equity premium)? Your answer must depend on whether you believe that the expected rate of return on the stock market is higher for longer-term investments. This would be a reasonable conjecture—after all, if risk-averse Treasury investors can expect a higher rate of return if they buy longer-term claims, why would risk-averse equity investors not also expect a higher rate of return if they buy longer-term claims? If the expected rate of return on the stock market is higher for longer-term projects, too, then any premium for longer-term investments could cancel out in the equity premium, and you could use the same equity premium regardless of how long-term your project is. Unfortunately, no one knows the answer. (After all, we don't know with great confidence even the short-term expected rate of return on the stock market.) My personal preference is to use the same (geometric) equity premium estimate, regardless of the duration of the project. Other CAPM users may come to a different conclusion.

A philosophical question of practical importance: Is the equity premium horizon-dependent?

Solve Now!

Q 13.12 What is today's risk-free rate for a 1-year project? For a 10-year project?

Q 13.13 If you can use only one Treasury, which risk-free rate should you be using for a project that will yield $5 million each year for 10 years?

13·4.C. Investment Projects' Market Betas

Finally, you must estimate your project's **market beta**. It measures how the rate of return of your project fluctuates with that of the overall market. Unlike the previous two inputs, which are the same for every project/stock in the economy, the beta input depends on your specific project characteristics: Different investments have different betas.

Unlike the risk-free rate and the equity premium, beta is specific to each project.

The Implications of Beta for a Project's Risk and Reward

Before we get to beta estimation, let's use some intuition on how market beta should relate the returns of individual stocks to those of the market. The market beta has an influence both on the expected return of projects and on the range of observed returns. Say the risk-free rate is 3%, the expected rate of return on the market is 8%, and therefore the equity premium is 5%. A stock with a beta of −1 would therefore have an expected rate of return of −2%. However, more than likely, the stock market's rate of return will not be *exactly* 8%. Let's consider just one positive and one negative market scenario (in addition to the most likely outcome of 8%) as a stand-in for market volatility. Figure 13.3 illustrates how a negative beta influences the mean and volatility of stock returns in different market conditions:

Beta creates both a mean rate of return, and an "amplification factor" of the market rate of return.

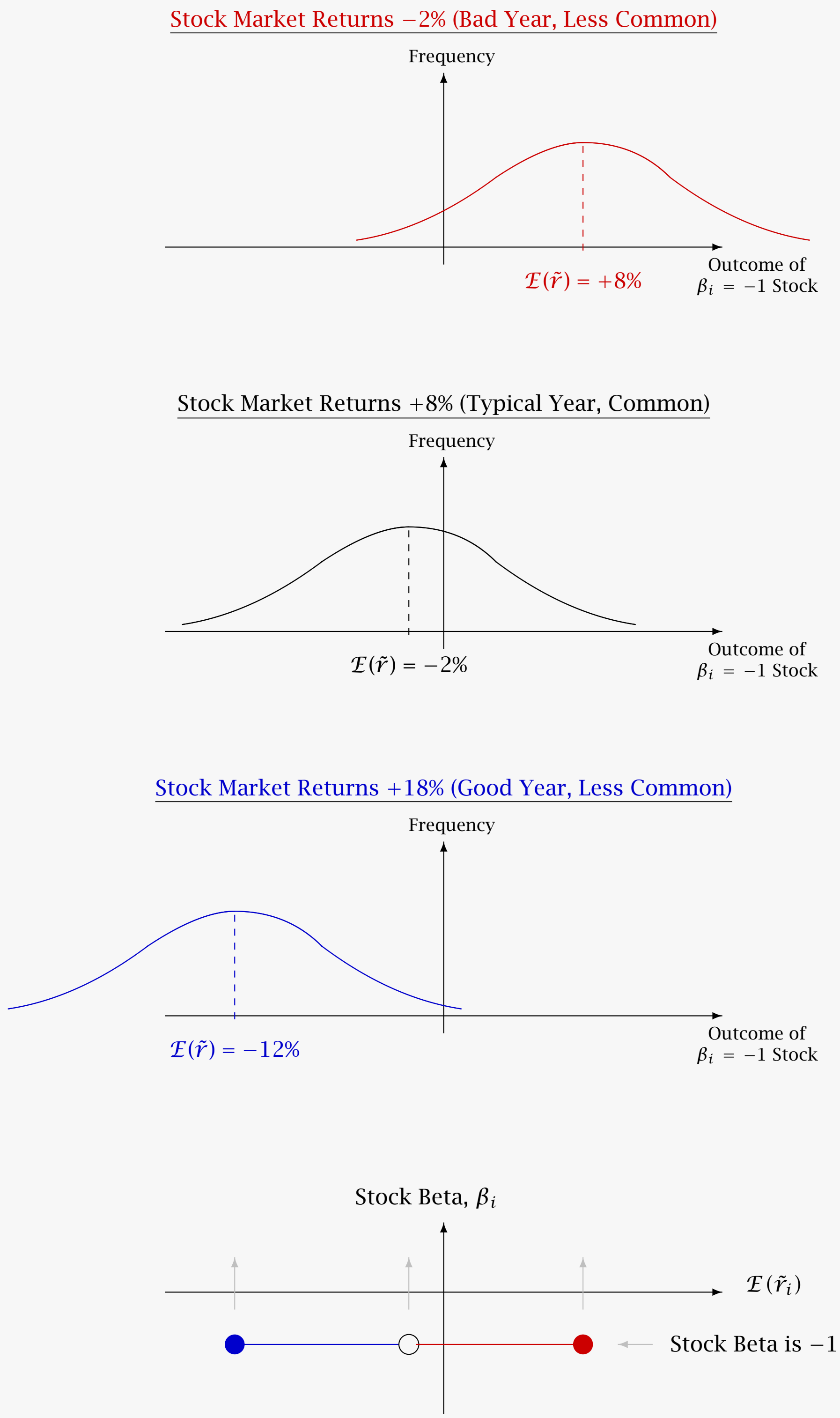
Bad year for the stock market: If the stock market were to drop by 10% relative to its mean of 8%—i.e., the market would return a rate of return of $\tilde{r}_M = (-2\%)$—then your $\beta = (-1)$ stock would not be expected to earn −2%, but $\mathcal{E}(\tilde{r}_i) = r_F + \beta_i \cdot [\tilde{r}_M - r_F] = 3\% + (-1) \cdot (-2\% - 3\%) = +8\%$.

Normal year for the stock market: If the stock market were to perform by happenstance 8%, just as expected, then your stock would be expected to earn its CAPM rate of return of $\mathcal{E}(\tilde{r}_i) = 3\% + (-1) \cdot (8\% - 3\%) = -2\%$.

Good year for the stock market: If the stock market were to outperform its expectation by 10% (for a rate of return of 18%), then your stock would be expected to earn a rate of return of $\mathcal{E}(\tilde{r}_i) = 3\% + (-1) \cdot (18\% - 3\%) = -12\%$.

In all cases, your stock would have some idiosyncratic risk, too, which would add "noise." These are just expected returns conditional on how the market performs—the actual return will be different, drawn from a distribution around these expected values. The bottom panel shows just the mean in these three scenarios.

Figure 13.3: Conditional Performance of a Stock with a Market Beta of $\beta_i = -1$



This figure shows that for a stock with negative beta of −1, the better the stock market performs, the worse this stock performs. In this plot, our stock obeys the CAPM with $\mathcal{E}(\tilde{r}_i) = 3\% + (-1) \cdot (\mathcal{E}(\tilde{r}_M) - 3\%)$.

Let's think about how stocks with other market betas would perform if the market underperformed, performed as expected, or outperformed. For example, consider a stock with a market beta of +2. If the stock market returns −2% (underperforms by 10%), then your $\beta_i = 2$ stock would return $\tilde{r}_i = 3\% + 2 \cdot (-2\% - 3\%) + \text{noise} = -7\% + \text{noise}$. Conversely, if the stock market were to increase by 10% relative to its mean—i.e. return an absolute +18%—you would expect your positive beta stock to do really well ($\mathcal{E}(\tilde{r}_i) = 3\% + 2 \cdot (18\% - 3\%) = 33\%$).

Let's consider different stocks the same way.

Figure 13.4 rotates the last panel in Figure 13.3, so that the market beta is on the X-axis, and that the expected rate of return is on the Y-axis. It then repeats the computations for stocks with different market betas, which shows how different beta stocks are expected to perform, conditional on whether the market beats its mean (of 8% by 10%, i.e., +18%), hits its mean (of 8%), or misses its mean (of 8% by 10%, i.e., −2%). You can see how the market beta determines both the stock's expected rate of return—the mean given by the CAPM—and how it dampens or amplifies the effect of the stock market performance on the stock. The latter is really just the definition of market beta—it measures how a project comoves with the stock market. The sign of the market beta determines whether the investment tends to move with or against the stock market. And it is of course the CAPM that posits how the expected rate of return should be increasing with the market beta.

Beta can be thought of as an amplifier of market movements.

Beta Estimation

How do you find good beta estimates? Depending on the project, this can be easy or difficult.

Richard is asking me: "Does the design treat these as heads or are these simply part of an unnumbered paragraph list? I can ask Paul if necessary—I haven't seen a design sample." A: I have no idea.

Market betas for publicly traded firms: For publicly trading stocks, finding a market beta for its equity is usually easy. Almost every financial website publishes them.

If publicly traded, it's easy: look it up or run regressions.

Market betas from a regression: You could also run the market model regression yourself. There is no mystery: The betas published on financial websites are really just estimated from historical time-series regressions, too. They do exactly what we did in Section 10.9·4.B: They compute the covariance, divide it by the variance, and perhaps do a little bit of shrinking.

↦ *Estimating betas from historical data, Section 9·4.B on Page 194*

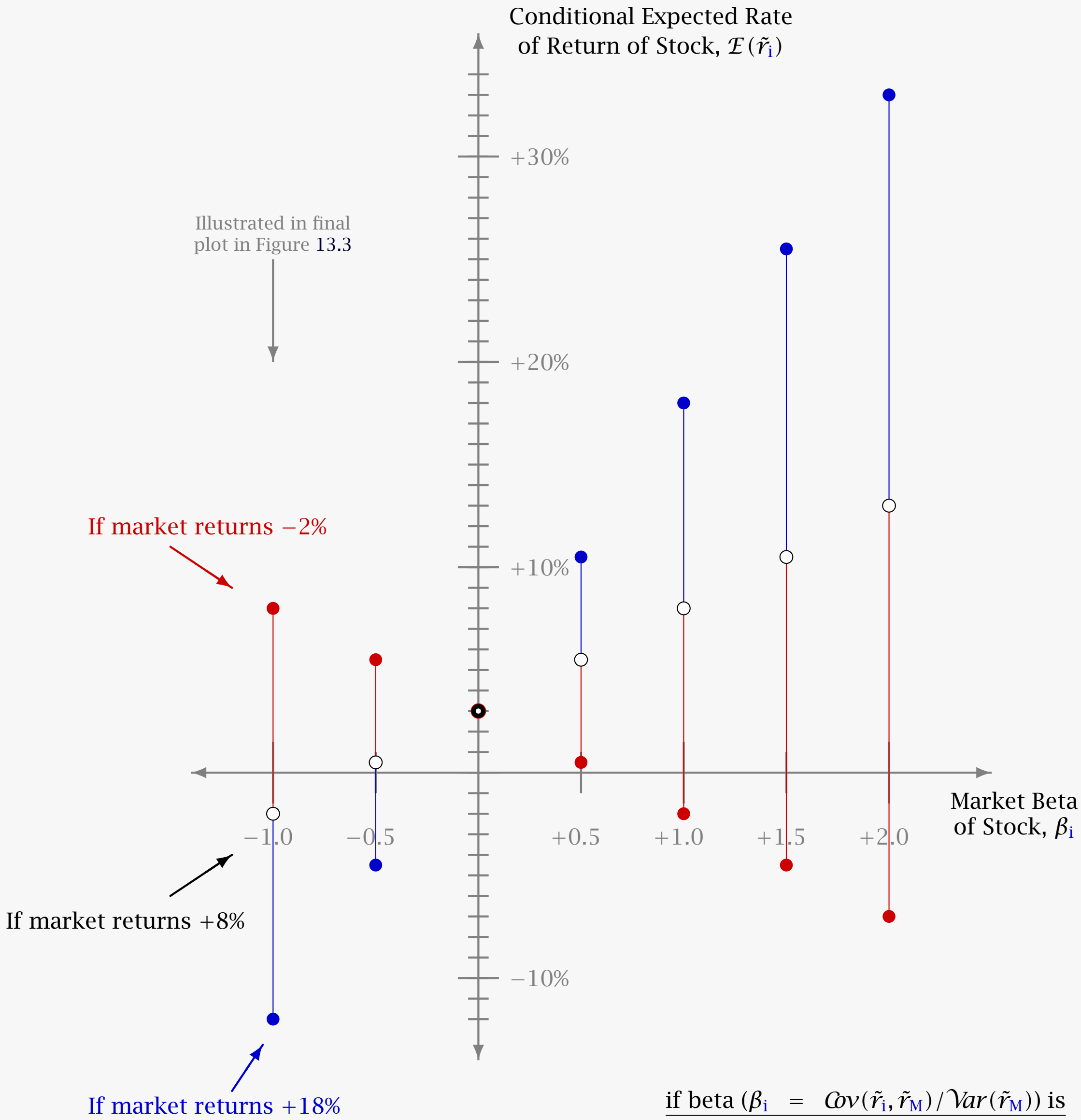
Market betas from comparables: One problem with the simple regression method is that individual betas are often very noisy. (Shrinking helps a little, though.) For example, think of a pharmaceutical company whose product happened to be rejected by the FDA. This would cause a large negative rate of return in one particular month. This month would now become a "statistical outlier." If the market happened to go up (down) this particular month, the company would likely end up having a negative (positive) market beta estimate—and this beta estimate would likely be unrepresentative of the future market beta. In the long run, such announcements would appear randomly, so beta would still be the right estimate—but by the time the long run happens, we may already be dead. To reduce estimation noise in practice, it is common to estimate not just the beta of the firm in question but also the beta of a couple of similar firms (comparables similar in size and industry, perhaps), and then to use a beta that reflects some sort of average among them.

If your project has no historical rate-of-return experience—perhaps because it is only a division of a publicly traded company or because the company is not publicly traded—you may have no choice other than this method of estimating a beta from comparable firms. (However, recall that the CAPM is only meaningful to begin with if your investors hold most of their wealth in the market portfolio.) For example, if you believe your new soda company project is similar to PepsiCo, you could adopt the asset beta of PepsiCo and use it to compute the CAPM expected rate of return. Realizing that firms that are smaller than PepsiCo, such as your own, tend to have higher betas, you might increase your beta estimate.

For private firms, use comparable, publicly-traded stocks.

Market betas based on economic intuition: If you really cannot think of a good publicly traded firm that you trust to be a good comparable, you may have to rely more heavily on your

Figure 13.4: The Effect of Market Beta on Stock Returns in Good and Bad Markets



	if beta ($\beta_i = \mathcal{C}ov(\tilde{r}_i, \tilde{r}_M)/\mathcal{V}ar(\tilde{r}_M)$) is			
	−1.0	0.0	+1.0	+2.0
The stock's expected rate of return	−2%	+3%	+8%	+13%
If market outperforms its mean of 8% by 10% ($\tilde{r}_M = +18\%$), then stock $\tilde{r}_i$ is expected to return	−2% − 10% = −12%	3% ± 0% = 3%	8% + 10% = 18%	13% + 20% = 33%
If market underperforms its mean of 8% by 10% ($\tilde{r}_M = -2\%$), then stock $\tilde{r}_i$ is expected to return	−2% + 10% = +8%	3% ± 0% = 3%	8% − 10% = −2%	13% − 20% = −7%

Each line represents the range of return outcomes for one stock (with one particular market beta) if the market rate of r were to be between −10% and +10%. The black circle is the unconditional expected rate of return (or conditional on the m turning in its expected performance of 8%)—i.e., points on the security market line $\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 3\% + 5$ The red solid circles show the expected rate of return conditional on a market rate of return of −10%. Stocks with neg beta are expected to perform well in this case. The blue solid circles show the expected rate of return conditional on a m rate of return of +10%. Stocks with a negative beta are expected to perform poorly in this case.

judgment. Think about how the rate of return of your project is likely to covary with the stock market. If you can make such a judgment, you can rearrange the CAPM Formula to obtain a beta estimate:

$$\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i \quad \iff \quad \beta_i = \frac{\mathcal{E}(\tilde{r}_i) - r_F}{\mathcal{E}(\tilde{r}_M) - r_F}$$

The right side of this formula helps translate your intuition into a beta estimate. You can ask such questions as "What rate of return (above the risk-free rate) will your project have if the stock market were to have +10% or –10% rate of return (above the risk-free rate)?" Clearly, such guess work is difficult and error-prone—but it can provide a beta estimate when no other is available.

