Chapter 4

Fourth rung: the Planets

> *[Kepler] had to realize clearly that logical-mathematical theorizing, no matter how lucid, could not guarantee truth by itself; that the most beautiful logical theory means nothing in natural science without comparison with the exactest experience. Without this philosophical attitude, his work would not have been possible.* Albert Einstein, foreword for "Johannes Kepler: Life and Letters", Baumgard and Callan, 1953.

If you have ever been told one piece of astronomy trivia, it is probably that the word "planet" comes from the Greek πλάνητες ἀστέρες (*planētes asteres*, wandering star) or sometimes just πλανῆται (*planētai*, wanderers). As the ancient Greeks studied the heavens, they observed that most of the stars seemed fixed into constant patterns, or *constellations*, in a great celestial sphere gently rotating around a single fixed axis (oriented along *celestial north*, which is very nearly where the north star Polaris is currently located). A handful of stars, however, seemed to follow their own unique path across the sky at their own speed. These *planētes asteres* did not wander into every quadrant of the sky, but traveled within a consistent band along with the Sun and Moon. This band was called the *ecliptic* (because it was in this band that eclipses happened) and in ancient Greece it happened to cross twelve constellations in the fixed heavenly sphere. These twelve constellations came to be known as the *Zodiac*, and to this day many people ascribe great meaning to which constellation the Sun was nearest at an individual's moment of birth. While the Sun steadily makes its way all around the ecliptic and through each of the twelve Zodiac constellations in a year, and the Moon wobbles through the same ecliptic in neatly predictable phases, the five troublesome *planētes asteres*, known now as the planets Mercury, Venus, Mars, Jupiter, and Saturn, move in a more complex pattern. Mostly, the planets moved in an east to west direction along the ecliptic, but occasionally they would slow and reverse direction, moving west to east in *retrograde motion* for a while before resuming their east to west travel again. More confusingly, each planet seemed to move at its own speed and have its own unique pattern of retrograde motion.

The planets' movements fascinated the ancient Babylonian astrologers as far back as the second millennium BCE. The motions of the heavenly bodies were considered important and powerful omens and so they kept careful records on cuneiform tablets, seeking patterns across generations that would allow them to make predictions of ominous events to come. Because their concern was primarily with their predictive power, Babylonian astronomy focused heavily on refining their calculations, with no apparent interest in the underlying mechanics of the cosmos. While they did not directly tackle the puzzles of the distance ladder, the Babylonian records gave rise to calendar and timekeeping systems so accurate that many elements continue to be used in our systems today.

In contrast to the Babylonians, early Greek astronomy was closely linked to their study of geometry and one of their earliest cosmological models came from a clever mathematician named Eudoxus. In answer to a question Plato posed on how to account for the apparent motions of the planets with uniform and orderly geometric motions, Eudoxus proposed a system of nested spheres which rotated at different speeds about different axes of rotation. Eudoxus' astronomical works have not survived to the present day except for a handful of quotes in later works, but he appears to have considered each planet independently with its own number of spheres (Mars, for example, had four). As he considered this a geometry problem, he was apparently satisfied by coming up with a solution that was merely plausible rather than predictive. His system was virtually impossible to make calculations with using the mathematics of the era, and woefully inaccurate when compared to any observational data. A generation later, his student Callippus attempted to improve the accuracy and ended up with some 34 nested spheres to account for all five planets, the Sun, the Moon, the distant fixed stars, and the central, rotating Earth. Aristotle went on to make further modifications, primarily adding "unrolling" spheres to cancel out the motion of an inner sphere from impacting the next one out.

It was at about this time that Greeks suddenly seemed to gain access to the Babylonian astrological records. One account claims that Alexander the Great, amid his conquests, ordered his historian to have the cuneiform records of the Babylonians translated and that this historian sent copies of the translation work to his great-uncle, who just happened to be Aristotle. While Aristotle did not abandon the Eudoxan spheres, the infusion of centuries of Babylonian records spurred the Greeks to seek more accurate geometric explanations. Aristarchus' work on heliocentrism is one example of that; however the only piece of that work that survives is concerned specifically with the relationship between the Earth and the Sun, so we are left somewhat in the dark about Aristarchus' thoughts on the movements of the other planets.

Apollonius of Perga, another brilliant Greek mathematician, provided the next refinement. Known today primarily for some of the earliest work on conic sections, Apollonius was able to simplify the Eudoxan model by introducing the concept of the *deferent* and *epicycle*. Epicycles can be hard to explain, but we can fortunately draw upon our modern heliocentric understanding of the solar system. to more easily picture what had to be very difficult to describe in Hellenistic Greece. As we are taught in school, the Earth orbits around the Sun, and the Moon in turn orbits around the Earth. The orbit of the Earth corresponds to one of Apollonius' deferents, and the smaller orbit of the Moon around the earth is an epicycle. If one then removes the physical Earth from this picture, so that the Moon now orbits an epicycle centered at some imaginary point that in turn traverses the deferent, then you have a pretty good idea of what an epicycle-based orbit would look like. The epicycle-based model of planetary motion did a better job of approximating retrograde motion than the Eudoxan spheres and was less computationally complex. The Greeks, in fact, were able to build clock-like mechanical devices to show the motions of the planets using epicycles. These devices were still incredibly complex, so much so that the surviving remains of one, called the *Antikythera Mechanism* and found in a shipwreck from 80 BCE over a century ago, is often accused of being a hoax to this day.

While he did not invent them, the astronomer most closely associated with epicycles is unquestionably Claudius Ptolemy. In his monumental work Μαθηματικὴ Σύνταξις (*Mathēmatikē Syntaxis*, the Mathematical Arrangement), now known as the *Almagest*, Ptolemy laid out his comprehensive geocentric astronomical model, using extensive use of epicycles (and, in some cases, epicycles of epicycles) to try to fit the observed motions of the planets into neat circular orbits. It was far from perfect, but the *Almagest* was so comprehensive that it dominated astronomical study for over a millennium, with generations of valiant astronomers trying to patch the cracks, typically through the application of still more epicycles. It was not until the fifteenth century, however, that a monk named Bessarion from the remnants of the Byzantine empire began laying the groundwork for a major shift in how we all see the universe.

There were several unlikely things that came together in the lifetime of Basilius Bessarion, virtually none of which have much to do with astronomy. He began his education in Constantinople which was not far from where he was born, and then continued it in Greece, where he was introduced to the Plato's writings and Pythagoras' numerical mysticism. He studied all the traditional areas of Greek learning, including astronomy, and he began his own collection of Greek writing which he brought back to Constantinople. He rose to power within the Eastern Orthodox Church, and when the Byzantine Emperor decided to try to reconcile the Orthodox Church in Constantinople with the Catholic church in Rome, he was chosen as a member of the delegation which traveled first to Ferrara in 1438 and then to Florence the following year. This reconciliation did not occur, but Bessarion was an ardent enough supporter of it that he found life in Greece following the Council of Ferrara-Florence uncomfortable. Fortunately, he had so impressed the Roman pope that he was subsequently made a cardinal in the Roman Catholic Church. From that point, Bessarion lived in Rome and Bologna for the remainder of his life. His court attracted Greek refugees and others interested in Greek learning, as he brought what had grown into the largest private library in Europe with him to Rome. Bessarion continued to seek out scholars and after a rival produced a particularly bad translation of Ptolemy's *Almagest* into Latin, Bessarion asked a German mathematician named Peuerbach and his student, Regiomontanus (he was born Johannes Müller but a fancy Latinized professional name had become the fashion), to write a critique and counter-translation. Peuerbach died partway through the work, but on his deathbed made Regiomontanus promise to complete it. Regiomontanus did as he promised, publishing *Epytoma in almagesti Ptolemei* and gaining Bessarion's patronage (and access to his library) in return. Bessarion had one more part to play in our story (and the larger story of the Renaissance): shortly before his death, he donated his entire library to the Republic of Venice. Consisting of nearly five hundred Greek and three hundred Latin manuscripts, the collection went on to become the core of the famous *Bilbioteca Marciana* (library of St. Mark) and established the region of northern Italy as a center of Greek learning for the next century.