Equity and Asset Betas Revisited

Don't use the equity beta to estimate your project's hurdle rate.

↦ *Asset and equity betas, Formula* **??** *on Page* **??**

It as important that you always distinguish between asset betas and equity betas. Let me remind you with an example. Assume that the risk-free rate is 4% and the equity premium is 5%. You own a \$100 million project with an asset-beta of 2.0 that you can finance with \$20 million of risk-free debt. By definition, risk-free debt has a beta of 0.0. To find your equity beta, write down the formula for your asset beta (firm beta):

$$\begin{array}{ccccc} & 20\% \cdot (0.0) & + & 80\% \cdot (\beta_{\text{Equity}}) & = 2.0 \\ \beta_{\text{Firm}} = & \left(\frac{\text{Debt value}}{\text{Firm value}}\right) \cdot \beta_{\text{Debt}} & + & \left(\frac{\text{Equity value}}{\text{Firm value}}\right) \cdot \beta_{\text{Equity}} & \end{array}$$

Solve this to find that your equity beta is 2.5. This is what you would find on Yahoo!*Finance*. You would not want to base your project hurdle rate on what you read on Yahoo. Such a mistake would recommend you use a hurdle rate of $\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 4\% + 5\% \cdot 2.5 = 16.5\%$. This would be too high. Instead, you should require your project to return $\mathcal{E}(\tilde{r}_i) = 4\% + 5\% \cdot 2.0 = 14\%$.

If you use comparables, first unlever them.

Conversely, if your project is private, you may have to find its hurdle rate by looking at public comparables. Let's presume you find a similarly sized firm with a similar business that Yahoo!*Finance* lists with a beta of 4. Remember that financial websites always list only the equity beta. The CAPM tells you that the expected rate of return on the equity is $4\% + 5\% \cdot 4 = 24\%$. However, this is not necessarily the hurdle rate for your project. When you look further on Yahoo!*Finance*, you see that your comparable is financed with 90% debt and 10% equity. (If the comparable had very little debt, a debt beta of 0 might have been a good assumption, but unfortunately, in this case, it is not.) Corporate debt rarely has good historical return data that would allow you to estimate a debt beta. Consequently, practitioners often estimate the expected rate of return on debt via debt comparables based on the credit rating. Say your comparable's debt is rated BB and say that BB bonds have offered *expected* rates of return of 100 basis points above the Treasury. (This might be 200 basis points *quoted* above the Treasury). With the Treasury standing at 4%, you would estimate the comparable's cost of capital on debt to be 5%. The rest is easy. The expected rate of return on your project should be

↦ *Credit ratings, Section 6·2.D on Page 126*

$$\begin{array}{rcccl} \mathcal{E}(\tilde{r}_{\text{Project}}) = & 90\% \cdot 5\% & + & 10\% \cdot 24\% & = 6.9\% \\ = & w_{\text{Debt}} \cdot \mathcal{E}(\tilde{r}_{\text{Debt}}) & + & w_{\text{Equity}} \cdot \mathcal{E}(\tilde{r}_{\text{Equity}}) & \end{array}$$

This would make a good hurdle rate estimate for your project.

Solve Now!

Q 13.14 According to the CAPM formula, a zero-beta asset should have the same expected rate of return as the risk-free rate. Can a zero-beta asset still have a positive standard deviation? Does it make sense for such a risky asset to offer no higher a rate of return than a risk-free asset in a world in which investors are risk averse?

Q 13.15 If you had representative historical project returns, how would you obtain the stock market beta?

Q 13.16 A comparable firm (with comparable size and in a comparable business) has a Yahoo!*Finance*-listed equity beta of 2.5 and a debt/asset ratio of 2/3. The debt is almost risk free.

(a) Estimate the beta for your firm if your projects have similar betas, but your firm will carry a debt/asset ratio of 1/3.

(b) If the risk-free rate is 3% and the equity premium is 2%, then what should you use as your firm's hurdle rate?

(c) What do investors demand as the expected rate of return on the comparable firm's equity and on your own equity?

Q 13.17 You own a stock market portfolio that has a market beta of 2.4, but you are getting married to someone who has a portfolio with 0.4. You are three times as wealthy as your future significant other. What is the beta of your joint portfolio?

Q 13.18 here? Assume that you can short. If your portfolio has a market beta of 0.6 and you can short a fund with a market beta of 1, what portfolio do you have to purchase to eliminate all market risk?

13·5 Empirical Evidence: Is the CAPM the Right Model?

Now you know how securities should be priced in a perfect CAPM world. What evidence would lead you to conclude that the CAPM is *not* an accurate description of reality? And does the CAPM seem to hold or not?

13·5.A. The SML If the CAPM Does Not Work

Q: What happens if a stock offers too much or too little expected rate of return? A: Investor stampedes.

What would happen from the CAPM's perspective if a stock offered more than its appropriate expected rate of return? Investors in the economy would want to buy more of the stock than would be available: Its price would be too low. It would be too good a deal. Investors would immediately flock to it, and because there would not be enough of this stock, investors would bid up its price and thereby lower its expected rate of return. Eventually, the price of the stock would settle at the correct CAPM expected rate of return. Conversely, what would happen if a stock offered less than its due expected rate of return? Investors would not be willing to hold enough of the stock: The stock's price would be too high, and its price would fall.

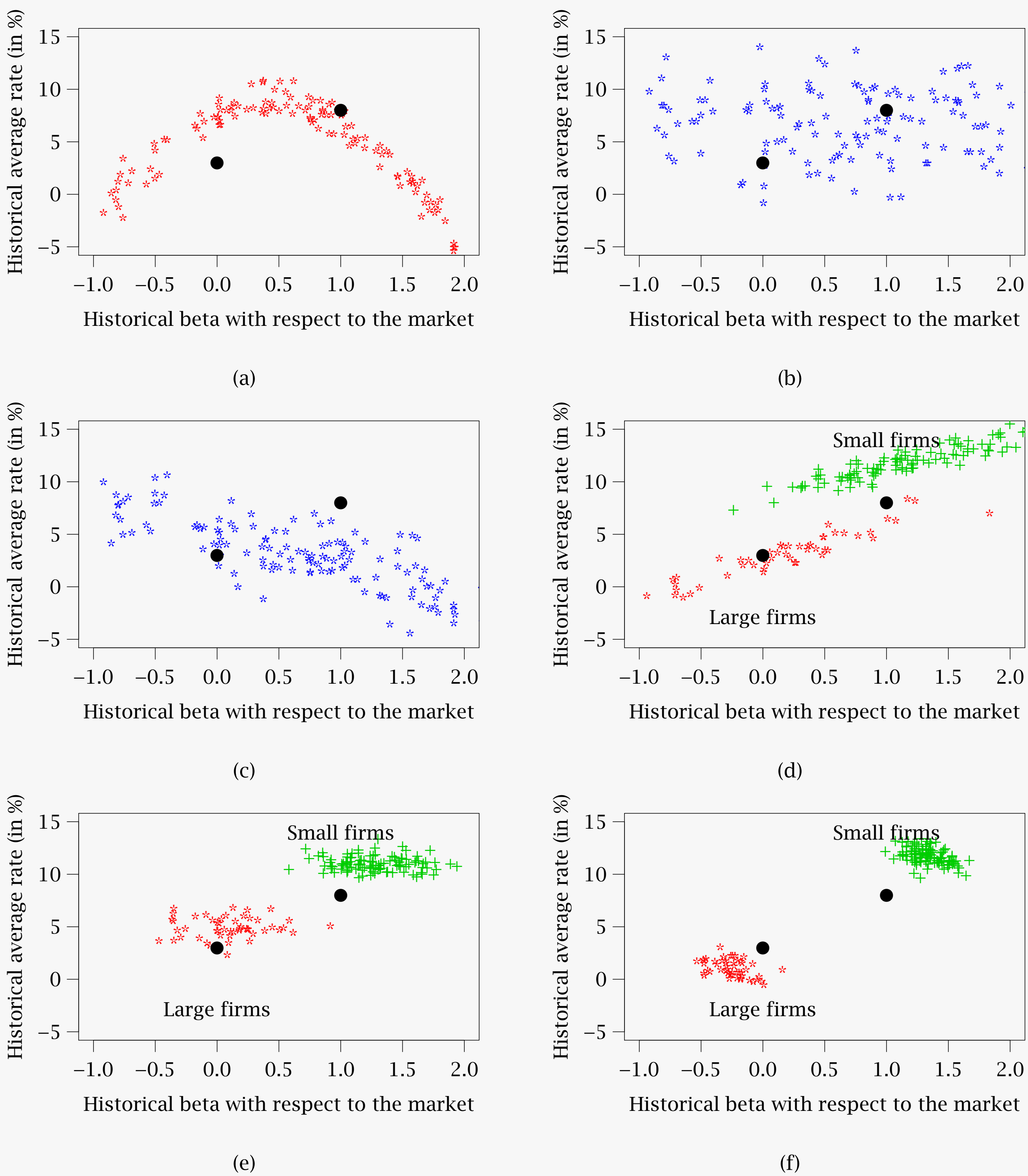
Assets not priced according to the CAPM do not allow you to make money for nothing. However, it could imply good deals.

Neither situation should happen in the real world—investors are just too smart. However, you must realize that if a stock were not to follow the CAPM formula, buying it would still be risky. Yes, such a stock would offer too high or too low an expected rate of return and thus be a good or a bad deal, attracting too many or too few investors chasing a limited amount of project—but it would still remain a risky investment, and no investor could earn risk-free profit.

Some security market lines if the CAPM does not hold.

Under what circumstances would you lose faith in the CAPM? Figure 13.5 shows what security market relations could look like if the CAPM did not work. In plot (a), the rate of return does not seem to increase linearly with beta if beta is greater than about 0.5. Because beta is a measure of risk contribution to your market portfolio, as an investor, you would not be inclined to add stocks with betas greater than 1 or 2 to your (market) portfolio—these stocks' risk contributions are too high, given their rewards. You would like to deemphasize these firms, tilting your portfolio toward stocks with lower betas. In plot (b), the rate of return seems unrelated to beta, but the average rate of return on the stock market seems quite a bit higher than the risk-free rate of return. In this case, you again would prefer to tilt your portfolio away from the overall market and toward stocks with lower betas. This would allow you to construct a portfolio that has lower overall risk and higher expected rate of return than the market portfolio. In plot (c), higher beta securities offer *lower* expected rates of return. Again, you should prefer moving away from your current portfolio (the market) by adding more of stocks with lower market betas.

Figure 13.5: The Security Market Line in non-CAPM Worlds



Each point is the historical beta and historical average rate of return on one asset. (The market and risk-free rate are noted with a fat dot.) In plots (d) through (f), small firms tend to have both higher market betas and rates of return.

In these figures, the security market line does not appear to be one line, depending only on market beta, just as the CAPM suggests. Therefore, if these patterns are not just statistical mirages, you should be able to invest better than just in the market: from the CAPM perspective, there are “great deal” stocks that offer too much expected return given their risk contributions to your (market) portfolio, which you would therefore want to overemphasize; and “poor deal” stocks that offer too little expected return, given their risk contribution, which you would therefore want to underemphasize.

Richard notes: “place fig after its text call-out.”

More security market lines if the CAPM fails with respect to a specific better alternative.

Plots (d) through (f) illustrate a distinction between small firm stocks and large firm stocks—categories that the analyst has to identify. In plot (d), even though each cluster has a positive relationship between beta and the expected rate of return, small firms have a different relationship than large firms. Yet, the CAPM says not only that market beta should matter, but that *market beta is all that should matter*. If you knew whether a firm is small or large, you could do better than you could if you relied only on the market beta. Rather than just holding the market portfolio, you would prefer tilting your portfolio toward small stocks and away from large stocks—for a given beta contribution to your portfolio, you would earn a higher reward in small stocks. Plots (e) and (f) show the same issue, but more starkly. If you could not identify whether a firm was small or large, you would conclude that market beta works—you would still draw a straight positive line between the two clusters of firms, and you would conclude that higher market-beta stocks offer higher rewards. But truly, it would not be beta at all that matters, but whether the firm is small or large. After taking into account what type the firm is, beta would not matter in plot (e), and even matter *negatively* in plot (f). In either case, as a financial investor, you could earn higher expected rates of return buying stocks based on firm size rather than based on beta.

Historical patterns could be spurious.

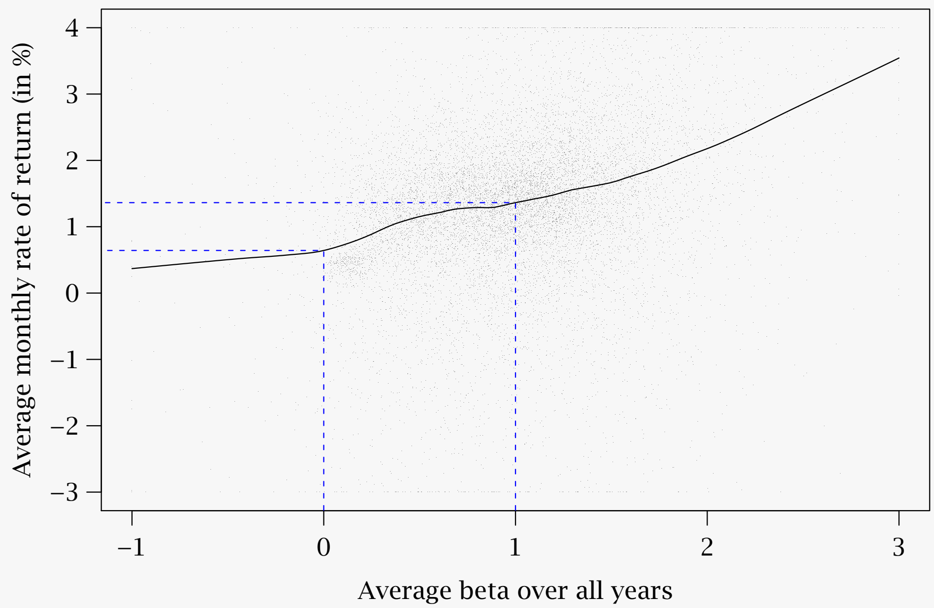
But be warned: Such relationships could also appear if your procedures to estimate beta or expected rates of return are poor—after all, when you plot such figures with real-world historical data, you do not have the true beta or true expected rates of return. Even if your statistical procedures are sound, statistical noise makes this a hazardous venture. In particular, in real life, although you can estimate market betas pretty reliably, you can only roughly estimate expected rates of return from historical rates of return.

13·5.B. The Historical Estimated SML

A model is a model—it ain't never perfect.

Before I let you in on the truth, please realize that a model is just a model—models are never perfect descriptions of reality. They can be useful within a certain domain, even if on closer examination they are rejected. For example, we do not live in a world of Newtonian gravity. Einstein's model of relativity is a better model—though it, too, is not capable of explaining everything. Yet no one would use Einstein's model to calculate how quickly objects fall. The Newtonian model is entirely appropriate and much easier to use. Similarly, planetary scientists use Einstein's model, even though we know it, too, fails to account for quantum effects—but it does well enough for the purposes at hand and there are as yet no better alternatives (even though string theory is trying hard).

Figure 13.6: Average Historical Rates of Return Against Historical Market Beta, 1970–2000.



This is the real empirical relationship between monthly betas and monthly average rates of return from 1970–2000. (The latter are not annualized.) The betas are with respect to the value-weighted stock market. Extreme observations were truncated at –1 and +3 for beta, and at –3% and +4% for monthly returns. The black line is "smoothed" to fit points locally, allowing it to show non-linearities. The dashed blue line indicates that this smoothed line suggests that "beta = 0" securities had approximate rates of return of 64 basis points per month, or about 8% per annum. The typical "beta = 1" securities had approximate rates of return of 136 basis points per month, or about 18% per annum. (Because these are arithmetic averages, you would have earned less than this in a buy-and-hold strategy.)