Four short months after Basilius Bessarion's death, Nicolaus Copernicus (to continue the fashion by Latinizing the Polish name Mikolaj Kopernik) was born. While we tend to think of him as the great Polish astronomer, Copernicus was a priest, diplomat, statesman, physician, and economist who managed to reshape our cosmological view in the free time around his his professional duties. His uncle, Lucas Watzenrode, Prince-Bishop of the semi-autonomous Polish region of Warmia, raised Copernicus and his siblings after the deaths of Copernicus' parents. Watzenrode set Copernicus onto a path of life as an important member of his court and the church bureaucracy, possibly even hoping he would succeed him as Prince-Bishop of Warmia. He provided for Copernicus' education, sending him to university in Poland and then to Bologna to study church law. Copernicus had already developed both an interest in astronomy and a lifelong love of books from his studies in Poland. In Bologna, Copernicus quickly fell into the circle of the Neoplatonist Greek scholars, so much so that he had to return some years later to eventually complete his law degree. He became a student of the Ferraran astronomer Domenico Novara who had himself been a student of Regiomontanus. During his time in Bologna, Copernicus quickly learned to read and write in Greek and delighted in adding Greek writings to his own rapidly growing library. From study of Regiomontanus' works, he not only became well acquainted with Ptolemy's *Almagest* (and the problems it presented), but also was introduced to the works of Aristarchus. He had already begun making some astronomical observations in Warmia and continued to make others in Bologna to test Ptolemy's theories. After three years of study, he returned to Warmia to duties within his uncle's court only to be shortly sent back, this time to Padua, to study as a physician. It should not surprise you to learn that in addition to his astronomical library, Copernicus also built a significant library of medical texts.

Copernicus never ventured to Italy again. He spent years making astronomical observations at night as he spent his days serving as secretary and physician to his uncle (and subsequent Prince-Bishops of Warmia), traveling in diplomatic delegations, assisting with the general political administration of Warmia, and offering medical consultations when needed. He wrote policy on monetary reform and only published one other book within his lifetime, a translation into Latin of 85 Byzantine poems.

He did, of course, write out his heliocentric model, initially in a small pamphlet written shortly after his return from Italy which he only circulated among close friends and never intended for publication. Following this, he made some key astronomical observations and calculations to better refine his model before writing his major work, *De revolutionibus orbium coelestium* (On the Revolutions of the Heavenly Spheres). The calculations he made helped him confirm the accuracy of the heliocentric model but also proved to be important to our story within a few short decades.

Because they were looking for predictable patterns, the Babylonians had worked out what is called the *synodic* or *apparent period* for each of the planets. This is simply the amount of time it takes for a planet to return to the same place in the sky. Copernicus used that knowledge to work out the sidereal periods of each of the planets. The sidereal period is the amount of time it takes for a planet to return to the same point in its orbit in space. Under the geocentric model, the sidereal period is very nearly insignificant but when the Earth is not in a fixed position, the sidereal periods of the other planets become much more important. Copernicus began with the sidereal periods of Mercury and Venus because confirming that their sidereal periods (88 days and 225 days respectively) were less than the 365 day period of the Earth's orbit would confirm their positions between the Earth and the Sun. The position of the outer three planets was not questioned, but calculating their sidereal periods was not difficult and so the sidereal periods of all five planets were included in *De revolutionibus*.

In his earliest drafts of *De revolutionibus*, Copernicus credited Aristarchus for the idea of heliocentrism; however this was removed from later drafts and the work was published without reference to him. While Aristarchus clearly was a source of inspiration for Copernicus, the Copernican model was developed to a level of detail far beyond what Aristarchus has laid out seventeen centuries before, and Copernicus had poured into it not only his own observations and calculations but also the ideas of many others he had spent his lifetime collecting. Despite his work being well-received by those close to him, he was reluctant to bring it to print for fear of facing scorn. Near the end of his life he reluctantly agreed to give the manuscript to a printer in Nuremberg for publication. Legend states that the final printed pages were placed into his hands on his deathbed and that he looked upon them peacefully before promptly departing the mortal world.

Initially, *De revolutionibus* was not met with scorn so much as with indifference. Due to its technical complexity and density, the first print run of 400 copies did not sell out. While a handful of mathematicians and astronomers embraced it enthusiastically many continued to rely on the Almagest for key astrological computations. Heliocentrism carried a whiff of heresy throughout the sixteenth century, but it was not until 1616, some seventy years after its publication, that *De revolutionibus* gained the dubious honor of being put on the Catholic church's heretical forbidden book list. Handily, those seventy years were precisely enough time to influence another brilliant mathematician who star was on the rise.

If you have ever taken a basic physics class, you will have learned that the German mathematician Johannes Kepler developed three fundamental laws of planetary motion in the early 17^{th} century. These laws are:

1. The orbit of a planet is an ellipse with the sun at one focus of the ellipse.

2. A line drawn between a planet and the sun sweeps an equal area within the ellipse in an equal period of time.

3. The square of the time it takes a planet to complete one rotation is directly proportional to the cube of the semi-major axis of the orbital ellipse.

Given what was already known from observation of the heavens, these principles probably seem far from obvious. What is also not obvious in the statement of these laws is that their discovery was not simply a matter of dry mathematical calculation but rather an astonishing tale of the fates of princes and kingdoms, thefts, betrayals, riches nearly beyond imaging, and an unlikely and too brief collaboration between two men who seemed to have virtually nothing in common. We have as central figures in our tale a wealthy and landed nobleman who seemed destined for high political leadership; and a promising young scholar of the newly burgeoning middle class. One considered a geocentric model the only sensible possibility; and the other was hopelessly smitten with the ideas of Copernicus and Aristarchus. One had a perfectly functional nose; and the other famously had the dubious distinction of having lost his in a duel over his mathematical prowess. Standing beside the humble Johannes Kepler we have his challenging collaborator, the brilliant and extravagant Danish astronomer Tycho Brahe; but we must begin somewhere and that is with Kepler himself.

Poor Johannes Kepler knew from an early age that what he wanted to do more than anything else was to, by his works, glorify God. He was born in 1571 to an innkeeper's daughter and a mercenary who had the misfortune of dying while Kepler himself was still a small child. He had a modest education, beginning with Latin grammar school, moving on to seminary and from there to the renowned theological school at the University of Tübingen in Germany where he studied with the intent of becoming a Lutheran minister. Kepler had an undeniable prowess for mathematics, apparent even in childhood, and at it was at Tübingen that this aptitude helped shaped his future. He studied under the instruction of the renowned mathematician and astronomer Michael Mästlin; and while Mästlin dutifully instructed his students in the Ptolemaic model of the heavens, he also introduced Kepler to Copernicus' daring new heliocentric model of planetary motion. During his time at university, Kepler embraced Copernicus' model and argued that it was not only the most reasonable theoretically, but also the one that best harmonized with theological constraints imposed by the Bible.

Upon graduation from Tübingen, Kepler sought a position within the Lutheran church; but unfortunately for Kepler's rather humble dream, his spectacular mathematical skill and the faintest whiff of heresy in his unorthodox writings led his professors to instead recommend him for a position teaching mathematics and astronomy at a school in the far away city of Graz, now in modern-day Austria.

Kepler was not particularly happy to leave the familiarity of Tübingen, but he dutifully taught mathematics, astronomy, rhetoric, and Virgil at Graz, and despite having a generally low regard for astrology as the "foolish little daughter of the respectable mother astronomy", produced several *Calendaria et prognostica* (horoscope calendars) during his years there. With an insight that likely had more to do with his mathematical abilities than the course of the stars, Kepler was able to make several predictions that came true within short order and quickly brought him some level of professional prestige and unexpected income.

It was in Graz, however, that he had an epiphany which led him to create a truly beautiful model of the solar system. Kepler already knew that the planets moved in some regular fashion, and wanted to fit this movement into an orderly progress about the Sun. He first tried to create a system in two dimensions, with the circular orbit of each planet layered between regular polygons. When he could not make this system align with observation, he expanded it into a third dimension, with the orbit of each of the six (known) planets arranged on the surface of a sphere that was itself nested inside one of the five *Platonic solids*.

The Platonic solids are extremely symmetric polyhedra, where each face is the same shape and size, and all the edges meet at the same angle. The first of these is the *tetrahedron*, where each of the four faces is a triangle. The next of these is the *cube*, which you probably already know has six square faces. In fact, if you are of the geekier sort, you may find the three remaining solids, the *octahedron* (eight faces), the *dodecahedron* (twelve faces), and the *icosahedron* (twenty faces) to be very familiar, as they (along with one non-Platonic 10-sided polyhedron) form the full set of dice for some popular table-top roleplaying games.

You can likely imagine just how deeply satisfying it would be to place a crystalline sphere for Saturn's orbit upon a delicate pedestal, gently set a perfect cube inside only to then open the cube, place another crystalline sphere inside for Jupiter's orbit and so continue all the way down until in the center of it all you place the golden orb of the Sun. Here was perfect symmetry, the wisdom of the ancient Greeks, and the loving hand of God himself all made manifest in stunning harmony. Kepler was understandably very excited about this. His first action was to write this new hypothesis down in a treatise he called the *Mysterium cosmographicum* (Cosmic Mystery). He reached out to his old professor Mästlin to help him publish the work and then, naturally, sent an unsolicited copy to virtually every influential astronomer and notable prince (and potential patron) in Europe. Fortunately for Kepler, printed works were still relatively difficult to come by, and spam hadn't been invented yet, so *Mysterium cosmographicum* was not only read but was in fact fairly well-received in most circles, and led to some wonderful correspondence with various luminaries of the age.

While recognition and praise for a job well done is tremendously satisfying, it rarely pushes science forward. So it should be unsurprising that it was not

Mysterium cosmographicum that fundamentally changed our understanding of the universe. Rather, it was the short and fairly congenial letter sent to Kepler by Tycho Brahe that said (heavily summarized) "You're wrong.".

Tycho Brahe, some twenty-five years Kepler's elder, was already a titan in the field of astronomy when he and Kepler began their correspondence. Born into the highest and most politically connected of Denmark's noble families, he was expected to study law and follow in the footsteps of his many relatives who had been Privy Councilors to the Danish king. He was sent to the University of Copenhagen as a teen and found himself fascinated by the study of astronomy, in no small part due to having the chance to observe a solar eclipse that occurred a day later than predicted. At the age of sixteen, he observed what is called a *Great Conjunction* - a conjunction of the planets Jupiter and Saturn - and noticed that the planetary tables predicting the event were off by two days from Copernicus' model and by a whole month from the more accepted Ptolemaic model. The young Brahe concluded that the problem with the models was a lack of observational data, and he began recording his observations of the planets and stars while still at university.

While his family encouraged him to travel and hoped he would be similarly entranced by law and politics, Brahe only fell further into scientific studies, adding alchemy and medicine to his areas of interest. Eventually resigning themselves to his fate, Brahe's family supported him in building his first observatory and laboratory on his uncle's estates at the dissolute Herrevad Abbey. It was here that Brahe observed a supernova and published his observations in his first significant work, *De stella nova* (The New Star). His new discovery brought him significant attention, and with it the freedom to lecture on astronomy, visit other observatories in Europe, and generally live the kind of flamboyant life that Brahe seemed to favor (his infamously missing nose was long gone, having been cut off during his university days in a drunken duel with a cousin at a holiday engagement party). Brahe made good use of his travels to recruit tradesmen and artisans for King Frederick II of Denmark to help build a new palace at Elsinore[1]. Upon his return to Denmark, the king, still wishing to keep Brahe as a key courtier, was quite keen to give him one of Denmark's more influential estates to rule over. Brahe demurred, preferring not to be distracted from his scientific studies by politics. When Frederick II realized that Brahe was planning to leave Denmark entirely to continue his studies, he offered him the island of Hven instead and sufficient funds to build whatever facilities he felt he needed to pursue research there.

Sandwiched between Denmark and Sweden and within easy sailing distance of Copenhagen, Hven had historically been a small freehold of farmers and boasted a single fishing village and one prominent church. As Lord of Hven, Tycho Brahe set about building not one, but two of what he called "observatories" (and what we would call "palaces"), along with a robust infrastructure to support them. After establishing a printing press to allow him to publish and

[1]Yes, *that* Elsinore, and there is some strong evidence than the moody and entirely too clever for his own good Danish student and princeling we call Hamlet was in part inspired by Tycho Brahe.

distribute his scientific works, Brahe even went so far as to build his own paper mill, growing frustrated with seemingly endless paper shortages at his press and having to send men out for months at a time to scour northern Europe for a reliable source of paper. He threw lavish parties, hosted many dignitaries for the Danish crown and spent astonishing sums of the king's money on research. He attracted a number of intellectuals and scientists to assist him in the development and construction of complex astronomical instruments and in the rigorous observation and recording of the movement of the planets, stars, and other significant celestial bodies in his observatories Uraniborg and Stjerneborg, as well as other alchemical and botanical experiments. Ever one to keep his royal patron content, Brahe also carefully composed lavish nativity horoscopes for the birth of each prince and princess of Denmark, as well as regular astrological calendars filled with thoughtful insights into politics, economics, and military movements for his king. As strange as it seems to us now, it was the value of Brahe's astrological work and prognostication that drove King Frederick II to invest so consistently and heavily in Brahe's scientific complex. At its height, Frederick II was spending approximately 2% of the GDP of Denmark - about the same percentage that the United States spent on the Apollo Moon landing project! - on the facilities at Uraniborg. Brahe used that wealth to not only invest in infrastructure and instruments, but to cultivate experimentation and publication among the promising students of the University of Copenhagen and others from Germany with contracts much like our modern post-doctoral research programs; he maintained close contact with his researchers as they dispersed across Europe and advanced their own careers further. He sent stacks of books to the great Frankfurt Book Fair for distribution across Europe and produced lavishly bound copies both for Frederick and for other significant European princes. And, most significantly to our story, every night he and his assistants painstakingly recorded the position of various planets, stars, and comets as they moved across the sky for fifteen solid years.