The dark but open secret is that this latter situation is pretty much the situation in which corporations find themselves—the CAPM is not really correct. However, we have no good all-around alternatives that are clearly better.

The Empirical Evidence: Where it Works

The empirical relation looks reasonably linear and upward sloping: good for the CAPM.

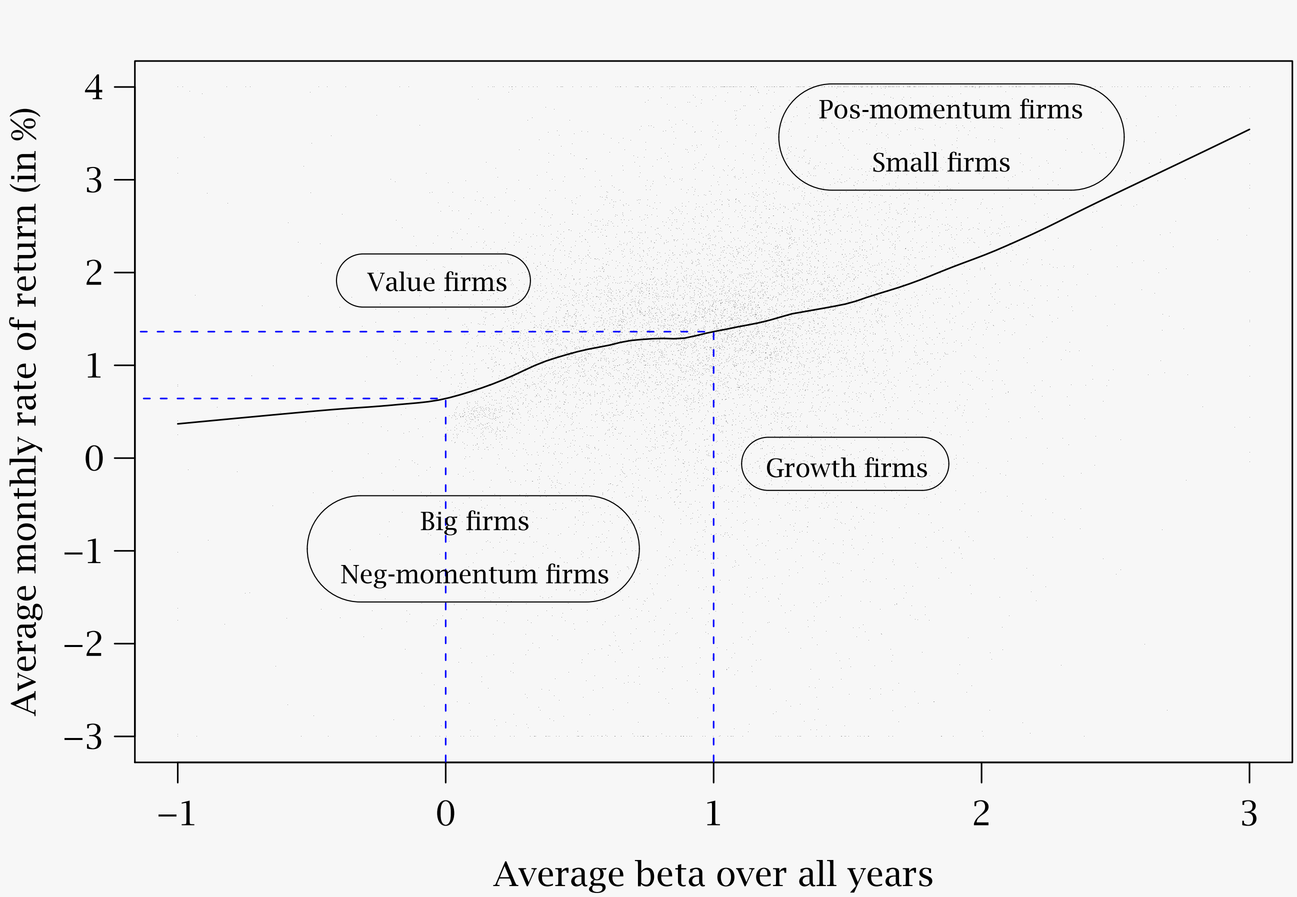
But let me first explain where the CAPM works and where it fails. What does the security market line look like in real life? Figure 13.6 plots the relationships from 1970 to 2003. The typical stock with a beta of 0 earned a rate of return of about 0.6% per month (8% per annum), while the typical stock with a beta of 1 (e.g., the stock market itself) earned a rate of return of about 1.5% per month (18% per annum). Not drawn in the figure, the average stock with a beta of 2 earned about 2.2% per month (30% per annum), and the average stock with a beta of 3 earned about 3.5% per month (50% per annum). (These annual returns are arithmetic averages—the geometric annual rates of return would have been lower.) You can see that these 34 years were a very good period for risky financial investments! Most important, from the perspective of the CAPM, the historical relationship between average rates of return and betas seems to have been reasonably close to linear, just as the CAPM suggests. If we stopped now, I would have advised you to conclude that the CAPM is a pretty good model.

The Empirical Evidence: Where it Fails

But this is deceptive: The CAPM fails against specific better alternatives.

But look back at Figure 13.5. The empirical evidence—that is, hundreds of academic papers—shows that the CAPM does not fail in the sense of the first three plots. Although the relation between market beta and average rates of return is roughly linear, the CAPM fails more in the sense of the last three plots: There are better alternative classifications of stocks. Although you cannot see this in Figure 13.6, the CAPM fails when stocks are split into groups based on different characteristics. The empirical reality is therefore closer to the latter three figures than it is to the idealized CAPM world.

Figure 13.7: Historical Firm Types Locations in Plot of Rates of Return against Historical Market Beta, 1970–2000.



Stocks with positive (negative) momentum have experienced unusually good (bad) returns over the last year, except the most recent month. Small firms and big firms are self-explanatory. Value firms are "boring" firms with high sales and accounting values, but low market values—such as Proctor&Gamble or RJR Nabisco. Growth firms are "exciting" firms with low sales, but high market values—such as Google or Apple.

Clusters of firms on the northwest contain value firms; clusters on the southeast contain growth firms.

The most damaging evidence may well be that firms that are classified as exciting "growth firms" (they have low sales and accounting values but high market value—the Google's of this world) generally underperform boring "value firms" (the opposite—the PepsiCo's of this world). Figure 13.7 illustrates roughly where these types of stocks tended to cluster in the graph of beta versus expected rates of return. The plot also notes some other stock characteristics that explain which firms had high average returns and which firms had low average returns. Moreover, if we take into account these other characteristics, market beta no longer seems to be too important, at all.

Unfortunately, although we can rationalize after the fact why these specific firm characteristics mattered for future returns, we really had no great reasons why they should have mattered in the first place. We also do not know whether these characteristics are themselves just stand-ins for something else that we have not yet found, or whether they matter in themselves. And not only do we finance experts not know for sure what characteristics should matter and why, but we also don't know how CFOs should operate in such a world. Should we advise managers to pretend that their firms are growth firms—because investors like this claim so much they are willing to pay a lot for shares of such firms—but that as soon as they have their investors' money, should they then invest it as if they were value firms? No one really knows. (On the plus side, it keeps ongoing academic finance research interesting.)

What now?

13·5.C. What Do You Do Now?

If the CAPM does not hold, why torture you with it? This is an easy question to answer. (The tougher question is how you can get estimates of appropriate hurdle rates for your projects in this case.)

Reasonable cost of capital estimates (often): Even though the CAPM is rejected, market beta is still often a useful cost-of-capital measure for a corporate finance manager.

Why? Look again at the last three plots in Figure 13.5. Let's presume it was just firm size that mattered to expected rates of return. If you have a beta of around 1.5, you are more than likely a small firm with an expected rate of return of 10% to 15%; if you have a beta of around 0, you are more than likely a big firm with an expected rate of return of 3% to 7%. Thus, beta would still provide you with a decent cost of capital estimate, even though it was not market beta itself that mattered, but whether your firm was large or small. (Market beta helped by indicating to you whether the firm was a growth or a value firm.) Admittedly, using an incorrect model is not an ideal situation, but the cost-of-capital estimates are often reasonable enough that corporate managers generally can live with them for purposes of finding a hurdle rate.

Beta can also work as a proxy for firm size.

This logic does not apply to value and growth firms: Value firms with higher expected rates of return do not have higher betas. Thus, you should not use the CAPM as a proxy to compute expected rates of return for projects (stocks) that are extreme value or growth firms—for such firms, the CAPM cost of capital estimates could be far off. Don't rely on them.

Avoid the CAPM for extreme growth or value firms.

Good intuition: The CAPM has impeccable intuition. It is a model that shines through its simplicity and focus on what *should* matter—diversification. It thereby often helps you to sharpen your thinking about what your corporate projects should offer your investors.

And let's not forget—the CAPM is easy to use, at least relative to the potential alternatives that you can learn about in chapter appendix 14·5.

↦ *CAPM alternatives, Section 14·5 on Page 343*

Alternatives—please stand up: If you cannot live with the fact that the CAPM is not perfectly correct, I really do not know what to recommend to you as a clearly better alternative. It takes a model to beat a model, and we really do not have an all-around good replacement for the CAPM. This is why we stick to the CAPM.

For example, one alternative model is to use the size and value/growth firm characteristics that I just mentioned as proxies for appropriate expected rates of return. But it is not even clear whether the higher returns for value firms reflect appropriate rewards for risk-taking that investors require (and which therefore should flow into a hurdle rate), or whether these firms earned superior returns because the stock market was not perfect (and which therefore need not flow into a hurdle rate). Imperfect markets are the subject of our next chapters.

Important: Everyone expects you to know the CAPM!

Everyone uses it: Table 13.1 shows that we are not alone: 73% of the CFOs reported that they always or almost always use the CAPM. (And CAPM use was even more common among large firms and among CFOs with an MBA.) No alternative method was used very often. Consequently, you have no choice but to understand the CAPM model well—*if you will work for a corporation, then the CAPM is the benchmark model that your future employer will likely use—and will expect you to understand well.* The CAPM is simply *the* standard. The CAPM is also used as a benchmark by many investors rating their (investment) managers, by government regulatory commissions, by courts in tort cases, and so on. It is literally the dominant, if not the only, universal model for the cost of capital.

Table 13.1: CFO Valuation Techniques

Cost of Capital—An Input Into NPV and Needed for IRR			
CAPM	(73%)	With caution	Chapter 13
Historical average returns	(39%)	Rarely	Chapter 9
Modified CAPM	(34%)	With Caution	Chapter 13
Backed out from Gordon Model	(16%)	Occasionally	Chapter 3
Whatever Investors Tell Us	(14%)	Occasionally	Chapter 2

Rarely means "usually no, and often used incorrectly." Source: Campbell and Harvey, 2001.

My personal opinion: use with caution in a corporate finance context for capital budgeting.

Let me infuse a bit more of my personal opinion now. Different academics draw different conclusions from the empirical evidence. Some recommend outright against using the CAPM, but most professors recommend "use with caution." I am among them. I would suggest that, as a student, your concern should be about the domain within which you should reasonably use the CAPM. Think about whether it is useful for your own cost of capital estimates, or whether the CAPM errors would likely to be too large to be useful.

Don't expect accuracy and don't use it for financial investing.

Here is what I would definitely warn against:

Accuracy: The CAPM is a poor model if precision is of the essence. If you believe that CAPM expected rates of return should be calculated with any digits after the decimal point, then you are deluded. Please realize that, at best, the CAPM can only offer expected rates of returns that are of the "right order of magnitude," plus or minus a few percentage points perhaps.

Actually, if accuracy and precision are important, you are thoroughly in trouble. We do not have *any* models that offer great accuracy. (Fortunately, it is often more important that you estimate value *better* than your competitors than that you estimate value very accurately. And always remember that valuation is as much art as it is science.)

Investment purposes: If you are not a corporate CFO looking for a project hurdle rate, but a financial investor looking for good investments from the universe of financial instruments, please don't use the CAPM. Although the CAPM offers the correct intuition that wide diversification needs to be an important part of *any* good investment strategy, there *are* still better investment strategies than just investing in the market index. Some are explained in the chapter appendix 14·5, more will be discussed in an advanced investments course.

And also please do not confuse the CAPM with the mean-variance framework discussed in the previous chapter. Mean-variance optimization is an asset-selection technique for your individual portfolio, and it works, regardless of whether the CAPM works or not.

IMPORTANT:

- Be aware of the strengths and weaknesses of the CAPM.
- The empirical evidence suggests that the CAPM is not a great model for predicting expected rates of return. This is especially so for extreme-value and extreme-growth firms, or firms having experienced very high or very low rates of returns recently.
- The CAPM is still the benchmark model in the real world. Every corporation uses it—and every corporation expects you to know it.
- The CAPM offers reasonably good estimates for the cost of capital (hurdle rate) in many but not all corporate settings.
- The CAPM never offers great accuracy.
- The CAPM is a decent model for corporate capital budgeting, but it is not a good model for a financial market investor. The CAPM speculates that the market portfolio is mean-variance efficient, but in real life, you can optimize your portfolio and choose portfolios closer to the mean-variance frontier.
- Mean-variance optimization works even if the CAPM does not.

Solve Now!

Q 13.19 Does the empirical evidence suggest that the CAPM is correct?

Q 13.20 If the CAPM is wrong, why do you need to learn it?

How Bad Are Mistakes

How Robust is the CAPM?

By now, you should realize that you will never perfectly know the required inputs for the CAPM. You can only make educated guesses. And even *after* the fact, you will never be sure—you observe only realized rates of returns, never expected rates of return. Exactly how robust are CAPM estimates with respect to errors in its inputs? Well, it depends on the inputs:

Where will we go wrong?

The risk-free rate: Errors in the risk-free rate (r_F) are likely to be modest. The risk-free rate can be considered to be almost known for practical purposes. Just make sure to use Treasuries that match the timing of your project cash flows.

Errors in the risk-free rate tend to be small.

Market beta: Reasonable beta estimates typically have some uncertainty, but good comparables can often be found in the public market. If due care is exercised, a typical range of uncertainty about beta might be about plus or minus 0.4. For example, if the equity premium is 3% and if you believe your beta is 2, but it is really 1.6 instead, then you would overestimate the appropriate expected rate of return by $2 \cdot 3\% - 1.6 \cdot 3\% = 1.2\%$. Although this level of uncertainty is not insignificant, it is tolerable in corporate practice.

Errors in beta estimates are meaningful but still tend to be modest.

Equity premium estimates: Reasonable equity premium estimates can range from about 2% per year to about 6% per year—a large range. *To date, there is no universally accepted method to estimate the expected rate of return on the market, so this disagreement cannot be easily settled with data and academic studies.* Unfortunately, reasonable differences of opinion in estimating the expected rate of return on the market can have a large influence on expected rate of return estimates. For example, assume the risk-free rate is 3%, and take a project with a beta of 2. The CAPM might advise this corporation that potential investors demand either an expected rate of return of 5% per year (equity premium estimate of 1%) or an expected rate of return of 19% per year (equity premium estimate of 8%), or anything in between. This is—to put it bluntly—a miserably large range of possible cost of capital estimates. (And this range does not even consider the fact that *actual* future project rates of return will necessarily differ from *expected rates of return*!) Of course, in the real world, managers who want to take a project will argue that the expected rate of return on the

Disagreement about the equity premium tends to be large—and this can make a big difference.

market is low. This means that their own project looks relatively more attractive. Potential buyers of projects will argue that the expected rate of return on the market is high. This means that they claim they have great opportunities elsewhere, so that they can justify a lower price offer for this project.

I repeat: *Use the CAPM as guidance, not as gospel.*

Model errors: What about the CAPM as a model itself? This error is difficult to assess. Perhaps a reasonable approach is to use the CAPM in a corporate context unless the firm is very unusual—an extreme value or growth or small firm, for example. Just remain aware that the model use itself introduces errors.

When you put NPV and the CAPM together, watch first for cash flow and equity premium errors.