In short, for a long time it was really good to be Tycho Brahe and an extraordinary opportunity to be able to visit and study at the bustling Uraniborg and witness the gears of science moving forward. Brahe envisioned that Uraniborg's observatories and laboratories and tireless research would long outlast him, carried forward under the purview of his young sons. Frederick II was happy to oblige him in that, even despite the mildly-troubling fact that Brahe had married a common woman, making his children not part of the nobility and unable to inherit his vast estates automatically as his heirs. Even Frederick II's death a decade or so after Uraniborg was established was not seen as a terrible setback, as Tycho Brahe had so many close relatives in the regency of the young king Christian IV that the future of his island seemed well assured. Once Christian IV took his throne, however, the young king showed that he was more interested in consolidating his power and limiting that of the Danish nobility. He also did not see the value in spending vast sums on scientific research that could instead be put into military spending for a country on the brink of war. While Tycho Brahe had charmed the nobility and many academics, he had not fostered a great love in the hearts of the native Hven families who had been put to work building his elaborate research center, and he had inadvertently run afoul of other researchers at the University of Copenhagen who resented his unprecedented access to resources and money. A handful of scandals and a conveniently timed riot of the common folk outside the door of Brahe's Copenhagen mansion gave the king an opportunity to confiscate the island and send Tycho Brahe frantically scrambling to pack as many of his instruments and as much of his research as he could and depart Denmark hastily before he lost even more.

Brahe was forced into exile with nothing of particular value beyond his family and a large catalog of unpublished astronomical data. As one does as exiled nobility, he spent the next couple years shuffling from castle to castle across Europe, relying on the hospitality of any friend of sufficient rank and wealth to support his entourage in suitable fashion. As hope of reconciliation with Christian IV faded, Brahe called in every favor and reached out to every contact trying to find a connection to a new patron who could support his continued research and enable him to publish his precious astronomical data.

It was in the midst of this uncertainty that Johannes Kepler's *Mysterium cosmographicum* and accompanying letter reached Brahe. He read the book with interest but it had the misfortune of arriving from the courier with another short work by the German mathematician Nicolaus Reimus Ursus. Brahe was well-acquainted with Ursus — they had a bitter rivalry that went back to Ursus' brief studies at Uraniborg. Ursus, also highly gifted mathematically, had little formal education as he had been born into the peasantry and worked for many years as a pig herder. He had fought to be taken seriously as a mathematics student and his time at Hven was cut short when another of Brahe's assistants caught him frantically copying down astronomical data and smuggling his copies in his pants. He'd taken Tycho Brahe's novel *geo-heliocentric model* of the universe, where the sun and moon revolved around the earth and the remaining planets revolved around the sun revolving around the earth, and published it as his own work. Now, in the introduction to the defense of geo-heliocentrism being Ursus' idea, Ursus had republished Kepler's accompanying *Mysterium* cover letter to him in which Kepler had praised Ursus lavishly.

Ever the diplomat, Brahe's letter back to Kepler was cordial and praised *Mysterium*, noting that it was very clever even if data he possessed showed it to be completely incorrect. He went so far as to invite Kepler to visit and discuss the ideas within the book before launching into a condemnation of Ursus. Kepler responded with a heartfelt apology for any insult that his praise of Ursus might have given. The two continued in correspondence on astronomical matters while Kepler taught at Graz and Brahe sought a patron, ultimately securing himself the position of court astronomer and mathematician to Rudolph II, the Holy Roman Emperor. Tycho Brahe was able to move his family to Prague where they were kept in the aristocratic style they were accustomed to. Rudolph II even granted him funds to construct a new observatory in Prague. It was from a position of much greater strength that Brahe was able to again reach out to Kepler and invite him to Prague to discuss the finer details of his *Mysterium* model.

For Kepler, the timing of this invitation could not be better. Religious unrest and sweeping militant Roman Catholicism were making Graz a very dangerous place for Protestants. Kepler had reached out again to his mentor Mästlin about the possibility of securing a faculty position at Tübingen, but there were simply no opportunities for as radical a thinker as Kepler was seen as in his homeland of Styria (in modern-day Austria) at that time. Faced with the strong possibility that his properties and household goods would be lost if he stayed in Graz, an extended visit in Prague offered him a reprieve from these earthly woes and a chance to focus on the puzzle that the heavens presented him.

While he arrived in Prague with a great deal of enthusiasm at the prospect of collaborating with Tycho Brahe, Kepler soon found the experience to be frustrating. During his time in Hven, Brahe had studied and heavily annotated Copernicus' works, and rejected Copernicus' heliocentric model in favor of his own geo-heliocentric model on the basis that his much more complete observational data was in contradiction to Copernicus' perfect circular orbits. Brahe had shared this concern with Kepler in his correspondence, and Kepler was eager to study Brahe's data and to find the eccentricities that highlighted the flaws Brahe saw in Kepler's model. He was disappointed to discover while there that Brahe, previously burned by Ursis and others, did not grant Kepler access to his catalog but rather would bring up a random astronomical point - "today something about the Apogee, tomorrow something about the knots of another planet" - at communal mealtimes.

Eventually, Brahe relented somewhat and suggested that perhaps Kepler would be content to study Mars, whose orbit with regular retrograde motions had been a particularly troublesome puzzle. Brahe himself had worked on the problem of Mars during his time in Hven and had not made satisfactory progress. Now he thought it a suitable challenge for the bright and highly motivated Kepler. Kepler was quick to agree, but because Brahe was so cautious with his data, Kepler estimated that it would take at least two years to sift through the great volume of material adequately to establish Mars' path. At this point, two very large egos got in the way of things and after heated disagreement, Kepler left Prague angrily and returned to Graz. It was some months before the two astronomers reconciled and a few months more found Kepler and his family exiled from Graz permanently and seeking a home and long-term contract in Brahe's service in Prague.

Now we would love to be able to tell you that what followed was a long and brilliant period of intense collaboration between these two astronomical giants whose mutual respect and driving curiosity lead to each to many wonderful discoveries, but life is inevitably messier than that. Kepler settled himself in to working on the problem of Mars' orbit, being cautiously fed one page of Brahe's observational data at a time. Kepler himself described Brahe as stingy with his data, lamented his inability to copy faster, and even went so far as to ask his mentor Mästlin to write to Brahe and ask for some data himself so that Mästlin could then share the precious intelligence back with Kepler. Meanwhile, Brahe, who preferred to always have several projects in the works at one time, proposed to Emperor Rudolph that he could prove his worth producing a new and much more accurate set of astrological tables. These *Rudolphine Tables* would chart the planetary motion through the constellations in great detail, allowing anyone who used them to draw up much more accurate horoscopes than they were able to using the current error-ridden charts that drew on Copernicus' scant data. With ongoing rumblings of war across Europe, Emperor Rudolph saw great value in better astrological predictions just as Frederick II had before him and was happy to fund compilation and publication of these tables by Brahe and his staff.