↦ *Errors in cash flows and discount rates, Section 4·1.A on Page 62*

You will often use the CAPM expected rate of return as your cost of capital in an NPV calculation. Here, you combine errors and uncertainty about expected cash flows with your errors and uncertainty in CAPM estimates. What should you worry about? Recall that in Section 4·1.A, you learned the relative importance of correct inputs into the NPV formula. The basic conclusion was that for short-term projects, getting the cash flows right is more important than getting the expected rate of return right; for long-term projects, getting both right is important. We just discussed the relative importance of getting the equity premium and the project beta right. Now recall that your basic conclusion was that the CAPM formula is first and foremost exposed to errors in the market risk premium (equity premium), though it is also somewhat exposed to beta estimates. Putting these two insights together suggests that for short-term projects, worrying about exact beta estimates is less important than worrying about estimating cash flows first and the appropriate equity premium second. For long-term projects, the order of importance remains the same, but having good equity premium estimates now become relatively more important. In contrast, in most cases, honest mistakes in beta, *given reasonable care*, are relatively less problematic.

Solve Now!

Q 13.21 Is the CAPM likely to be more accurate a model if the beta is very positive? One? Zero? Negative?

Q 13.22 To value an ordinarily risky project, that is, a project with a beta in the vicinity of about 1, what is the relative contribution of your personal uncertainty (lack of knowledge) in (a) the risk-free rate, (b) the equity premium, (c) the beta, and (d) the expected cash flows? Consider both long-term and short-term investments. Where are the trouble spots?

Anecdote: "Cost of Capital" Expert Witnessing

When Congress tried to force the "Baby Bells" (the split-up parts of the original AT&T) to open up their local telephone lines to competition, it decreed that the Baby Bells were entitled to a fair return on their infrastructure investment—with fair return to be measured by the CAPM. (The CAPM is either the de facto or legislated standard for measuring the cost of capital in many other regulated industries, too.) The estimated value of the telecommunication infrastructure in the United States is about $10 to $15 billion. A difference in the estimated equity premium of 1% may sound small, but even in as small an industry as local telecommunications, it meant about $1,000 to $1,500 million a year—enough to hire hordes of lawyers and valuation consultants opining in court on the appropriate equity premium. Some of my colleagues bought nice houses with the legal fees.
I did not get the call. I lack the ability to keep a straight face while stating that "the equity premium is exactly x point y percent," which was an important qualification for being such an expert. In an unrelated case in which I did testify, the opposing expert witness even explicitly criticized my statement that my cost of capital estimate was an imprecise range—unlike me, he could provide an exact estimate!

13·6 Summary

The chapter covered the following major points:

- The CAPM provides an "opportunity cost of capital" for investors, which corporations can use as the hurdle rate (or cost of capital) in the NPV formula. The CAPM formula is

  $$\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i$$

  Thus, there are three inputs: the risk-free rate of return, the expected rate of return on the stock market, and the project market-beta. Only the latter is project specific.
- The line plotting expected rates of return against market-beta is called the security-markets line (SML).
- Use a risk-free rate that is similar to the approximate duration or maturity of the project.

↦ *Duration c on Page 107*

- There are a number of methods to estimate market-beta. For publicly traded firms, it can be computed from stock return data or obtained from commercial data vendors. For private firms or projects, a similar publicly traded firm can often be found. Finally, managerial scenarios can be used to estimate market-betas.
- The expected rate of return on the market is often a critical input, especially if market-beta is high—but it is difficult to guess.
- The CAPM provides an expected rate of return, consisting of the time-premium and the risk-premium. In the NPV formula, the default-risk and default-premium works through the expected cash flow numerator, not the cost of capital denominator.
- Corporations can reduce their risk by diversification—but if investors can do so as easily, diversification per sé does not create value.
- To value a project, corporations should not use the cost of capital (market-beta) applicable to the entire firm, but the cost of capital (market-beta) applicable to the project. However, because the effort involved can be enormous, you should use individual, project-specific costs of capital primarily when it makes a difference.
- Certainty equivalence is discussed in the appendix. You must use the certainty-equivalence form of the CAPM when projects are purchased or sold for a price other than their fair present market-value.

Appendix

A Application: Certainty Equivalence

How can you value a project if you do not know the efficient price today?

As I noted earlier, the CAPM is called an asset-pricing model—but then it is presented in terms of rates of return, not prices. What if you wanted to know the value of an investment asset with an uncertain rate of return? Put differently, if I asked you how much in cash you would be willing to pay for an asset, you would find it difficult to use the CAPM to tell me your "certain price" today that would leave you indifferent. This price is called the **certainty equivalent**.

The fact that you don't yet know how to answer this question turns out to have one perplexing consequence, which leaves you with one important and difficult conceptual issue best illustrated with a brainteaser: What is today's value of a gift expected to return \$100 next year?

a. Valuing Goods Not Priced at Fair Value

At a price of zero, is the appropriate cost of capital in the CAPM formula infinite? No!

Start with this puzzle: How do you even compute the beta of the gift's rate of return with the rate of return on the stock market? The price is \$0 today, which means that your actual rate of return will be infinite! But you clearly should be able to put a value on this gift. Indeed, your intuition should tell you that this cash flow is most likely worth a little less than \$100, the specifics depending on how the gift's cash flow covaries with the stock market. But, how do you compute this value? The solution to this puzzle is that the *price* of the gift may be \$0 today, but its *present value* today (PV) is not—and it is the latter, i.e., the fair value, that is used in the CAPM, not the former. (For the rest of this section, assume that all expectations and covariances are from time 0 to time 1.)

IMPORTANT:

- The CAPM works only with expected rates of return that are computed from the *true* perfect market asset values today and the true perfect market expected value tomorrow.
- If either the price today or the value next period is not fair, then you cannot compute an expected rate of return and assume that it should satisfy the standard CAPM formula, $\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i$.

The CAPM works only on fair prices in perfect markets.

Of course, in a perfect and efficient market, what you get is what you pay for ($P = \text{PV}$), so this issue would never arise. But, if you buy an asset at a better or worse deal ($P < \text{PV}$ or $P > \text{PV}$), for example, from a benevolent or malevolent friend, then you can absolutely not use such a price to compute the expected rate of return in the CAPM formula. The same applies to $\mathcal{E}(\tilde{P})$: the expected value tomorrow must be the true expected value, not a sweetheart deal value at which you may let go of the asset, nor an excessive price at which you can find a desperate buyer. If it is, you cannot use the CAPM formula.

Here is how to rearrange the CAPM formula into the certainty equivalence (CEV) formula.

Now, return to the question of how to value a gift. The specific computational problem is tricky: If you knew the present value today, you could compute a rate of return for the cash flow, then from the rate of return you could compute the project beta, which you could use to find the discount rate to translate the expected cash flow back into the present value (supposedly the price) today. Alas, you do not know the price, so you cannot compute a rate of return. To solve this dilemma, you must use an alternative form of the CAPM formula.

IMPORTANT: The **certainty equivalence form** rearranges the CAPM formula into:

$$\text{PV} = \frac{\mathcal{E}(\tilde{P}) - \lambda \cdot \mathcal{C}ov(\tilde{P}, \tilde{r}_M)}{1 + r_F} \quad \text{where } \lambda \equiv \frac{\mathcal{E}(\tilde{r}_M - r_F)}{\mathcal{V}ar(\tilde{r}_M)} \tag{13.2}$$

where PV is the price today and $\tilde{P}$ is the price next period.

If there is only one future cash flow at time 1, then $\tilde{P}$ is this cash flow, and the rate of returns are from time 0 to time 1. As before, we need the risk-free rate and an estimate of the equity premium. Let's work with a risk-free rate of 3% and an expected equity premium is 5%. In addition, we need the volatility of the stock market. Let's assume for our example's sake that the standard deviation is 20%. This means that the variance is $20\% \cdot 20\% = 0.04$, and therefore that lambda is $0.05/0.04 = 1.25$. You could now value projects as

It gives the price (not the rate of return) today.

$$\text{PV} = \frac{\mathcal{E}(\tilde{P}) - 1.25 \cdot \mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}})}{1+3\%} = \underbrace{\frac{\mathcal{E}(\tilde{P})}{1+3\%}}_{\text{as-if-risk-free}} - \underbrace{\left(\frac{1.25}{1+3\%}\right) \cdot \mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}})}_{\text{risk discount}} \tag{13.3}$$

The name "certainty equivalence" is apt. The first form in Formula A.1 shows that, after you have reduced the expected value of the future cash flow ($\mathcal{E}(\tilde{P})$) by some number that relates to the cash flow's covariance with the market, you can then treat this reduced value as if it were a perfectly certain future cash flow and discount it with the risk-free rate. The second form in Formula 13.3 shows that you can decompose the price (present value) today into an "as-if-risk-free" value that is discounted only for the time premium (with the risk-free rate), plus an additional risk premium (discount) that adjusts for any covariance risk with the stock market.

The covariance between the future value $\tilde{P}$ and the rate of return on the market is related, but not identical to, the project's market beta. It is *not* the covariance of the project's *rate* of return with the market rate of return, either. It is the covariance of the project's *cash flow* with the market rate of return, instead.

Watch out: The covariance here is related to, but not exactly equal to, the market beta.

With the certainty equivalence formula, you can now begin thinking about how to value your \$100 expected gift. Assuming that the risk-free rate is 3% per annum, and that the lambda is the aforementioned 1.25,

Example: Find the gift's price if the covariance is 0.

$$\text{PV} = \frac{\$100 - 1.25 \cdot \mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}})}{1+3\%}$$

$$\text{PV} = \frac{\mathcal{E}(\tilde{P}) - \lambda \cdot \mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}})}{1+r_{\text{F}}}$$

If you believe that the gift's payout does not covary with the rate of return on the market, so $\mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}}) = 0$, then

$$\text{PV} = \frac{\$100 - 1.25 \cdot 0}{1+3\%} = \frac{\$100}{1+3\%} \approx \$97.09$$

$$\text{PV} = \frac{\mathcal{E}(\tilde{P}) - \lambda \cdot \mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}})}{1+r_{\text{F}}}$$

Now let's see what the value is if you believe that your windfall does covary with the market. How can you estimate your cash flows' covariance with the rate of return of the stock market? You need to write down some scenarios, and then compute the covariance. This is easiest to understand in an example. Let's assume that you believe that if the market goes up by 28%, your gift will be \$200; if the market goes down by 12%, your gift will be \$0. Further, you also believe these two outcomes are equally likely.

Example: Find the gift's price if the covariance is positive.

	Bad	Good	Mean	*Var*	*Sdv*
Prob:	1/2	1/2			
Stock Market	-12%	+28%	8%	4%	20%
Our Windfall	\$0	\$200	\$100	$\$^2$ 10,000	\$100

I have chosen the stock market characteristics to match the example above. That is, the expected rate of return on the market is 8%, and its variance is $[(28\% - 8\%)^2 + (-12\% - 8\%)^2]/2 = 0.04$. Now you can use the covariance formula to compute the average product of deviations from the means. This is

↦ *Covariance computation, Formula ?? on Page ??*

$$\mathcal{Cov}(\tilde{P}, \tilde{r}_{\text{M}}) = \frac{(\$200 - \$100) \cdot (28\% - 8\%) \ + \ (\$0 - \$100) \cdot (-12\% - 8\%)}{2} = \$20$$

$$= \frac{\text{Sum of all } [\tilde{P}_{\text{outcome j}} - \mathcal{E}(\tilde{P}_{\text{outcome j}})] \cdot [\tilde{r}_{\text{M,outcome j}} - \mathcal{E}(\tilde{r}_{\text{M}})]}{N}$$

Lambda is still 1.25, and you can now use the certainty equivalence formula to value your expected windfall of \$100 next year. The gift is worth

$$\text{PV} = \frac{\$100 - 1.25 \cdot \$20}{1 + 3\%} = \frac{\$75}{1 + 3\%} \approx \$72.82$$

$$\text{PV} = \frac{\mathcal{E}(\tilde{P}) - \lambda \cdot \mathcal{Cov}(\tilde{P}, \tilde{r}_M)}{1 + r_F}$$

This is a lot less than the \$97.09 it would be worth if it did not covary with the market.

An alternative way to write the CEV formula.

There are two more ways to rearrange the certainty equivalence formula. The first changes the cash flow covariance into a cash flow regression beta. You can do this by using the formula

$$b_{\tilde{P},\tilde{r}_M} = \frac{\$20}{0.04} = \$500$$

$$= \frac{\mathcal{Cov}(P_1, \tilde{r}_M)}{\mathcal{Var}(\tilde{r}_M)}$$

This $b_{\tilde{P},\tilde{r}_M}$ is the slope of a regression line in which the future cash value (not the rate of return) is the dependent variable. You can now use a third certainty equivalence form, which gives the same result

$$\text{PV} = \frac{\$100}{1 + 3\%} - \left[\frac{5\%}{1 + 3\%}\right] \cdot \$500 \approx \$72.82$$

$$\text{PV} = \frac{\mathcal{E}(\tilde{P})}{1 + r_F} - \left[\frac{\mathcal{E}(\tilde{r}_M) - r_F}{1 + r_F}\right] \cdot b_{\tilde{P},\tilde{r}_M}$$

A final form is really more like the original CAPM. It translates the cash flow regression beta back into the ordinary CAPM beta, which we all love. To do this, use the formula

$$\beta_P \approx \frac{\$500}{\$72.82} \approx 6.867$$
$$\beta_P = \frac{b_{\tilde{P},\tilde{r}_M}}{\text{PV}} \qquad (13.4)$$

Of course, you usually do not know the \$72.82 price to begin with, which is why this is a less useful form. You can now compute the value as

$$\text{PV} = \frac{\$100/1.03}{1 + \left(\frac{8\% - 3\%}{1.03}\right) \cdot 6.85} \approx \$72.82$$

$$\text{PV} = \frac{P_1/(1 + r_F)}{1 + \left[\frac{\mathcal{E}(\tilde{r}_M) - r_F}{1 + r_F}\right] \cdot \beta_i}$$

I find this CAPM form sometimes intuitively useful. It says that after you have discounted the project by the risk-free rate, you discount it a second time using $\left[\frac{\mathcal{E}(\tilde{r}_M) - r_F}{1 + r_F}\right] \cdot \beta_i$ as your second interest rate.

DIG DEEPER

Knowing the fair price of \$72.8155, you can check that you have really just worked with the CAPM formula. The project will either provide a rate of return of $\$200/\$72.8155 - 1 \approx 174.67\%$, or a rate of return of -100%, for an expected rate of return of **37.33%**. Let's confirm this:

Ordinary market beta: The market beta computed with rates of return is

$$\beta_i \equiv \beta_{i,M} \equiv \frac{\mathcal{Cov}(\tilde{r}_i, \tilde{r}_M)}{\mathcal{Var}(\tilde{r}_M)} \approx \frac{\frac{(+174.67\% - 37.33\%) \cdot (+28\% - 8\%) + (-100\% - 37.33\%) \cdot (-12\% - 8\%)}{2}}{\frac{(+28\% - 8\%)^2 + (-12\% - 8\%)^2}{2}} \approx \frac{0.274667}{0.04} \approx 6.867$$

Ordinary CAPM expected rate of return: The CAPM formula states that the expected rate of return, given this beta of 6.867, should be

$$\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 0.03 + (0.08 - 0.03) \cdot 6.867 \approx 0.3733$$

which is indeed what we computed as our average between 174.67% and –100%.

Solve Now!

Q 13.23 Although you are a millionaire, keeping all your money in the market, you have managed to secure a great deal: If you promise to go to school (which costs you a net disutility worth $10,000 today), then your even-richer Uncle Vinny will buy you a Ferrari (expected to be worth $200,000), provided his business can afford it. He is an undertaker by profession, so his business will have the money if the stock market drops, but not if it increases. For simplicity, assume that the stock market drops in 1 year out of every 4 years. When it does, it goes down by -10%; if it does not, it goes up by 18%. (Write it out as four separate possible state outcomes to make your life simpler.) The risk-free rate is 6%. What is your uncle's deal worth to you?

b. Application: The CAPM Hurdle Rate for a Project With Cash Flow History Only

Your task: Find the opportunity cost of capital of a privately held corporation.

Here is your first professional consulting assignment: You are asked to advise a privately held firm on its appropriate cost of capital. The owners of this firm are very wealthy and widely diversified, so that their remaining portfolio is similar to the market portfolio. (Otherwise, our investor's opportunity cost of capital may not be well represented by the CAPM—and, therefore, the calculations here will not be relevant for a typical cash-strapped entrepreneur, whose portfolio would *not* be similar to the market portfolio.) To make this a more realistic and difficult task, this firm is either privately held or only a division of a publicly held firm, so you cannot find historical public market values and so that there are no obvious publicly traded comparable firms. Instead, the firm hands you its historical annual cash flows:

Year	1999	2000	2001	2002	2003	2004	Average
S&P 500	+21.4%	−5.7%	−12.8%	−21.9%	+26, 4%	+9.0%	+2.7%
Cash flows	$8,794	$5,373	$8,397	$6,314	$9,430	$9,838	$8,024

In an ideal world, this is an easy problem: You could compute the value of this firm every year, then compute the beta of the firm's rate of return with respect to the market rate of return, and plug this into the CAPM formula. Alas, assessing annual firm value changes from annual cash flows is beyond my capability. You can also not assume that percent changes in the firm's cash flows are percent changes in the firm's value—just consider what would happen to your estimates if the firm had earned zero in one year. All this does not let you off the hook: What cost of capital are you recommending? Having only a time series of historical cash flows (and no rates of return) is a very applied and not simply an obscure theoretical problem, and you might first want to reflect on how difficult it is to solve this problem without the certainty equivalence formula.

We are assuming independent cash flows. In real life, be careful. Be very careful.

First, we have to make our usual assumption that our historical cash flows and market rates of returns are representative of the future. However, here we have to make a much bigger assumption. It could be that your cash flows in one year are not a draw from the same distribution of cash flows, but that they also say a lot about your future cash flows. For example, a lousy year could induce the firm to make changes to raise cash flows. Or a great year could signal the beginning of more great years in the future. If either is the case, our naive application of the CEV method fails. (Instead of using a cash flow, you would have to use the expected value of the firm next year—a very difficult task in itself.) Let me repeat this:

> **Big Warning:** In the way we are now using our CEV approach on historical cash flow data, we are assuming that historical cash flows are independent draws that inform you about the distribution of future cash flows. This means that there should no autocorrelation—any year's cash flow should not be any more indicative of next year's cash flow than any other's. More sophisticated techniques could remedy this shortcoming, but we do not have the space to cover them, too.

Here is a heroic attempt to value this private firm.

Under this cash flow assumption, we begin by computing the beta of the firm's cash flows with respect to the S&P 500. This is easier if we work with differences from the mean:

Year	1999	2000	2001	2002	2003	2004	Average
Demeaned S&P 500	+0.187	−0.084	−0.155	−0.246	+0.237	+0.063	0
Demeaned Cash Flows	+$770	−$2,651	+$373	−$1,710	+$1,406	+$1,814	$0

To compute the covariance of the S&P 500 returns with our cash flows, we multiply these and take the average (well, we divide by $N-1$, because this is a sample, not the population, but it won't matter in the end),

$$\mathcal{C}ov_{\text{CF},\tilde{r}_M} = \frac{(+0.187)\cdot(+\$770)+(-0.084)\cdot(-\$2,651)+\cdots+(+0.063)\cdot(+\$1,814)}{5} \approx \$235.4$$

and compute the variance

$$\mathcal{V}ar(\tilde{r}_M) = \frac{(+0.187)^2+(-0.084)^2+\cdots+(0.063)^2}{5} \approx 0.0373$$

The cash flow beta is the ratio of these,

$$b_{\text{CF,M}} = \frac{\mathcal{C}ov_{\text{CF},\tilde{r}_M}}{\mathcal{V}ar(\tilde{r}_M)} = \frac{\$235.4}{0.03734} \approx \$6,304$$

Now substitute the inputs into the CEV formula.

The historical mean cash flow was \$8,024. We still need an assumption of a suitable equity premium and a suitable risk-free rate. Let's adopt 4% and 3%, respectively. In this case, the value of our firm would be

$$\text{PV} = \frac{\$8,024}{1+3\%} - \left[\frac{4\%}{1+3\%}\right]\cdot\$6,304 \approx \$7,791 - \$245 \approx \$7,546$$
$$= \frac{\mathcal{E}(\tilde{\text{CF}})}{1+r_F} - \left[\frac{\mathcal{E}(\tilde{r}_M)-r_F}{1+r_F}\right]\cdot b_{\tilde{\text{CF}},\tilde{r}_M}$$

The certainty equivalence formula tells us that because our firm's cash flows are correlated with the market, we shall impute an additional risk discount of \$245. We can translate this into a cost of capital estimate—at what discount rate would we arrive at a value of \$7,546?

$$\$7,546 = \frac{\$8,024}{1+\mathcal{E}(\tilde{r})} \Rightarrow \mathcal{E}(\tilde{r}) \approx 6.3\%$$
$$\text{PV} = \frac{\mathcal{E}(\text{CF})}{1+\mathcal{E}(\tilde{r})}$$

We now have an estimate of the cost of capital for our cash flow for next year. We can also translate this into an equivalent returns-based market beta, which is

$$3\% + 4\%\cdot\beta_{i,M} = 6.3\% \Rightarrow \beta \approx 0.8$$
$$r_F + [\mathcal{E}(\tilde{r}_M)-r_F]\cdot\beta_{i,M}$$

↦ *Translation to standard market beta, Formula 13.4 on Page 326*

Of course, you could have used Formula 13.4 instead: With a present value of \$7,546, the cash flow beta of \$6,304 divided by 7,546 would have yielded the same ordinary beta estimate of 0.8.

Are we close?

Now I can reveal who the firm in this example really was—it was IBM. Because it is publicly traded, we can see how our own estimate of IBM's cost of capital and market beta would have come out if we had computed it from IBM's annual market values. Its rates of return were

Year	1999	2000	2001	2002	2003	2004	Average
IBM's Rate of Return	+17.5%	−20.8%	+43.0%	−35.5%	+20.5%	+7.2%	+5.3%

If you compute the market beta of these annual returns, you will find an estimate of 0.7—very close to the estimate we obtained from our cash flow series. (For IBM, this is a fairly low estimate. If we used monthly cash flows or monthly stock returns, we would obtain a higher market-beta estimate.)

Solve Now!

Q 13.24 A firm reported the following cash flows:

Year	1999	2000	2001	2002	2003	2004	Average
S&P 500	+21.4%	−5.7%	−12.8%	−21.9%	+26.4%	+9.0%	+2.7%
Cash Flows	+\$2,864	+\$1,666	−\$1,040	+\$52	+\$1,478	−\$962	+\$676

(Note that the cash flows are close to nothing in 2002 and even negative in 2004, the latter preventing you from computing percent changes in cash flows.) Still assuming an equity premium of 4% and a risk-free rate of 3%, what cost of capital would you recommend for one year of this firm's cash flows?

[No keyterm list for capmrecipe-g.]

End of Chapter Problems

Q 13.25 Write down the CAPM formula. What are economy-wide inputs, what are project-specific inputs?

Q 13.26 The risk-free rate is 6%. The expected rate of return on the stock market is 8%. What is the appropriate cost of capital for a project that has a beta of 2?

Q 13.27 The risk-free rate is 6%. The expected rate of return on the stock market is 10%. What is the appropriate cost of capital for a project that has a beta of –2? Can this make economic sense?

Q 13.28 Draw the SML if the true expected rate of return on the market is 6% per annum and the risk-free rate is 2% per annum. How would the figure look if you were not sure about the expected rate of return on the market?

Q 13.29 A junk bond with a beta of 0.4 will default with 20% probability. If it does, investors receive only 60% of what is due to them. The risk-free rate is 3% per annum and the risk premium is 5% per annum. What is the price of this bond, its promised rate of return, and its expected rate of return?

Q 13.30 What would it take for a bond to have a larger risk premium than default premium?

Q 13.31 A corporate zero bond promises 7% in 1 year. Its market beta is 0.3. The equity premium is 4%; the equivalent T-bond rate is 3%. What is the appropriate bond price today?

Q 13.32 Explain the basic schools of thought when it comes to equity premium estimation.

Q 13.33 If you do not want to estimate the equity premium, what are your alternatives to finding a cost of capital estimate?

Q 13.34 Explain in 200 words or less: What are reasonable guesstimates for the market risk premium and why?

Q 13.35 Should you use the same risk-free rate of return both as the CAPM formula intercept and in the equity premium calculation, or should you assume an equity premium that is independent of investment horizon?

Q 13.36 Should a negative-beta asset offer a higher or a lower expected rate of return than the risk-free asset? Does this make sense?

Q 13.37 An unlevered firm has an asset market beta of 1.5. The risk-free rate is 3%, the equity premium is 4%.

(a) What is the firm's cost of capital?
(b) The firm refinances itself. It repurchases half of its stock with debt that it issues. Assume that this debt is risk free. What is the equity beta of the levered firm?
(c) According to the CAPM, what rate of return does the firm have to offer to its creditors?
(d) According to the CAPM, what rate of return does the firm have to offer to its levered equity holders?
(e) Has the firm's weighted average cost of capital improved?

Q 13.38 Look up betas on Yahoo!*Finance* today, and compare them to those in Table **??** on Page **??**:

(a) How does the beta of Intel today compare to its earlier estimate from May 2008? Was its beta stable (over time)?
(b) How does the beta of AMD today compare to its earlier estimate from May 2008? Was its beta stable?
(c) AMD is a much smaller firm than Intel. How do their betas compare?

Q 13.39 A comparable firm (in a comparable business) has an equity beta of 2.5 and a debt/equity ratio of 2. The debt is almost risk free. Estimate the beta for your firm if projects have constant betas, but your firm will carry a debt/equity ratio of 1/2. (Hint: To translate a debt-to-equity ratio into a debt-to-asset ratio, make up an example.)

Q 13.40 A Fortune 100 firm is financed with $15 billion in debt and $5 billion in equity. Its historical levered equity beta has been 2. If the firm were to increase its leverage from $15 billion to $18 billion and use the cash to repurchase shares, what would you expect its levered equity beta to be?

Q 13.41 The prevailing risk-free rate is 5% per annum. A competitor to your own firm, though publicly traded, has been using an overall project cost of capital of 12% per annum. The competitor is financed by 1/3 debt and 2/3 equity. This firm has had an estimated levered beta of 1.5. What are they using as their equity premium estimate?

Q 13.42 Apply the CAPM. Assume the risk-free rate of return is the yield on 5-year bonds. Assume that the market's expected rate of return is 3% per year above this. Download monthly rate of return data on the 30 stocks in the Dow-Jones index from 2000 on. (If need be, ignore dividends.)

➤ What were the historical average monthly rate of returns?

- What were the historical levered market betas?
- What were the historical market betas, adjusted (shrunk) toward 1 by averaging with 1?
- How do these estimates compare to the market-beta estimates of the financial website from which you downloaded the data?
- Does it appear as if these stocks followed a CAPM-like relationship?
- What were the historical asset market betas? (Consider the debt-equity ratio as given by Yahoo!*Finance*. Assume, somewhat incorrectly, that the debt promised 100 basis points above the risk-free rate and that debt betas are 0.)
- What were the historical asset expected rates of return?
- Does it appear as if these firms followed a CAPM-like relationship?

Q 13.43 Draw some possible security markets relations which would not be consistent with the CAPM.

Q 13.44 Does the empirical evidence suggest that the CAPM is correct?

Q 13.45 Why do you need to understand the CAPM?

Q 13.46 Under what circumstances is the CAPM a good model to use? What are the main arguments in favor of using it? When is it not a good model?

Q 13.47 Explain the kinds of projects for which it is important to get accurate equity premium estimates?

Q 13.48 Although you are a millionaire, keeping all your money in the market, you have managed to secure a great deal: If you give your even-richer Uncle Vinny $20,000 today, he will help you buy a house, expected to be worth $1,000,000, if his business can afford it. He is a stock broker by profession, so his business will have the money if the stock market increases, but not if it drops. For simplicity, assume that the stock market drops in 1 year out of every 4 years. When it does, it goes down by -10%; if it does not, it goes up by 18%. (Write it out as four separate possible state outcomes to make your life simpler.) The risk-free rate is 5%. What is your Uncle's promise worth at market value?

Q 13.49 Your corporate division had the following net cash flows:

Year	1999	2000	2001	2002	2003	2004	2005
S&P 500	+21.4%	−5.7%	−12.8%	−21.9%	+26.4%	+9.0%	+2.7%
Cash Flows	+$2,000	$0	$0	$0	+$2,500	+$1,000	+$500

Assume that the risk-free rate is 1% per annum. Use the certainty equivalence concept to answer the following questions:

- What should be a reasonable value approximation for this corporate division?
- What should be the cost of capital for this corporate division?

24 "Solve Now" Answers

1. You can use the CAPM formula here: $4\% + (16\% - 4\%) \cdot 1.5 = 22\%$.

2. With $r_F = 4\%$ and $\mathcal{E}(\tilde{r}_M) = 7\%$, the cost of capital for a project with a beta of 3 is $\mathcal{E}(\tilde{r}) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 4\% + (7\% - 4\%) \cdot 3 = 13\%$.

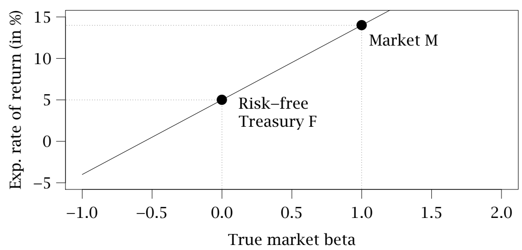
3. With $r_F = 4\%$ and $\mathcal{E}(\tilde{r}_M) = 12\%$, the cost of capital for a project with a beta of 3 is $\mathcal{E}(\tilde{r}) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 4\% + (12\% - 4\%) \cdot 3 = 28\%$.

4. With $r_F = 4\%$ and $\mathcal{E}(\tilde{r}_M) = 12\%$, the cost of capital for a project with a beta of -3 is $\mathcal{E}(\tilde{r}) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 4\% + (12\% - 4\%) \cdot (-3) = -20\%$. Yes, it does make sense that a project can offer a negative expected rate of return. Such assets are such great investments that you would be willing to expect losses on them, just because of the great insurance that they are offering.

5. No—the real-world SML is based on historical data and not true expectations, and thus more like a scatterplot of historical risk and reward points through which we hope a straight upward-sloping line fits.

6. Write down the CAPM formula and solve $\mathcal{E}(\tilde{r}_i) = r_F + [\mathcal{E}(\tilde{r}_M) - r_F] \cdot \beta_i = 4\% + (7\% - 4\%) \cdot \beta_i = 5\%$. Therefore, $\beta_i = 1/3$. Note that we are ignoring the promised rate of return.

7. note: newly drawn.



8. The equity premium, $\mathcal{E}(\tilde{r}_M) - r_F$, is the premium that the stock market expects to offer on the risky market above and beyond what it offers on Treasuries.

9. It does not matter what you choose as the per-unit payoff of the bond. If you choose \$100, you expected it to return \$99.
   (a) Thus, the price of the bond is PV $= \$99/(1 + (3\% + 5\% \cdot 0.2)) \approx \$95.19$.
   (b) Therefore, the promised rate of return on the bond is $\$100/\$95.19 - 1 \approx 5.05\%$.
   (c) The risk-free rate is 3%, so this is the time premium (which contains any inflation premium). The expected risk premium is 1%. The remaining 1.05% is the default premium.

10. The cost needs to be discounted with the current interest rate. Since payment is up front, this cost is \$30,000 now! The appropriate expected rate of return for cash flows (of your earnings) is $3\% + 5\% \cdot 1.5 = 10.5\%$. You can now use the annuity formula to determine the PV if you graduate:

$$\frac{\$5,000}{10.5\%} \cdot \left[1 - \left(\frac{1}{1 + 10.5\%}\right)^{40}\right] = \$47,619 \cdot 98.2\% \approx \$46,741.46$$

With 90% probability, you will do so, which means that the appropriate risk-adjusted and discounted cash flow is about \$42,067.32. The NPV of your education is therefore about \$12,067.32.

11. An estimate between 2% and 8% per year is reasonable. Anything below 0% and above 10% would be unreasonable. For reasoning, please see the different methods in the chapter.

12. Use the 1-Year Treasury rate for the 1-year project, especially if the 1-year project produces most of its cash flows at the end of the year. If it produces constant cash flows throughout the year, a 6-month Treasury rate might be more appropriate. Because the 10-year project could have a duration of cash flow flows much shorter than 10 years, depending on use, you might choose a risk-free Treasury rate that is between 5 and 10 years. Of course, it would be even better if you match the individual project cash flows with individual Treasuries.