After some months of this routine of Kepler picking the Mars observations from the greater body of data page by page, Brahe became very suddenly and dramatically ill with a bladder ailment following a long formal banquet. He suffered great pain, fell into a feverish delirium, and spent his last days pleading with Kepler to not let all his work have been in vain, and making him promise to publish the Rudolphine Tables and promote Brahe's geo-heliocentric model over Copernicus' heliocentrism. Within two days of Brahe's sudden death in 1601, Kepler found himself appointed his successor as Rudolph's Imperial Astronomer and Mathematician. Rudolph announced his intention to purchase all of Brahe's instruments and research data and at last gave Kepler unrestricted access to Brahe's research so that he might finish up the Rudolphine Tables as he had promised Brahe on his deathbed.

Finally, Kepler felt he had everything he needed to sort out Mars' orbit and show that it agreed with the tidy circular orbits of the Copernican model. As he really dug into the data this time, he quickly ran into two significant problems. The first was that now that he had Brahe's observations of Mars laid out neatly before him, it was very apparent that the path that Mars took through the night sky did not fit the circular orbits he expected. The second problem was that while Rudolph was quite keen to keep Brahe's research materials, they did still technically belong to Brahe's heirs and the family was very much aware that their patriarch's life's work was not in their hands.

The responsibility for dispersing Brahe's estate ultimately fell to his son-in-law, Frans Tengnagel van de Camp. Tengnagel was a diplomat and minor noble who had come to Hven to research, and Brahe had used him as a prestigious agent and courier to deliver the most opulent copies of his publications to various dignitaries. He had married Brahe's daughter Elisabeth a few months before and travelled with his new bride to his family estates where she had given birth to Tycho Brahe's first grandchild only a month before Brahe's death. The following year Tengnagel and Elisabeth returned to Prague to find the family in desperate need of money and selling off what few copies of Brahe's published works they had to cover their living expenses. While Rudolph had promised a large sum of money, and interest on that sum, for Brahe's instruments and research materials, he had not only not paid Brahe's wife that money owed, but had fallen behind on paying Brahe's pension prior to his death. Tengnagel immediately set about putting things to rights and this included confronting Kepler and demanding Brahe's observational data back. Kepler, annoyed that he was in danger of again losing the precious data while the Mars problem remained unsolved, and generally scornful of Tengnagel's ability as an astronomer and his insistence on promoting Brahe's geo-heliocentric model, very grudgingly gave back almost all of Brahe's data. He did not expect Tengnagel to notice that all the information relating to Mars was conspicuously absent. Whatever Kepler thought of Tengnagel's astronomical abilities, Tengnagel was not stupid enough to overlook an entire planet, and what resulted was a spectacular squabble over Brahe's intellectual property rights that any modern movie studio would find uncannily familiar. Both sides incessantly demanded more money from the other, mostly due to the fact that Emperor Rudolph did a terrible job of paying either what he promised them. Kepler had little interest in the Rudolphine Tables while the Mars problem remained unresolved, Rudolph had no idea why Mars was at all important, and Tengnagel knew he could not complete and publish the Rudolphine Tables without the Mars data. He reluctantly came to realize that he could not complete them without Kepler's astronomical and mathematical knowledge as well. Kepler, meanwhile, was convinced that Tengnagel was hoarding Brahe's data in the hopes of making significant astronomical discoveries on his own and getting them to press himself before Kepler would be able to do so, and believed Tengnagel was deliberately throwing up roadblocks to Kepler's own work to slow him down. Suffice it to say that nobody got much of anything done for a couple years until finally Rudolph's Imperial Confessor, Johannes Pistorius, mediated a delicate agreement that gave Kepler access to all of Brahe's data in exchange for his assistance in completing the Rudolphine Tables for publication.

Kepler had now endured years of religious persecution, monetary woes, a complete inability to achieve recognition from the institution he loved most, and fights over data access and publication rights; but nobody except Kepler seemed to care very much that he still had not sorted out the problem of the Mars orbit. He had gone from being a darling of astronomical thought across Europe to somewhat minor court functionary of the Holy Roman Empire. At last he had unfettered access to the data he needed, but he was constantly hampered by a lack of time. The Emperor expected that Kepler would appear at court routinely, for at least an hour a day. Rudolph valued these astrological consultations despite Kepler's low regard for and general lack of interest in astrology. Kepler complained bitterly to far flung mathematical colleagues about his inability to get anything done and did not seem to suspect that he was in the midst of the most signficant work of his entire career.

In the advancement of not only science, but pretty much any endeavor, there seem to be two kinds of inspiration. The first strikes suddenly and unexpectedly, flowing through you intensely, leaving you shaky and teary-eyed and bubbling with energy and with a work that is astonishingly complete, beautiful, and unaccountably easy to bring forth. It was this kind of inspiration that Kepler had experienced in writing his *Mysterium cosmographicum* and it unquestionably seemed to come to him directly from God to his own humble soul for reasons he could not explain.

The other kind of inspiration is slow and gradual, such that you almost might not realize it growing within you. It is as though the problem before you is a chestnut, hard and unyielding and hammer as you might at it, it will not break open for you. However, if instead you take the chestnut and soak it, with time and patience you will find it gradually softens and softens until at last you lift it from the water and it falls open in your hand. Time grants you what force could not attain and you can easily examine the elements of value that were locked inside before. Mars was Kepler's chestnut and as frustrating as his time in Prague was, it was gradually softening the problem until he was able to tease it apart with a method so simple it could only be seen as genius.

As we have seen, since the observations of the Babylonians and Greeks, it was known that the planets lie (basically) in a single two-dimensional plane (the ecliptic). The Earth lies in this plane as well, which means that when looking at the planets in the night sky, every visible planet will appear to fall along a straight line. While a formal system of polar coordinates had not been defined yet, the ancient Greeks understood that to explain where in space something is, in two dimensions[2] you need only an angle (*ecliptic longitude* in astronomical terms) and a distance. The records on Mars' position in the sky from Copernicus and earlier astronomers gave ecliptic longitudes but were very sporadic, sometimes here, sometimes there, with large gaps between the observations. Brahe did what no one in the world had done before, recording the declinations and right ascensions of all the planets every single night (weather permitting) for fifteen consecutive years. This gave Kepler the ecliptic longitudes of Mars whenever it was visible, which he needed, but gave no distances, which was the other thing Kepler needed to define where Mars was in space. Kepler had another problem with this data. While Brahe firmly believed that the Earth was fixed as the center of the visible universe, Kepler's heliocentric model meant that all the observations of Mars from the earth were from a moving vantage point. It didn't really help that Kepler didn't know what the orbit of the Earth was either.

Kepler was stuck with two unknowns, the Earth's orbit and Mars' orbit and, at best, half a piece of data, the ecliptic longitude information, which gave the relative positions between the Earth and Mars for fifteen years. With too many unknowns and not enough equations it seemed like an impossible problem to solve. Kepler also realized that he would not be able to work out the orbit of Mars until he had figured out the orbit of the Earth. The earth's motion kept changing our vantage point for observations of Mars so without knowing where in space we were, we couldn't figure out where another moving object was relative to us.