13. The duration of this cash flow is around or a little under 5 years. Thus, a 5-year zero Treasury would be a reasonably good guess. You should not be using a 30-day, a 30-year, or even a 10-year Treasury. The 10-year Treasury would have too much of its payments as principal repayment at the end of its 10 year term.

14. Yes, a zero beta asset can still have its own idiosyncratic risk. And, yes, it is perfectly kosher for a zero-beta asset to offer the same expected rate of return as the risk-free asset. The reason is that investors hold gazillions of assets, so the idiosyncratic risk of the zero-beta asset will just diversify away.

15. You could run a time-series regression of project rates of return on the stock market.

$$\tilde{r}_p = \alpha + \beta \cdot \tilde{r}_M$$

(PS: Although not discussed, a slightly better regression would be $\left(\tilde{r}_p - r_F\right) = \alpha + \beta \cdot (\tilde{r}_M - r_F)$.) The statistical package (or spreadsheet) would spit out the project beta. The digging-deeper note explains that if you want to improve the forward-looking estimate, you might want to shrink this beta toward 1.

16. This is an asset beta versus equity beta question. Because the debt is almost risk-free, we can use $\beta_{\text{Debt}} \approx 0$.
   (a) First compute an unlevered asset beta for your comparable with its debt-to-asset ratio of 2 to 3. This is $\beta_{\text{Asset beta}} = w_{\text{Debt}} \cdot \beta_{\text{Debt}} + w_{\text{Equity}} \cdot \beta_{\text{Equity}} = (2/3) \cdot 0 + (1/3) \cdot 2.5 \approx 0.83$. Next, assume that your project has the same asset beta, but a smaller debt-to-asset ratio of 1 to 3, and compute your own equity beta: $\beta_{\text{Asset beta}} = w_{\text{Debt}} \cdot \beta_{\text{Debt}} + w_{\text{Equity}} \cdot \beta_{\text{Equity}} \approx 0.833 = (1/3) \cdot 0 + (2/3) \cdot \beta_{\text{Equity}} \Rightarrow \beta_{\text{Equity}} = 1.25$.
   (b) With an asset beta of 0.83, your firm's asset hurdle rate should be $\mathcal{E}(\tilde{r}_i) = 3\% + 2\% \cdot 0.83 \approx 4.7\%$.
   (c) Your comparable's equity expected rate of return would be $\mathcal{E}(\tilde{r}_{\text{Comps Equity}}) = 3\% + 2\% \cdot 2.5 = 8\%$. Your own equity's expected rate of return would be $\mathcal{E}(\tilde{r}_{\text{Our Equity}}) = 3\% + 2\% \cdot 1.25 = 5.5\%$

17. Your combined happy-marriage beta would be $\beta_{\text{combined}} = (3/4) \cdot 2.4 + (1/4) \cdot 0.4 = 1.9$.

18.

$$\beta_i = w_p \cdot 0.2 + (1 - w_p) \cdot 1 = 0 \quad \Rightarrow \quad w_p = 1.25$$

If your wealth is \$100, you want to short \$25 in the fund to increase your own portfolio holdings to \$125. The net portfolio has zero market beta.

19. No, the empirical evidence suggests that the CAPM does not hold. The most important violation seems to be that value firms had market betas that were low, yet average returns that were high. The opposite was the case for growth firms.

20. Even though the CAPM is empirically rejected, it remains the benchmark model that everyone is using in the real world. Moreover, even if you do not trust the CAPM itself, at the very least it suggests that covariance with the market could be an important factor.

21. The CAPM should work very well if beta is about 0. The reason is that you do not even need to guess the equity premium if this is so.

22. For short-term investments, the expected cash flows are most critical to estimate well (see Section 4·1.A on Page 62). In this case, the trouble spot (d) is really all that matters. For long-term projects, the cost of capital becomes relatively more important to get right, too. The market betas and risk-free rates are usually relatively low maintenance (though not trouble free), having only modest degrees of uncertainty. The equity premium will be the most important problem factor in the cost of capital estimation. Thus, the correct answer for long-term projects is (b) and (d).

23. This is a certainty equivalence question. Although it is not a gift per se, you cannot assume that \$10,000 is a fair market value, so that you can compute a rate of return of 1,900%—after all, it is your Uncle trying to do something nice for you. There are four outcomes:

Prob:	1/4	1/4	1/4	1/4	
	Crash	No-Crash	No-Crash	No-Crash	Mean
Stock Market	-10%	+18%	+18%	+18%	11%
Ferrari	\$200,000	\$0	\$0	\$0	\$50,000

Plug this into the formula and find $\mathcal{C}ov(\tilde{P}, \tilde{r}_M) = 1/4 \cdot [\$150,000 \cdot (-21\%) + (-\$50,000) \cdot (+7\%) + (-\$50,000) \cdot (+7\%) + (-\$50,000) \cdot (+7\%)] = -\$10,500$. We also need to determine the variance of the market. It is $\mathcal{C}ov(\tilde{r}_M, \tilde{r}_M) = [(-21\%)^2 + (+7\%)^2 + (+7\%)^2 + (+7\%)^2]/4 \approx 147\%\%$ (which incidentally comes to a standard deviation of 12% per annum, a bit low.) With the risk-free rate of 6%, lambda (λ) in Formula A.1 is $(11\% - 6\%)/147\%\% \approx 3.4$.

You can now use the certainty equivalence formula: The expected value of the Ferrari gift is \$50,000. If it were a safe payoff, it would be worth $\$50,000/1.06 \approx \$47,169.81$. Because you get more if the rest of your portfolio goes down, the Ferrari gift is actually great insurance for you. You value it $3.4 \cdot (-\$10,500)/1.06 \approx \$33,679$ above its risk-free equivalent of \$47,169.81: This Ferrari is therefore worth \$80,849.06. You have to pay \$10,000 today, of course, so you have managed to secure a deal for \$70,849.06.

24. First, compute the deboned cash flows: Jacqui: You can transpose table to fit it.

Year	1999	2000	2001	2002	2003	2004	Average	Variance
S&P 500	+21.4%	−5.7%	−12.8%	−21.9%	+26.4%	+9.0%	+2.7%	373.4%%*
Cash Flows	+\$2,864	+\$1,666	−\$1,040	+\$52	+\$1,478	−\$962	+\$676	
Demeaned S&P 500	+18.7%	−8.4%	−15.5%	−24.6%	+23.7%	+6.3%	0%	
Demeaned Cash Flows	+\$2,188	+\$98.7	−\$1,716	+\$624	+\$802	−\$1,638	\$0	
Cross-Product	\$408.36	-\$83.46	\$266.60	\$153.79	\$189.73	-\$102.67	\$166.47*	

The asterisk reminds you that I divided both the average cross-product and the variance by 5 rather than 6 to reflect the fact that this is a sample and not the population. The cash flow beta is about $\$166.47/373.4\%\% \approx \$4,455.55$. We now have the inputs to use our formula:

$$\text{PV} \approx \frac{\$676}{1+3\%} - \left[\frac{4\%}{1+3\%}\right] \cdot \$4,455.55 \approx \$657 - \$173 \approx \$484$$

$$= \frac{\mathcal{E}(\tilde{P})}{1+r_F} - \left[\frac{\mathcal{E}(\tilde{r}_M) - r_F}{1+r_F}\right] \cdot b_{\tilde{P},\tilde{r}_M}$$

This suggests a cost of capital of about $\mathcal{E}(C_{1\ \text{year}})/P_0 - 1 \approx \$667/\$473 - 1 \approx 41\%$. It turns out that this firm was Sony. This cost of capital estimate seems far too high. This is probably because the cash flow beta of Sony was way too high in relation to the ordinary CAPM market beta of Sony. Our CEV calculations did not do well in assessing value, probably because Sony's cash flows were far more volatile than its value.

All answers should be treated as suspect. They have only been sketched and have not been checked.

CHAPTER 14§

The CAPM: The Theory and its Limits

Where does the CAPM equation come from?

This chapter appears in the Survey text only.

THIS chapter explains the CAPM theory in detail *and* its shortcomings. Actually, because we have already discussed all the necessary ingredients—mostly in Chapter 12 which is absolutely necessary background for this chapter—this chapter is surprisingly simple. Really.

14·1 The Theory

14·1.A. The Logic and Formula

Here are the underlying mathematical facts, which were already explained.

We have already covered every necessary aspect of the CAPM theory in earlier chapters:

- In Section 8·3.D, we learned that the portfolio held by investors in the aggregate is the (market-capitalization) value-weighted portfolio.
- In Section 12·3, we learned that the combination of Mean-Variance Efficient portfolios is itself MVE.
- In Section 12·5, we learned that each stock i in a MVE-portfolio must offer a fair expected rate of return for its risk contribution to E. Formula 12.11 states that for each and every security i in the MVE portfolio E, this means that

$$\mathcal{E}(\tilde{r}_E) \;=\; r_F \;+\; [\mathcal{E}(\tilde{r}_E) - r_F] \cdot \beta_{i,E}$$

Here is the CAPM Theory.

The CAPM is a theory that has only one hypothesis and only one implication.

CAPM Hypothesis: The CAPM hypothesis is that each and every investor chooses an MVE portfolio.

Investors may not hold MVE portfolios.

Unlike the method for choosing weights for an MVE portfolio—which are just mathematical optimization techniques and which have to be correct—the CAPM is a real theory that may not hold in practice: investors might have different investment objectives and therefore not hold MVE portfolios.

CAPM Implication: If the CAPM hypothesis is correct, then the mathematical implication is that the aggregate value-weighted market portfolio is also MVE.

The CAPM could be wrong in practice.

If the CAPM hypothesis is not correct, there is no particular reason to expect the value-weighted portfolio to lie on the MVE Frontier. Instead, the value-weighted market portfolio could have risk-reward characteristics that place it far inside the MVE Frontier.

This is it: the Capital-Asset Pricing Model, which won the 1990 Nobel Prize in Economics.

IMPORTANT: The CAPM conjectures that all investors purchase MVE portfolios. As a necessary mathematical consequence, the value-weighted market portfolio is also MVE.

OK, let's do with algebra and logic.

Put altogether again, the logic of the CAPM is:

Mathematical Fact: To enter an MVE Frontier portfolio E, each stock in E has to offer an appropriate reward for its risk. This formula, which relates its expected rate of return to its covariation with E, is

$$\mathcal{E}(\,\tilde{r}_i\,) \;=\; r_F + [\mathcal{E}(\,\tilde{r}_E\,) - r_F] \cdot \beta_{i,E}$$

Stocks that offer expected rates of return higher than suggested by this formula would generate too much aggregate investor demand; stocks that offer lower expected rates of return would generate too little aggregate investor demand.

CAPM Theory Assumption: Every investor purchases an MVE portfolio.

$\Longrightarrow$ **Implication:** The value-weighted market portfolio is MVE.

$\Longrightarrow$ **Implication:** that

$$\mathcal{E}(\,\tilde{r}_i\,) \;=\; r_F + [\mathcal{E}(\,\tilde{r}_M\,) - r_F] \cdot \beta_i$$

In this case, the market portfolio consisting only of risky securities must be the tangency portfolio T.

DIG DEEPER

This is the "Sharpe-Lintner" CAPM. There is also a "Fisher Black" CAPM, in which there is no risk-free rate of return. Everything works the same, except that the risk-free rate is replaced by the constant a. All stocks in the economy follow the CAPM formula

$$\mathcal{E}(\tilde{r}_i) = a + [\mathcal{E}(\tilde{r}_M) - a] \cdot \beta_i \tag{14.1}$$

with respect to the value-weighted market portfolio M.

14·1.B. Some Odds and Ends

The market portfolio is theoretically clear, but practically difficult to identify.

- The theoretically correct market portfolio in the CAPM is the market-capitalization value-weighted portfolio, consisting of all possible investment assets in the economy, not just stocks in the U.S. stock market.

  However, in traditional investments use, the assumed goal is to pick the best portfolio among publicly traded stocks. It is as if the CAPM theory was restated to conjecture that investors seek to optimize the risk/reward relationship only within their portfolios of publicly traded domestic stocks. In this case, the proper market portfolio would be the market-capitalization value-weighted U.S. stock market portfolio. However, even this portfolio is difficult to obtain every day. Fortunately, the value-weighted U.S. stock market portfolio has high correlation with other broad stock market indexes, such as the S&P500. Thus, the S&P500 is often used as a reasonably good substitute.

  Although this is reasonable use for CAPM testing and corporate CAPM application, it is *not* good investment advice: you should definitely diversify across more assets than just domestic stocks. International stocks, commodities, real-estate, and even education represent other investment classes that are readily available to join smart investors' portfolios.

  In the real world, different investors have different investment opportunities. For example, you can invest in your house or you can invest your education and reap the rewards—but I cannot invest in *your* house and *your* education. The CAPM ignores this issue altogether. But if you have too much invested in some untraded assets (education or your specific house), you might not want the same kind of stocks to diversify your educational investment risk away. That is, you might not want stocks to minimize your portfolio variance only. Instead, you might want to purchase relatively more stocks that go up when your other untraded assets go down, and vice versa.

Mutual Funds facilitate holding the market portfolio.

- Transaction costs would be high enough to prevent ordinary small investors from themselves purchasing the widely diversified portfolio prescribed by the CAPM. Mutual Funds (Section 8·3.B), however, have made this relatively easy.

If the market portfolio is not MVE, then the CAPM does not hold and the CAPM formula cannot be used.

- What if the market portfolio does not sit on the MVE Frontier? Then there will be stocks for which the CAPM Formula 14.1 does not hold. And then it should not be assumed that the CAPM formula is the appropriate relation predicting stocks' expected rates of return.

  Indeed, then anything else could conceivably explain higher or lower required rates of return. For example, instead of firms with higher market betas, it may be firms with higher P/E ratios or firms with older CEOs or firms containing the letter "Z" that may end up having to offer higher expected rates of return in order to induce investors to willingly purchase and hold them in equilibrium. Naturally, such a situation would make it very difficult for corporations to determine the appropriate cost of capital that their projects should offer.

Higher beta implies higher expected rate of return.

- Because $\mathcal{E}(\tilde{r}_M) > r_F$, the intuition of the CAPM formula is that stocks with higher stock-market betas have to offer higher expected rates of return. Stocks with high betas are less helpful in reducing the risk of an investor who already holds the market portfolio.

From the perspective of a widely diversified investor, a tiny bit of zero-beta stock is just as good as investment in the risk-free security.

➤ A risky project that has a beta of zero need only offer an expected rate of return no higher than the risk-free rate itself. This is because each investor would only hold a tiny amount of this security. Having one cent in the risk-free rate or in a security whose return is expected to be the same and with no correlation to the rest of the portfolio is practically the same. In the real and naturally not perfectly CAPM world, this is not exactly correct—but it is often still a reasonably good approximation.

Solve Now!

Q 14.1 What is the main CAPM hypothesis, and what is the main CAPM implication?

Q 14.2 Write down the CAPM formula without looking at the text. You must memorize it!

Q 14.3 Under what circumstances is the market-beta a good measure of risk?

Q 14.4 What can plotting both the stock-market portfolio and the risk-free rate into the MVE Frontier graph tell you about the model and about the security markets line?

Q 14.5 What can checking whether every single security follows the CAPM formula (i.e., that every single possible investment is on the security markets line in a graph of expected rate of return against beta) tell you about the MVE Frontier (a graph of expected rate of return against standard deviation)?

Q 14.6 If all but one security are right on the security-markets line, is the market portfolio MVE?

Q 14.7 Is the CAPM market portfolio the value-weighted or the equal-weighted portfolio?

Q 14.8 Can the S&P500 be used as a proxy for the market?

Q 14.9 In the formula, why do higher beta stocks offer higher expected rates of return?

Q 14.10 If there is a risk-free security, does the stock market portfolio have to be the tangency portfolio for the CAPM to hold, or can it be any portfolio on the mean-variance frontier of risky assets?

Q 14.11 Is it possible for a risky stock to offer an expected rate of return that is less than the risk-free rate?

14·2 Does the CAPM Hold?

14·2.A. Listing All the CAPM Assumptions

Let's judge the CAPM Model Assumptions.

We have sneaked in the CAPM assumptions one at a time, perhaps to make them appear more palatable. It is excusable if they have slipped your mind by now. To help you judge how likely it is that the CAPM assumptions are reasonably satisfied in the real world, it is worthwhile to repeat them all at once. These are all the assumptions that we have used to conjecture that each and every investor buys an MVE portfolio:

Market Assumptions: ➤ The market is perfectly competitive.
- ➤ There are no transaction costs.
- ➤ There are no taxes.
- ➤ The investment opportunity choices are identical for each investor. Therefore, the value-weighted market portfolio is identical for all investors. It consists of all assets that can be invested in. It includes such assets as real estate, bonds, international markets, etc.
- ➤ The previous assumption means that are no untraded assets that only some, but not other investors can hold. That is, you cannot have your own house, or your own children, or your own education, or your own labor income, or your own executive stock options.

Informational Assumptions: ➤ Financial markets *are* efficient: as we will discuss in Chapter 16, this means that the market does not ignore information in the setting of financial prices, which an investor could use to outperform the market securities line. You cannot use public information to pick stocks better than the market can.

➤ There is no *actual* inside information that allows some investors to pick stocks better than the market can.

➤ There is no *perceived* inside information. That is, investors do not *believe* that they are able to pick stocks better than the market can.

➤ All investors share the same "opinion" about security expected rates of return and security variance/covariances.

➤ All model parameters are perfectly known:

- The expected rate of return on each and every stock (including the market portfolio) is known.
- All covariances (and thus betas) are known.
- If you as a researcher want to test the CAPM, i.e., whether other variables matter, you must also know possible alternative return factors (e.g., firm size) that investors can use.

➤ There are no managerial agency problems (i.e., managers enriching themselves) that can be changed if a large investor holds more of the particular stock with agency problems.

Preference Assumptions: ➤ Investors care only about their portfolio performance. They do not care about other characteristics of their holdings (e.g., socially responsible corporate behavior).

➤ Investors like mean and dislike variance. This fully describes their portfolio preferences. (It does assume that investors do not care, e.g., about skewness of their portfolio's future investment rate of return.)

↦ *Skewness 11·1 on Page 240*

➤ Agents maximize their portfolio performance independently each period. (Although this may appear innocuous, it implies that investors do not choose their portfolios to insure themselves against changes in future investment opportunities. You cannot buy stocks that you believe to pay off more in a recession, because you believe then will be a good time to double up investment.)

➤ Investors do not care about how their portfolios perform relative to other variables, including variables related to their own personal characteristics (except for risk-tolerance). For example, investors must not care about earning a different rate of return in states in which they find themselves ill or states in which inflation is high.

The CAPM is just a helpful model, not perfect reality.

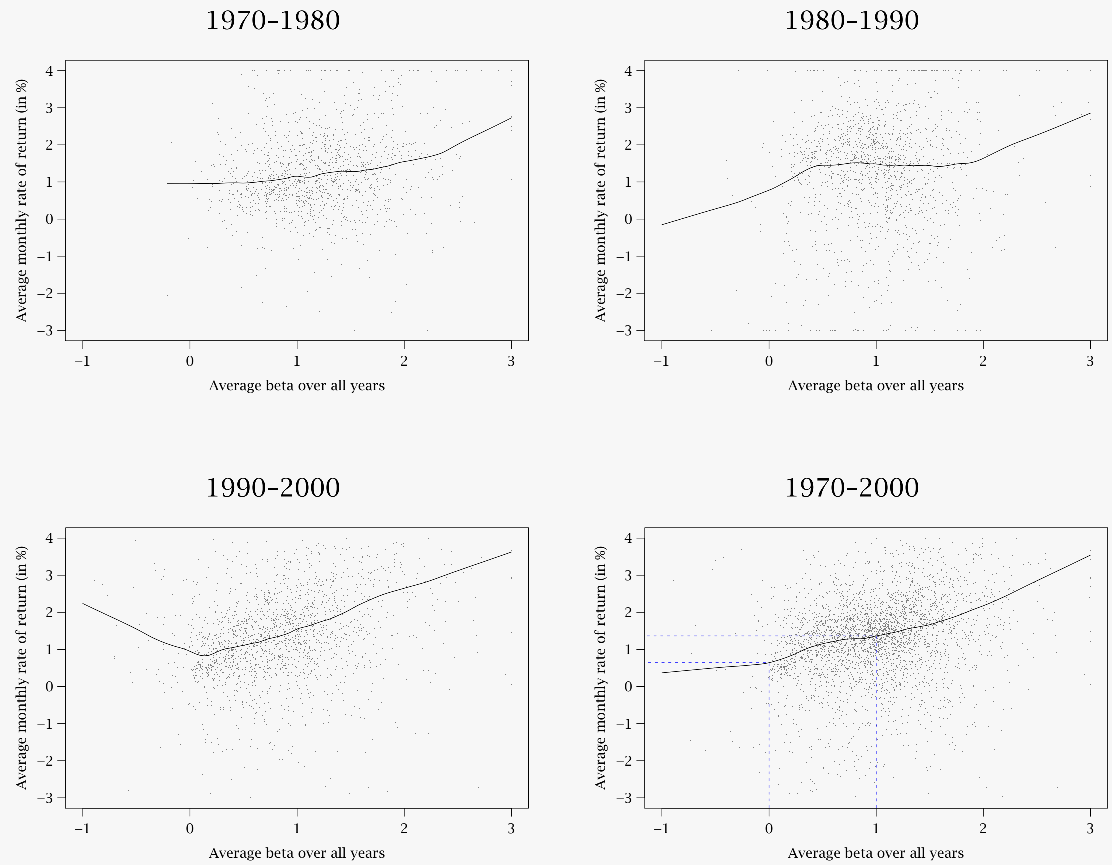
If these assumptions hold, the CAPM will hold. Of course, it could also be that the value-weighted market portfolio just happens to lie on the MVE frontier, in which case the CAPM security markets line would just happen to be a line by accident. Fortunately, the market need not lie *perfectly* on the MVE frontier, nor must the assumptions hold *perfectly* in order for the CAPM to be a useful model. After all, the CAPM is *only* a model, not reality. Only mathematics works the same in theory and in practice—economic models do not. The real question is whether the CAPM assumptions are sufficiently badly violated to render the CAPM a useless model—and in what context.

14·2.B. Is the CAPM a good representation of reality?

You do not hold the stock market portfolio, therefore the CAPM is incorrect. True, but...

In real life, we know that some investors purchase portfolios other than MVE portfolios. This should not come as a big surprise: chances are that you yourself are one of these investors, even if you are only a small investor. Therefore, in the strictest sense, we already know that the CAPM does not hold. But, again, the CAPM is only a model. The question is whether the CAPM simplification of reality is helpful in understanding the world, i.e., whether the model is sufficiently close to reality to be useful. Perhaps most of the real big investors "who matter" do invest in portfolios sufficiently similar to MVE portfolios, so that the value-weighted market portfolio is on the MVE Frontier, meaning that the CAPM "roughly" works.

Figure 14.1: Average Historical Rates of Return Against Historical Market Beta.



The web chapter on empirical asset pricing examines, among other issues, noise and non-stationarity problems when testing investments models.

The empirical relationship is graphed.

Unfortunately, the empirical evidence suggests that the market portfolio does not lie too closely on the MVE Frontier, that the security markets line is not perfectly linear, and that expected security returns relate to other characteristics in addition to beta. Figure 14.1 plots average monthly returns against stock betas. Care must be exercised when viewing these figures, because there are few stocks with betas below 0 and above 2. Therefore, the smoothed lines are not too reliable beyond this range. In the 1970s, the relationship between beta and average (monthly) return was generally upward-sloping, though not perfectly linear. In the eighties, although stocks with betas below 0.5 and 1.5 had similar average rates of return, stocks outside the 0.5 to 1.5 beta range (not too many!) had a clear positive relationship. In the 1990s, firms with positive betas showed a nice positive relationship, though the (few) stocks below $\beta = 0$ did not.

The most interesting plot is the overall graph, plotting the relationships from 1970 to 2000. The typical stock with a beta of 0 earned a rate of return of about 8% per annum, while the typical stock with a beta of 1 (i.e., like the market) earned a rate of return of about 18% per annum. Not drawn in the figure, the average stock with a beta of 2 earned about 217 basis points per month (30% per annum), and the average stock with a beta of 3 earned about 354 basis points per month (50% per annum). These 30 years were an amazingly good period for financial investments! The figure shows also how there was tremendous variability in the investment performance of stocks. More importantly from the perspective of the CAPM, the relationship between average rate of return and beta was not exactly linear, as the CAPM suggests, but it was not far off. If we stopped now, we would conclude that the CAPM was a pretty good model.

But look back at Figure 13.5. The empirical evidence is not against the CAPM in the sense of the first three plots (linearity)—it is against the CAPM in the sense of the last three plots (better alternative classifications). Although we cannot see this in Figure 14.1, the CAPM fails when we split these stocks into groups based on different characteristics—and academics seem to come up with about five new different characteristics per year. The empirical reality is somewhat closer to the latter three figures than it is to the idealized CAPM world. This implies that market beta seems to matter *if* we do not control for certain other firm characteristics. For example, some evidence suggests that firms that are classified as "small growth firms" by some metrics generally underperform "large value firms"—but neither do we really know why, nor do we know what we should recommend a corporate manager should do about this fact. Maybe managers should pretend that they are growth firms—because investors like this so much they are willing to throw money at growth firms—but then act like value firms and thereby earn higher returns?! But we still have another "little problem" (irony warning)—we finance academics are not exactly sure what these characteristics are, and why they matter.

But this is deceptive—the CAPM fails against specific better alternatives.

14·2.C. Professorial Opinions on the CAPM

Different academics draw different conclusions from this evidence—and it should leave you with a more subtle perspective than we academics would like. See, we professors would be happy if we could tell you that the CAPM is exactly how the world works, and then close the chapter. We would probably even be happy if we told you that the CAPM is definitely *not* the right model to use, and then close the chapter. But it is not so simple. Yes, some professors recommend outright against using the CAPM, but most professors still recommend "use with caution." Now, before I give you my own view, be aware that this is just my own assessment, not a universal scientific truth. In any case, you positively need to be aware of the advantages and disadvantages of the CAPM, and know when to use it and when not to use it.

You need to know pros and cons.

My own personal opinion is that although the CAPM is likely not to be really true, market beta is still a useful cost-of-capital measure for a corporate finance manager. Why so? Look at the last three plots in Figure 13.5 again: If you have a beta of around 1.5, you are more than likely a growth firm with an expected rate of return of 10% to 15%; if you have a beta of around 0, you are more than likely a value firm with an expected rate of return of 3% to 7%. Thus, beta would still provide you with a decent cost of capital estimate, even though it was not beta itself that mattered, but whether the firm was a growth or a value firm. (Market beta helped in indicating to you whether the firm was a growth or a value firm in the first place.) Admittedly, using an incorrect model is not an ideal situation, but the cost-of-capital errors are often reasonable enough that corporate managers generally can live with them. And the fact is, if corporations cannot live with these errors, we really do not know what to recommend as a better alternative to the CAPM!

My personal opinion.

In sum, although the CAPM formula is mistaken, it continues to be the dominant model for the following reasons:

Why the CAPM continues to be and should be used.

1. Although the market portfolio does not seem to lie directly on the MVE Frontier, it does not appear to be *too* far away.
2. The CAPM provides good intuition for the characteristics that should determine the expected rates of return that have to be offered by stocks: stocks that help diversified investors achieve lower risk should be sellable even if they offer lower expected rates of return.
3. For many purposes and many stocks, the CAPM provides reasonable expected rates of return, not too far off from those gleaned from more complex, difficult, and/or arbitrary models.
4. There is no good alternative model that is either simpler to use or convincingly better in providing appropriate expected rates of return for investments.
5. The CAPM is *the* gold standard for cost-of-capital estimations in the real world. It is in wide use. Understanding it (and its shortcomings) is therefore crucial for any student with an interest in finance.

In contrast, my advice to an investor would be *not* to use the CAPM for investment portfolio choices. There are better investment strategies than just investing in the market, although wide diversification needs to be an important part of *any* good investment strategy.

Another interesting question is *why* the CAPM fails. In my own opinion, most investors do not invest scientifically. They buy stocks that they believe to be undervalued and sell stocks that they believe to be overvalued. Why do even small and relatively unsophisticated investors—often relying on investment advice that qualifies as the modern equivalent of hocus-pocus—believe they are smarter than the financial markets, whilst even professional investors have difficulties beating the market? This is perhaps the biggest puzzle in finance.

Although people's tendencies to categorize choices might have given us some hope that people at least maximize MVE within the domain of their own pure market investment portfolios, perhaps subject to their idiosyncratic estimates of expected rates of return, the evidence suggest that investors are not even internally consistent. Even among the stocks for which they hold opinions about high or low expected rates of return, investors tend not to choose securities to reach their own best MVE Frontiers.

Unfortunately, if every investor behaves differently, it is also not clear what the expected rate of return on a particular investment really should be or has to be—other than that it need not necessarily be that suggested by the CAPM. It could be high, it could be low, it could be anything.

This phenomenon is the domain of behavioral finance, eventually to be further expounded in a web chapter.

14·2.D. Why not Optimization instead of the CAPM?

What are our alternatives to the CAPM? Investors have good, although tedious alternatives...

Why could we not just rely on optimal portfolio theory (Chapter 12) instead of on the CAPM? From the perspective of an investor, this is a feasible and probably better choice. The only drawback is that instead of believing that the market portfolio is MVE, i.e., that a part of the investor's portfolio should be allocated to it, the investor has to solve the more general problem of where the MVE Frontier is. This is tedious, but possible. Even so, in many situations, i.e., when not too much money is at stake, even though the stock market portfolio is not exactly MVE, it is probably still close enough for most investors to just choose it anyway in order to avoid the pain of computation. Nevertheless, the CAPM should not be used for investment purposes where more than $100 million is at stake. There may be better investment strategies. Again, for smaller investment portfolios, the advice of purchasing a combination of the value-weighted market portfolio (including international stocks and other assets) and a risk-free rate is probably reasonably close to an MVE portfolio.

...while corporations really have few or no alternatives.

From the perspective of a firm, however, relying on optimal portfolio theory is not a feasible choice. Corporate managers need to be able to compute what expected rate of return investors would demand for a particular project. If they do not know what investors want in equilibrium, managers are in a guessing game. Take the CAPM for what it is useful for: it is a model that provides decent guesstimates of appropriate cost of capital for non-financial projects, in which the exact discount rate is not too critical. Do not use it if a lot of money can be lost if the discount rate is misestimated by a percent or two.

Solve Now!

Q 14.12 List as many assumptions about the CAPM that you can recall. Which ones are most problematic?

14·3 Portfolio Benchmarking with the CAPM?

Can the CAPM Formula be beaten in a CAPM world?

The performance evaluation of investment managers is an example of how problematic the application of the CAPM formula can be. The typical CAPM use in **portfolio benchmarking** has investors compute the beta of portfolios held by their investment managers, and check if the manager's portfolio return over the year beat the securities market line. Unfortunately, if the CAPM holds, the security markets line cannot be beaten. After all, the definition of the MVE Frontier is that it is the set of portfolios that offers the best possible combination of risk and return. If the CAPM does not hold, then the security markets line has no reason for living. It would be totally ad-hoc. Strictly speaking, the CAPM cannot be used for portfolio benchmarking.

There is a good intuitive reason to use the CAPM, anyway.

Nevertheless, the CAPM formula gives good intuition on what constitutes good performance contribution to a widely diversified investor. It also works if the fund manager with the presumably better information and stock picking ability is too small to move prices. This explains why the use of the CAPM formula for performance evaluation continues—despite all its problems.

14·4 Summary

The chapter covered the following major points:

- The logic of the CAPM is that if every investor holds a mean-variance efficient portfolio, then the (value-weighted) market portfolio is mean-variance efficient. This in turn means that stocks follow the linear relationship between the expected rate of reward and market-beta (the security markets line).
- The CAPM relies on many assumptions. It is only a model.
- The CAPM is a useful simplification in certain contexts (such as capital budgeting), because it is intuitive, easy to use, and often gives a reasonable enough cost-of-capital estimate.
- The CAPM is not a reasonable description of reality in many other contexts. It should not be used for stock investing purposes.
- There are no good alternatives to the CAPM. This is why the CAPM, with all its faults, is still the predominant model in most corporate contexts.

14·5 Theory: CAPM Alternatives!?