In between trundling off to give Emperor Rudolph his daily horoscope, Kepler came up with the ingenious idea that the way to get himself unstuck in this problem was, ironically, to stick something else in place. The sun was already a fixed point in his system and with a second fixed point, he would be able

[2]In three dimensions, you need two angles (traditionally given by astronomers in *equatorial coordinates*, with the two angles known individually as *declination* and *right ascension*) and a distance to completely specify a location, but knowing that all the planets lay in a single plane allowed one to replace these two angles with a single angle, namely the ecliptic longitude which was a certain combination of the declination and right ascension that took into account the tilt of Earth's axis of rotation.

to figure out the position of the earth. In mathematics, the *angle-angle-side theorem* states that if you know two angles of a triangle and the side next to them, you know the entire triangle, and can therefore work out the location of one vertex relative to the other two. This technique, known as *triangulation*, was known since Phoenician times and is used today in our GPS systems. In Kepler's system, with the sun fixed and the earth moving, the only remaining object in this system that he could easily fix in place was Mars itself.

To fix Mars in place, Kepler needed some information he knew had already been worked out by Copernicus. The Babylonians had found the amount of time that it took Mars to return to the same place in the sky, called the *synodic* or *apparent period* (it appears in the same place relative to the earth but the earth has also moved in that time). This is 780 days. Copernicus knew that the earth went around the sun once every 365 days, which meant that the angular velocity of the earth was 1/365 rotations/day and the difference in angular velocity between the earth and Mars was 1/780 rotations/day. By subtracting the difference in angle of velocity of the two planets from the earth's angle of velocity, Copernicus arrived at 687 days as the sidereal period of Mars, or the actual time it took Mars to orbit the sun and return to the exact same point in space. Kepler likely did not say "Bingo!" because that word hadn't been coined yet, but this was precisely what Kepler needed. Bingo!

Kepler now only needed to look at Mars' position relative to the earth in intervals of 687 days, something Copernicus was never able to do because of the sporadic nature of the observations of Mars in his day. Kepler, however, had Tycho Brahe's fifteen years of consistent planetary observations. From his other calculations related to what was to become his second law, Kepler already suspected that the orbits of the planets were not circular and that the Sun was not in the center of the orbit. He had put quite a lot of effort into both an elliptical shape, which he then set aside as being too computationally complex, and an egg shape, which just did not work at all. To define the shortest path as an ellipse rather than a circle, Kepler would need a minimum of four defined points. Brahe's years of data would have ideally yielded approximately eight data points, but between Mars not being visible at night some portion of the year and the sky above the island of Hven not being perfectly cloudless at all times, Kepler ended up with five fixed points to give a good approximation the orbit of the Earth and produced an orbit that was just very barely elliptical. By picking another fixed position for Mars, he could produce a new set of points and further refine Earth's orbit, confirming that these points fell upon the ellipse he had found. He continued in this manner (complaining bitterly about having to repeat his calculations some seventy times) until he had reasonably defined the Earth's orbit. Kepler could then use the same method, fixing the Earth at one point along its orbit around the Sun, to define Mars' orbit also. The orbit of Mars, very helpfully, is the most eccentric (which is to say, the least circular) elliptical shape which made Kepler's computational results very clear.

Kepler didn't have an army of assistants to do the mathematical calculations for him and even he found the repeated calculations to be tedious, so he took the elliptical orbits of the Earth and Mars and extrapolated out that all orbits are elliptical in shape and that the Sun is located at one of the two focii of the ellipse. Kepler's lifelong love of geometry meant that he intuitively understood various properties of the ellipse, but in case you have forgotten or never knew, the focii are two unique points in the center of the ellipse where the sum of the distance from both points to any point on the edge is always the same. If you think of a circle as a kind of ellipse where both focii are at the same point, precisely in the circle's center, then you can hopefully see how it made perfect sense to Kepler that if the Sun would be in the center of a circular orbit, it must be at one focus of an elliptical orbit.

This idea of all planets having an elliptical orbit with the sun at one focus became Kepler's first law of planetary motion and was published in his book *Astronomia nova* (The New Astronomy) together with his second law, which he had already worked out. The second law relates to the speed at which planets seemed to move and to reach his conclusion, Kepler again had to go to the observational data. If the position of the Sun was averaged out across observations, it suggested that the Sun (really the Earth) moved at a constant velocity, however if the actual position of the Sun in the sky at each observation was considered independently, it opened up the possibility that the planetary motion was not uniform. Again, Kepler made many calculations and, as he himself noted, a considerable number of mistakes which seemed to miraculously cancel each other out. He arrived at a conclusion that the velocity of the a planet around the Sun changes, moving fastest when it is nearest the Sun and slowest when it is farthest away. Since a planet is always the same distance from the center of a circle (where Kepler believed the Sun to be), this was what gave him the first thought that perhaps the orbits of the planets were not circular at all. He realized that since this change in velocity was inversely proportional to the distance from the Sun, that he should be able to measure the change in planetary position over a set amount of time and add up all those pie-slice areas into a whole orbit to figure out the orbital shape. If you are well-acquainted with calculus, this will look very much like integrating around the ellipse, and that is, in fact, how close Kepler came to developing calculus to solve this problem. When the calculations became unwieldy enough that he felt it necessary to set the problem aside (and ultimately had to rediscover the ellipse through the triangulation method), he moved forward with the conclusion that at least you could define the change in velocity by noting that an equal area of the orbit shape was covered in an equal time. This second law of Kepler is a bit harder to visualize than the first, but imagine you have baked an elliptical pie and you cut it into twelve equal pieces (taking the same amount of time to cut each slice, of course), with the tip of each slice ending at one focus (where the Sun is). If you arrange the pie slices on another plate, some will be short and fat and others will be long and skinny and when you offer them to twelve children, a spectacular fight will undoubtedly break out over who got the biggest piece no matter how many times you assure them that each piece is perfectly equal.

Interestingly enough, Kepler's second law was not the only time he nearly invented calculus to solve a problem and the other occasion he did so nicely highlights what an incredibly useful tool calculus is and how inevitable its development was for humanity. Kepler was quite keen to marry again following the death of his first wife and when he found a suitable young lady, elaborate wedding festivities were enthusiastically planned. In the process of buying wine for the wedding feast, Kepler was horrified by the typical winesellers' method of calculating how much wine was in a barrel to figure out its price. The winesellers would insert a stick through a small hole in the side of the barrel all the way to the opposite corner of the barrel lid and then based on how much of the length of stick fit in the barrel, give a rough calculation of the volume of wine. Kepler was convinced that the shape of the barrel affected the volume of wine it could hold (which it does) and set out to calculate what barrel shape would hold the largest volume of wine. To figure this out, Kepler sort of brute-forced an integration of circular barrel slices across the length of the barrel to determine what barrel dimensions would hold the least and most wine. To his embarrassment, Kepler discovered that the barrel shape the wine merchants used was actually the one that held the most possible wine, so after his lengthy calculations he went back and purchased a sufficient amount of wine. By all contemporary accounts, the wedding feast was lovely and Kepler and his new wife enjoyed a happy marriage and loving family that he greatly treasured the remainder of his life.