In a survey in 2007, about 75% of all finance professors recommended the CAPM for use in a corporate capital budgeting context. About 5% recommended the so-called APT. And 10% recommended the so-called Fama-French factors. Not surprisingly, these two alternative models have some advantages but perhaps more disadvantages relative to the CAPM from a capital budgeting perspective—if it were otherwise, we would have deserted the CAPM. (Forms of these models clearly work better for financial investment purposes, though.) It is impossible to explain these models fully in a first corporate finance course, but I want to give you at least a sketch.

14·5.A. The Arbitrage Pricing Theory (APT) and Intertemporal CAPM (ICAPM)

For understanding the analogy later, first recap the CAPM.

The first alternative is an extension of the ordinary CAPM, called the ICAPM. The second is called the **arbitrage pricing theory** (**APT**). In practical use, the two are indistinguishable, so I will just treat them as one and the same model here. Let's think back as to how the CAPM works:

1. The CAPM asks you to measure how each stock's rate of return moves together with the overall stock market rate of return. This is its market beta.
2. The model's intuition is that investors dislike stocks that move together with the stock market and like stocks that move against the stock market.
3. The CAPM tells you the exact formula by which you should receive a higher average rate of return for firms that expose you to a lot of covariation with the stock market. It may be

   $$\mathcal{E}(\tilde{r}_i) \;=\; 4\% \;+\; 5\% \cdot \beta_{i,\text{M}}$$

   where the second subscript reminds you that this beta measures a stock's sensitivity with respect to the market.

You might care about economy-wide factors other than the rate of return on the stock market.

Now let's assume that stocks differ not only in how they move with or against the stock market but also in how they move with or against other economic factors, say, the oil price. You might care about oil price changes because your business may do poorly if energy costs rise. Therefore, if you can find a stock that increases in value when oil prices rises, you would consider this stock as good insurance against bad business—just as you consider a stock that goes up when the market goes down as good insurance against market downturns in the CAPM framework. (If you are in this situation, chances are that you would really like to hold stocks like Exxon or Chevron.)

Different stocks can have different exposures to these other economy-wide factors.

How can you measure whether a stock goes up or down with the oil price? Simple—you get this measure the same way that you get a measure of whether a stock goes up or down with the stock market. For each stock, you run a time-series regression, in which the independent variable is not the rate of return on the stock market but the oil price change:

$$\tilde{r}_i \;=\; a + \beta_{i,\text{Oil Price Change}} \cdot (\text{Oil Price Change})$$

This gives you for each stock a beta that measures how its rate of return moves with oil price changes. A stock that has a very large $\beta_{i,\text{oil price change}}$—say 5—would go up a lot if the oil price increases—think Exxon. A stock that has a negative $\beta_{i,\text{oil price change}}$—say -3—would go down when the oil price increases—think United Parcel Service (which has to pay more for gas when the oil price increases).

And here is the analogy: assets with higher exposures (betas) to these factors have to offer higher expected rates of returns.

Would you be willing to pay more for a stock that acts as an insurance against oil price increases? If your livelihood is adversely affected by oil price changes, then the answer is probably yes. The more important question is whether this is also the attitude of most investors in the market. If it is, then a stock like Exxon, which has a high $\beta_{i,\text{oil price change}}$, would be more desirable. Such a stock would not have to offer as high a rate of return as another stock that has a low $\beta_{i,\text{oil price change}}$. The APT then gives you a formula that relates the oil-price-change beta (and other betas like it) to the expected rate of return on a stock—something like

$$\mathcal{E}(\tilde{r}_i) \;=\; 4\% \;+\; 5\% \cdot \beta_{i,\text{M}} \;-\; 3\% \cdot \beta_{i,\text{oil price change}}$$

You can now use the formula the same way you used the CAPM formula. To recap, the APT works like the CAPM but allows more than just one beta (and just one risk premium):

1. The APT asks you to measure for each stock how it moves with respect to factors (like the oil price) that you decide on. This gives you, for each stock, a set of market betas, one for each factor.
2. The intuition is that investors like stocks that have high or low betas with respect to these factors. (The sign depends on investors' preferences.)
3. The APT tells you the exact formula by which you should receive a higher average rate of return for firms that expose you to bad covariation with respect to the factors that matter.

What are the APT Factors?

Common APT models use as factors interest rate changes, GDP changes, bankruptcy risk, the returns of growth stocks, and the returns of small firms. Each stock then has a beta with respect to these factors. And an APT formula relates the average rate of return to these betas.

APT flexibility is both good and bad.

Unfortunately, the APT is even harder to use than the CAPM. The good news is that it allows you to specify that investors care about factors other than the overall stock market. You then use the beta of your project with respect to the market to determine the appropriate expected rate of return. The bad news is that it allows you to specify that investors care about factors other than the overall stock market. The problem is that the APT does not give you any guidance on what these factors should be. What factors do academics recommend? Sorry, there is no consensus of what the best APT factors are. So the APT's flexibility is both a blessing and a curse.

Canned usage is easy *if* you pay for it. However, estimates can vary widely.

Most commonly, corporations rely on third-party vendors who have developed such APT models. This way, they get at least a second opinion on their average cost of capital. (This is rarely done for individual projects, even though we know that costs of capital should be computed project by project.) The APT vendor reports the factors that they like to use (the market beta, and the oil price change in our example) and the "premiums" (4%, 5%, –3% in our example), and estimates your firm's betas with respect to these premiums. You can then multiply the factors with the premiums to obtain an alternative measure for the cost of capital. Alas, there is no guarantee that any one particular APT model is the right model. In fact, two APT vendors can easily derive completely different cost of capital estimates. You have to judge which one is better. In other words, use the APT at your own risk.

Solve Now!

Q 14.13 Explain how the APT model is similar to but more general than the CAPM.

14·5.B. The Fama-French-Momentum (-And-More) Model

The Fama-French factors plus momentum.

While the ICAPM and APT developed out of a tradition of theoretical models with empirical applications, another set of models has come out of a tradition of empirical research. The most prominent empirical regularities right now seem to be the following:

1. **Momentum:** Stocks tend to perform better if they have had high stock returns over the previous 12 months, but excepting the most recent month. (Omitting this last month is very important.) Firm's own momentum is a very robust predictor, although this is not exactly what the model that I am explaining would be using.
2. **Value:** Stocks tend to perform better if they have high accounting book value of equity (explained in Chapters 18 and 19) divided by the market value of equity. Firms that fit this criterion are called **value firms**, while firms that have higher market values than accounting book values are called **growth firms**. A typical value firm is "boring," like the diaper vendor Procter&Gamble. A typical growth firm is "exciting," like Google or Apple. In the long run, the superior stock return performance of value firms relative to growth firms has been a very robust relationship, too—even though there were some periods when it did not hold, first and foremost during the late 1990s.
3. **Size:** There is some evidence that smaller firms perform better than larger firms. The role of firm size is not as strong and robust as the two preceding effects.

The latter two regularities are usually called the **Fama-French factors**, because it was Eugene Fama and Ken French who investigated them most thoroughly. The first was suggested as an addition by Mark Carhart. Please don't think that these three empirical regularities are the only ones. There are literally dozens more (accounting accruals and net issuing activity are particularly noteworthy). However, these three factors are quite prominent. (For more determinants of average rates of return, you really have to read an investments text book.)

Use of the Model in a Corporate Context

Finding exposures—like APT exposures.

How can you use this model in a corporate context? Let me sketch how this would work. Ken French posts the historical rates of returns for the equity premium (which he calls XMKT) and the three other factors on his website at Dartmouth. They are:

XMKT: The equity premium is the average rate of return on the stock market net of the risk-free rate. The average rate of return on XMKT (from 1927 to 2006) was about 8.5%.

UMD (up-minus-down): The momentum net portfolio is the average rate of return on firms having done well over the last 12 months minus the average rate of return on firms having done poorly (both omitting 1 month). The average rate of return on this portfolio was about 8.9%. (Also note how big this rate of return is for a zero investment portfolio!)

HML (high-minus-low): The "value" portfolio is the average rate of return on stocks with high accounting book value relative to market value. The "growth" portfolio is the same for stocks with the opposite characteristics. The average rate of return on the net portfolio was about 4.6%.

SMB (small-minus-big): The "small firm" portfolio is the average rate of return on stocks of small firms. The "large firm" portfolio is the same for large firms. The average rate of return on the net portfolio was about 3.8%.

You would first run a time-series regression of your own project's (i's) historical rates of returns *net of the risk-free rate* on the four time-series:

$$\tilde{r}_i - r_F = a_i + b_i \cdot \text{XMKT} + c_i \cdot \text{UMD} + d_i \cdot \text{HML} + e_i \cdot \text{SMB} + \text{noise}$$

Note that the abnormal return of 3% should *not* repeat.

Now let's say that your regression package estimated your project's coefficients to be $a = 3\%$, $b = 2$, $c = 0$, $d = 0$, and $e = 0$. Well, then your particular stock behaves almost like a CAPM stock with a market beta of 2, because your reduced Fama-French model is

$$\mathcal{E}(\tilde{r}_i) - r_F = 3\% + 2 \cdot \mathcal{E}(\text{XMKT}) = 3\% + 2 \cdot [\mathcal{E}(\tilde{r}_M) - r_F] \tag{14.2}$$

Note that the risk-free rate intercept is already on the right-hand side, so your 3% estimated intercept would be an excess rate of return that your stock has earned historically, above and beyond what the Fama-French model would have suggested. You would therefore also not expect this extra 3% rate of return to repeat.

Here is how you would use the model to find a hurdle rate for one particular project.

What would be a good hurdle rate for your project? If you believe the future equity premium to be 5% and the Treasury risk-free rate to be 4%, then you would expect your stock's rate of return to be

$$\begin{aligned} \mathcal{E}(\tilde{r}_i) - 4\% &= 2 \cdot 5\% \\ \mathcal{E}(\tilde{r}_i) - r_F &= \beta_{i,\text{XMKT}} \cdot \mathcal{E}(\text{XMKT}) \end{aligned}$$

The model suggests an expected rate of return of $\mathcal{E}(\tilde{r}_i) = 4\% + 2 \cdot 5\% = 14\%$ for your project. Note how the Fama-French application omits the 3% from Formula 14.2 here—the reason, as just noted, is that the 3% was an unusual rate of return that you would not expect to repeat. Note that, instead of using your 5% guess about the future equity premium, you could have used the historical average rate of return on XMKT. From 1927 to 2006, it was 8.9%. In this case, you would have required your project to earn a rate of return of $5\% + 2 \cdot 8.9\% \approx 23\%$.

Ok, here comes a useful application of the model. The above project was just for practice.

Now let's choose another project. Let's say you estimate coefficients $a = 3\%$, $b = 0.5$, $c = -1$, $d = 2$, and $e = -2$ for this one. Again, you would need some estimates of the future average rate of return for the four Fama-French factors, just as you needed an estimate for the future average rate of return for the equity premium. Remember how we agonized about the equity premium? You really should agonize equally about all risk premium estimates. However, for lack of a good source and great intuition, most people just use the historical average rates of return, mentioned above. If you buy into the hypothesis that the historical averages are good predictors of the future premiums, you would then estimate your project's appropriate expected rate of return to be

$$\begin{aligned}\mathcal{E}(\tilde{r}_i) - r_F &= 0.5 \cdot \mathcal{E}(\text{XMKT}) + (-1) \cdot \mathcal{E}(\text{UMD}) + 2 \cdot \mathcal{E}(\text{HML}) + (-2) \cdot \mathcal{E}(\text{SMB}) \\ &= 0.5 \cdot 8.5\% + (-1) \cdot 8.9\% + 2 \cdot 4.6\% + (-2) \cdot 3.8\% \\ &= -3.05\%\end{aligned}$$

With a risk-free rate of return of 5%, you would set a hurdle rate of about 5% − 3% = 2%.

Practical notes, especially with respect to momentum.

Some final notes: firm size and firm market beta are sufficiently highly correlated that in most practical capital budgeting applications, you can ignore firm-size and rely on market beta alone (or the opposite). Moreover, momentum is such a short-term phenomenon that it is usually irrelevant for longterm capital budgeting purposes. Relying on 1-year momentum for a cost of capital estimates for 10-year investments in a corporate context does not make sense. This is why UMD is often excluded from this model in a corporate context. This leaves two factors—the firm's value/growth positioning and its market beta—as good inputs in a practical capital budgeting model. Moreover, this form of the model does not do justice especially to momentum, which is more of an idiosyncratic effect than a factor exposure to UMD. A better model would work with firms' own characteristics rather than these factor exposures. (In an APT context, one could then view these characteristics of stocks as picking up firms' betas to some factors. Of course, other researchers believe that these are not really betas, but more a reflection of market inefficiencies, the subject of Chapter 16 on Page 389.) This is all too telegraphic, of course. You should really consult an investments text to learn how to do this better.

Solve Now!

Q 14.14 Assume that you ran a time-series regression with your project on the Fama-French factors and found the following:

$$\mathcal{E}(\tilde{r}_i) - r_F = (-2\%) + (1.3) \cdot \text{XMKT} + (0.1) \cdot \text{UMD} + (-1) \cdot \text{HML} + (-0.1) \cdot \text{SMB}$$

What would the Fama-French-Momentum model suggest you use as the hurdle rate for this project? Recall that $\mathcal{E}(\text{XMKT}) \approx 8.5\%$, $\mathcal{E}(\text{UMD}) \approx 8.9\%$, $\mathcal{E}(\text{HML}) \approx 4.6\%$, and $\mathcal{E}(\text{SMB}) \approx 3.8\%$? Assume that the prevailing risk-free Treasury offers 3%.

Deeply Dug Appendix

[No keyterm list for capmtheory-g.]

End of Chapter Problems

Q 14.15 What are the APT factors?

Q 14.16 What are the Fama-French-Momentum factors?

Q 14.17 Assume that you ran a time-series regression with your project on the Fama-French factors and found the following:

$$\mathcal{E}(\tilde{r}_i) - r_F = (12\%) + (0.3) \cdot \text{XMKT} + (0.3) \cdot \text{UMD} + (-0.5) \cdot \text{HML} + (-0.5) \cdot \text{SMB}$$

What would the Fama-French-Momentum model suggest you use as the hurdle rate for this project?

14 "Solve Now" Answers

1. Because each and every investor purchases an MVE portfolio, the value-weighted market portfolio is MVE.
2. See Formula 14.1
3. It is a measure of risk contribution to an investor's portfolio, if this portfolio is the market portfolio.
4. If the market portfolio lies on the MVE Frontier, then the CAPM holds and all securities must lie on the security markets line.
5. Whether the stock market portfolio is on the MVE Frontier.
6. No.
7. The value-weighted portfolio.
8. Maybe yes, maybe no—but it certainly is *commonly* used as a proxy for the value-weighted domestic stock market portfolio.
9. Because $\mathcal{E}(\tilde{r}_M) > r_F$: the stock market is assumed to have a higher expected rate of return than risk-free Treasury bonds.
10. It has to be the tangency portfolio.
11. Yes, if this stock helps exceptionally well to diversify the market portfolio (beta is negative).

12. See Subsection 14·2.A.

13. The APT is almost like a multifactor version of the CAPM. Whereas in the CAPM, everything depends on the one factor that is the rate of return on the stock market, in the APT there can be multiple factors—such as the rate of return on the stock market, the rate of return from investing in oil, and so on. Both models then say that assets that are more exposed to these risks have to offer higher expected rates of return. Unlike the CAPM, the APT does not necessarily assume that the rate of return on the stock market is one factor. It also does not assume that there is an optimal market portfolio, in which all investors should invest.

14. The Fama-French-Momentum model suggests

$$\begin{aligned}\mathcal{E}(\tilde{r}_i) - r_F &= (+1.3) \cdot \mathcal{E}(\text{XMKT}) + (+0.1) \cdot \mathcal{E}(\text{UMD}) + (-1) \cdot \mathcal{E}(\text{HML}) + (-0.1) \cdot \mathcal{E}(\text{SMB}) \\ &\approx (+1.3) \cdot 8.5\% + (+0.1) \cdot 8.9\% + (-1) \cdot 4.6\% + (-0.1) \cdot 3.8\% \\ &\approx 6.96\% \approx 7\%\end{aligned}$$

This is a rate quoted *above* the risk-free rate. Thus, your appropriate cost of capital (hurdle rate) would be $3\% + 7\% = 10\%$.

All answers should be treated as suspect. They have only been sketched and have not been checked.