Duties at the the Imperial Court and general lack of funds meant that it took Kepler some years to publish *Astronomia nova*, but he continued his work of trying to make sense of the universe along with shuffling from city to city looking for a nobleman willing to make good on the salary the Holy Roman Emperor owed him. In 1619, Kepler published a vast treatise called *Harmonices mundi* (The Harmonies of the World). It is within these pages that we find buried what we call his third law, that the squares of orbital periods of the planets are proportional to the cube of their average distance from the Sun. If this seems like a somewhat odd relation to have found, it is worth noting that this result was not the central point of *Harmonices mundi* at all. While the "harmony" part immediately brings to mind music and *Harmonices mundi* does have some sections that jump through rather elaborate hoops to try to relate astronomical proportions to musical intervals, the "harmony" Kepler wrote of referred more to an order, pattern, and aesthetic sense of content that he very firmly believed the universe must have, being the masterful work of a thoughtful God. The entire book is really something of a love-letter to geometry. Kepler works very hard to begin with first geometrical principles and slowly build to a mathematical model for movements of the planets. Along the way, he not only considers what music the planetary orbits should make, but the mathematical theory of how they exert an astrological impact on the souls and even the weather of Earth. The early chapters of *Harmonices mundi* are one of the first mathematical works ever written on the subject of *tessellations* - ways to tile a plane with copies of one or more shape in a repeating pattern. In addition, the book goes into considerable depth regarding polygons (2-dimensional multi-sided shapes) and polyhedra (3-dimensional multi-sided shapes) and his thoughtful and detailed consideration on these subjects calls back to his excitement over the Platonic solids and their importance in the arrangement of the solar system. It is unsurprising that in the writing of this work, Kepler put a great deal of time looking at every proportional relation he could think of to see if there was some significance there, and thus his third law emerged with dozens of other geometrical siblings of rather less note.

In fact, none of what we consider Kepler's most significant works today received much acclaim in his lifetime. Most other astronomers, even his good friend Galileo Galilei, regarded the three laws and the mathematical model of our solar system they make possible them to be only interesting curiosities about the heavens. Kepler continued to be plagued by a lack of funds and political unrest, fleeing one town after another to avoid persecution for his staunch Lutheranism and trying to collect the salary owed him from no less than three Holy Roman Emperors. He became so desperate for support that he finally threw all his efforts into getting Tycho Brahe's Rudolphine Tables published, some 15 years after Emperor Rudolph had died. It was this work that generated considerable excitement as the stunning accuracy of the tables enabled other astronomers to put it to immediate use. Now they could write what they felt were much better horoscopes making sense of the chaos that was then sweeping across Europe.

Kepler himself spent his last years again writing horoscopes for a steady source of income. Though he had radically reshaped our model of the heavens, he went to his grave believing that if his beloved nested Platonic solids and crystalline spheres did not literally hang in space, that their proportions were laid across the skies, fundamentally shaping our solar system. Other astronomers and mathematicians quietly mourned his death but it was not until an ambitious Englishman named Isaac Newton set about proving Kepler's principles that the world began to realize that Kepler had given us something of far more gravity than even Tycho Brahe's observational masterpiece.

Mathematical notes

The modern mathematical description of the solar system models of Ptolemy, Copernicus, Brahe, and Kepler is greatly assisted by the concept of *Cartesian coordinates*, introduced by René Descartes in 1637, several decades after the works of Brahe and Kepler. In this system, every location in a plane such as the ecliptic is represented by a pair (x, y) of real numbers x, y, which measure the displacements (in some fixed unit of length, such as kilometers, miles, or the Astronomical Unit) from some arbitrarily fixed origin $(0, 0)$ in this plane with respect an arbitrarily chosen pair of perpendicular axes. Such a pair of numbers (x, y) is also known as a (two-dimensional) *vector*.

The heliocentric model of Copernicus is the easiest to describe, if one assumes perfectly circular orbits centered at the Sun. In this model, the Sun is placed at the origin $(0, 0)$ of the ecliptic plane. The Earth moves in a constant circular motion in this plane around the Sun; mathematically, its location at any given time t can be described using modern trigonometry by the formula

$$(r_E \cos(\omega_E t + \theta_E), r_E \sin(\omega_E t + \theta_E)) \tag{4.1}$$

at a given time t, where r_E is the radius of the circular orbit (that is, the distance from the Earth to the Sun, which is the *Astronomical Unit*), ω_E is the *angular velocity*, and θ_E is a phase offset that is of little physical significance (it depends on the choice of axes in the Cartesian coordinates). For sake of discussion we shall assume that the angular velocity ω_E is positive (corresponding to counterclockwise circular motion) rather than negative (corresponding to clockwise circular motion). Trigonometry tells us that the sine and cosine functions are periodic with period 2π:

$$\cos(\theta + 2\pi) = \cos(\theta); \quad \sin(\theta + 2\pi) = \sin(\theta). \tag{4.2}$$

From this and a little algebra, we see that whenever the time t is advanced by an amount $\frac{2\pi}{\omega_E}$, then the circular motion (4.1) will return to where it started, and the Sun will appear to return to its original position (with respect to the fixed stars) as seen from the Earth. This happens once a year (approximately 365 days); thus

$$\frac{2\pi}{\omega_E} \approx 365.25 \text{ days} \tag{4.3}$$

and so the orbital velocity of the Earth can be computed as

$$\omega_E \approx \frac{2\pi}{365.25 \text{ days}} \approx 0.017 \text{ days}^{-1}. \tag{4.4}$$

Similarly, the position of Mars in the Copernican model at a given time t would be given by the formula

$$(r_M \cos(\omega_M t + \theta_M), r_M \sin(\omega_M t + \theta_M)) \tag{4.5}$$

for some numbers r_M, ω_M, θ_M. The angular velocity ω_M of Mars required more effort to compute than ω_E, since all observations were from the perspective of the Earth rather than of Mars. But observe from (4.2) that whenever the phases $\omega_E t + \theta_E$ and $\omega_M t + \theta_M$ differ by a multiple of 2π, thus

$$\omega_E t + \theta_E = \omega_M t + \theta_M + 2k\pi$$

or equivalently

$$(\omega_E - \omega_M)t + (\theta_E - \theta_M) = 2k\pi \tag{4.6}$$

for some integer k, then the position of Mars is a scalar multiple of the position of Earth:

$$(r_M \cos(\omega_M t+\theta_M), r_M \sin(\omega_M t+\theta_M)) = \frac{r_M}{r_E}(r_E \cos(\omega_E t+\theta_E), r_E \sin(\omega_E t+\theta_E)).$$

Geometrically, this means that the Sun, Earth, and Mars become in *opposition*, with Mars positioned on the exact opposite location to the Sun as seen from the Earth. From (4.6) we see that whenever one starts from an opposition, and then advances the time t by an amount $\frac{2\pi}{\omega_E - \omega_M}$, then one returns to another opposition (with k replaced by $k+1$, reflecting the fact that the Earth has "lapped"

Mars by one orbit). From ancient Babylonian observations these oppositions were known to occur once every 780 days (this is known as the *synodic* period of Mars), so

$$\frac{2\pi}{\omega_E - \omega_M} \approx 780 \text{ days}.$$

Combining this with (4.4) one can soon compute the orbital velocity of Mars in the Copernican model,

$$\omega_M \approx \frac{2\pi}{365.25 \text{ days}} - \frac{2\pi}{780 \text{ days}} \approx 0.0091 \text{ days}^{-1}$$

with the true period (also known as the *sidereal period*) of Mars is given by

$$\frac{2\pi}{\omega_M} \approx 687 \text{ days}.$$

Approximately 106 days before or after an opposition of Mars, one can observe that Mars is in *quadrature*, which means that it makes a right angle with the Sun when viewed from the Earth. Since the angle between Mars and Earth when viewed from the Sun is equal to zero at opposition, and completes a full revolution of 2π radians once every 780 days, at quadrature this angle becomes $\frac{2\pi}{780} \times 106 \approx 0.854$ radians (or about 49 degrees). From trigonometry, we can then compute the ratio r_M/r_E of the distances between Mars and Earth as

$$\frac{r_M}{r_E} = \frac{1}{\cos 0.854} \approx 1.52$$

and so Mars is about 1.52 Astronomical Units away from the Sun in the Copernican model. See Figure 4.1. More sophisticated calculations of this type can compute the ratio between the distances to Mars and Earth when Mars appears in other locations relative to the Sun than in quadrature. Kepler discovered that this ratio varied slightly at such locations, leading him to conclude that the Copernican model was not perfectly accurate; eventually he was able to use these sorts of calculations to work out a better description of the orbits of Earth and Mars, leading to his famous laws of planetary motion.

Tycho Brahe modified the Copernican model to an equivalent model, now known as the *Tychonic model*, which placed the Earth at the origin $(0,0)$ rather than the Sun, but nevertheless gave the same observational predictions as the Copernican model. To oversimplify[3] matters, the modification was simply to subtract the Copernican position (4.1) of the Earth from every object in the Solar System using the vector subtraction law $(a,b) - (c,d) := (a-c, b-d)$, thus for instance the Sun would now be placed at

$$(0,0) - (r_E \cos(\omega_E t + \theta_E), r_E \sin(\omega_E t + \theta_E)) \tag{4.7}$$

[3]Here we neglect a separate modification of the Tychonic model, which was to make the entire Solar System also rotate around the Earth once a day, instead of having the Earth rotate on its axis once a day.

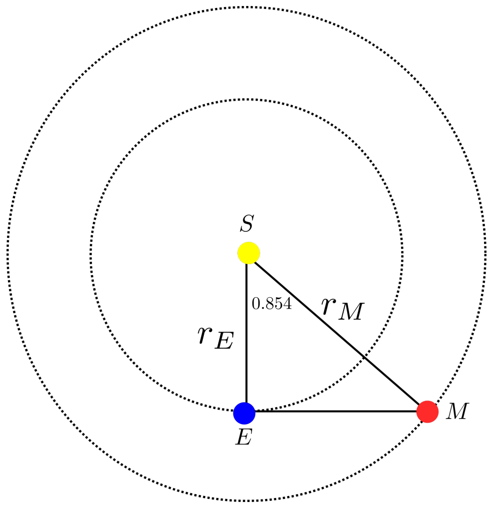

Figure 4.1: The relative positions of the Sun, Earth, and Mars at quadrature.

rather than at the origin $(0, 0)$, and Mars would now be placed at

$$(r_M \cos(\omega_M t + \theta_M), r_M \sin(\omega_M t + \theta_M)) - (r_E \cos(\omega_E t + \theta_E), r_E \sin(\omega_E t + \theta_E)). \tag{4.8}$$

In Tychonic model, the Sun would now revolve around the Earth in a perfect circular motion (4.7), while the position (4.8) of Mars is formed by superimposing an additional rotation (4.5) of Mars around the Sun.

One can also view the motion (4.8) in a different way: instead of starting with the Sun's position (4.7) and adding the original rotation (4.5) on top of it, one could start with the circular motion described by (4.5) (which is no longer occupied by any celestial body), and add the additional motion $-(r_E \cos(\omega_E t + \theta_E), r_E \sin(\omega_E t + \theta_E))$ on top of it; thus Mars would now be thought of as revolving around an uninhabited point in space which in turn orbits the Earth. The equivalence of these two perspectives can be viewed as a consequence of one of the most fundamental properties of vector arithmetic, namely the commutativity $v + w = w + v$ of vector addition.

The Ptolemaic model is similar to the Tychonic model, with the main difference being that the location of planets such as Mars is scaled by an additional positive scalar factor λ_M from the position (4.8) in the latter model, giving a location of the form

$$\lambda_M(r_M \cos(\omega_M t+\theta_M), r_M \sin(\omega_M t+\theta_M)) - \lambda_M(r_E \cos(\omega_E t+\theta_E), r_E \sin(\omega_E t+\theta_E)).$$

As the operation of multiplying a vector by a positive scalar does not affect its orientation, such a scaling would not affect predictions of where Mars would appear in the night sky. In the Ptolemaic model, a planet such as Mars would then rotate around an uninhabited point $\lambda_M(r_M \cos(\omega_M t + \theta_M), r_M \sin(\omega_M t + \theta_M))$

in space, which traced out a large circle known as a *deferent*. The smaller circle around this point that Mars moved along was known as an *epicycle*. There were additional complications to account for the cycle of day and night, and also Ptolemy ended up shifting some of the circles to not quite be centered at the Earth in order to try to match observational data more perfectly; as such, the model ended up being quite complicated, and eventually abandoned after Kepler's version of the Copernican model became accepted by astronomers. However, a basic insight of the Ptolemaic model still persists in modern mathematics, namely that any periodic motion can be described (at least approximately) by superimposing one or more circular motions atop each other, much as the epicycle motion was superimposed upon the deferent motion. Nowadays, this insight is captured by the modern theory of *Fourier series*, which we will not discuss here.

The three laws of Kepler were eventually demonstrated by Isaac Newton to be consequences of his famous laws of motion, and particularly the inverse square law of gravitation, which asserts that the gravitational force exerted on an object is inversely proportional to the square of the distance to that object. The full derivation of Kepler's laws from Newton's laws requires a certain amount of calculus and will not be reproduced in full here. Instead, we will mention just two aspects of the derivation. Kepler's second law can be derived from the law of conservation of angular momentum in Newtonian mechanics; this law makes an orbiting body spin faster as it gets closer to the center of its rotation, much as an ice skater spins faster when pulling her arms closer to her body; this increased speed compensates perfectly for the reduced amount of area covered by the orbit when it is close to the center, leading to the second law. Kepler's third law is a bit tricky to demonstrate in full, but one can anticipate it by the following back-of-the-envelope calculation. Suppose that a planet stays at a distance roughly R from the Sun, and takes a period T to perform a full orbit. The distance traversed by this orbit is then proportional to R, so the velocity of the planet is proportional to R/T. After a half-period $\frac{1}{2}T$ of the orbit, the velocity should reverse in orientation; to achieve this, the acceleration exerted on the orbit needs to be proportional to $(R/T)/(\frac{1}{2}T)$, which is proportional to R/T^2. On the other hand, the inverse square law asserts that the force exerted on the planet is proportional to $1/R^2$; by Newton's famous second law, the acceleration must also be proportional to $1/R^2$. We conclude that R/T^2 is proportional to $1/R^2$, which after some algebra implies that T^2 is proportional to R^3, which aligns with Kepler's third law.